TETRAGONAL MODULAR CURVES $X_1(M,N)$

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ABSTRACT. In this work, we determine all the modular curves $X_1(M,N)$ which are tetragonal.

0. Introduction

A smooth projective curve $X$ defined over an algebraically closed field $k$ is called $d$-gonal if it admits a map $\phi: X \to \mathbb{P}^1$ over $k$ of degree $d$. If the genus $g \geq 2$ and $d = 2$, then $X$ is called hyperelliptic. We will say that $X$ is tetragonal for $d = 4$.

For positive integers $M|N$, consider the congruence subgroup $\Gamma_1(M,N)$ of $\text{SL}_2(\mathbb{Z})$ defined by

$$\Gamma_1(M,N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N, M \mid b \right\}.$$ 

Then the modular curve $X_1(M,N)$ corresponding to $\Gamma_1(M,N)$ is related to moduli problems of elliptic curves containing a subgroup isomorphic to

$$\mathbb{Z}/M\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z}.$$ 

The author, Kim, and Park [9] get the determination of all the tetragonal modular curves $X_1(1,N)$ and $X_1(2,N)$ which play a central role in determining the structure occurs infinitely often as the torsion of the elliptic curves over quartic number fields.

In this paper, we finish determining all the tetragonal modular curves $X_1(M,N)$. It gives useful information for the torsion subgroups of the elliptic curves over the number fields of higher order.

For $X$ with the genus $g(X) \geq 2$ if one has a map $\phi: X \to C$ of degree 2 onto an elliptic curve $C$ (respectively a hyperelliptic curve $C$), then $X$ is called bielliptic (respectively $b$-hyperelliptic).

Our main result is as follows:

Theorem 0.1. The following are equivalent:

(a) $X_1(M,N)$ is tetragonal.

\[ \begin{array}{l}
\end{array} \]

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(b) $X_1(M, N)$ is rational, elliptic, hyperelliptic, bielliptic or bihyperelliptic. Explicitly these $(M, N)$ are:

- rational: $(1, N = 1 - 10, 12), (2, N = 2, 4, 6, 8), (3, N = 3, 6, 4, 4), (5, 5)$;
- elliptic: $(1, N = 11, 14, 15), (2, N = 10, 12), (3, 9), (4, 8)$;
- hyperelliptic: $(1, N = 13, 16, 18)$;
- bielliptic: $(1, N = 17, 20 - 22, 24), (2, N = 14, 16), (3, 12), (4, 12), (5, 10), (7, 7), (8, 8)$;
- bihyperelliptic: $(2, 18)$.

Remark 0.2. Since a rational curve has maps of degree 2 and 4 to a rational curve, rational and hyperelliptic curve are tetragonal. Since an elliptic admits a map of degree 2 to a rational curve, elliptic and bielliptic curves are also tetragonal. From the definition of a bihyperelliptic curve, we know that it is tetragonal.

1. Preliminaries

1.1. Modular curves $X_\Delta(N)$

Let $\Delta$ be a subgroup of $(\mathbb{Z}/N\mathbb{Z})^*$ which contains $-1$. Let $X_\Delta(N)$ be the modular curve defined over $\mathbb{Q}$ associated to the congruence subgroup

$$\Gamma_\Delta(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \mid a \in \Delta, N \mid c \right\}.$$ Note that for $\Delta = \{ \pm 1 \}$ this is just $X_1(1, N)$. From now on we denote $X_1(1, N)$ by $X_1(N)$.

Conjugating the group $\Gamma_1(M, N)$ with the matrix $\left( \begin{smallmatrix} 1 & 0 \\ 0 & \frac{1}{N} \end{smallmatrix} \right)$ we obtain a birational map from $X_1(M, N)$ to $X_\Delta(MN)$ with

$$\Delta = \{ \pm 1, \pm (N + 1), \pm (2N + 1), \ldots, \pm ((M - 1)N + 1) \}.$$ For $d|N$, let $\pi_d$ be the natural projection from $(\mathbb{Z}/N\mathbb{Z})^*$ to $(\mathbb{Z}/d, N/d)\mathbb{Z}^*$, where $\{d, N/d\}$ is the least common multiple of $d$ and $N/d$.

**Theorem 1.1** ([8]). The genus of the modular curve $X_\Delta(N)$ is given by

$$g(X_\Delta(N)) = 1 + \frac{\mu}{12} - \frac{\nu_2}{4} - \frac{\nu_3}{3} - \frac{\nu_\infty}{2},$$

where

$$\mu = N \cdot \prod_{p|N} \left( 1 + \frac{1}{p} \right) \cdot \frac{\varphi(N)}{|\Delta|},$$

$$\nu_2 = \left| \{ (b \text{ mod } N) \in \Delta \mid b^2 + 1 \equiv 0 \text{ mod } N \} \right| \cdot \frac{\varphi(N)}{|\Delta|},$$

$$\nu_3 = \left| \{ (b \text{ mod } N) \in \Delta \mid b^2 - b + 1 \equiv 0 \text{ mod } N \} \right| \cdot \frac{\varphi(N)}{|\Delta|},$$

$$\nu_\infty = \sum_{d|N \ \text{d > 0}} \frac{\varphi(d) \cdot \varphi \left( \frac{N}{d} \right)}{|\pi_d(\Delta)|}.$$
By using the above genus formula and the birational map from $X_1(M,N)$ to $X_{\Delta}(MN)$, one can calculate the genus of $X_1(M,N)$.

1.2. Abramovich’s bound

Let a smooth projective curve $X$ be $d$-gonal where $d$ is smallest for such $d$. Then $d$ is called the gonality of $X$ and is denoted by $\text{Gon}(X)$. The best general lower bound for the gonality of a modular curve seems to be the one that is obtained in the following way.

Let $\lambda_1$ be the smallest positive eigenvalue of the Laplacian operator on the Hilbert space $L^2(X_\Gamma)$ where $X_\Gamma$ is the modular curve corresponding to a congruence subgroup $\Gamma$ of $\Gamma(1)$. Let $D_\Gamma$ be the index of $\pm \Gamma$ in $\Gamma(1)$ and $d_\Gamma$ the gonality of $X_\Gamma$. Abramovich [1] shows the following inequality:

$$\lambda_1 D_\Gamma \leq 24d_\Gamma.$$ 

Using the best known lower bound for $\lambda_1$, due to Henry Kim and Peter Sarnak, as reported on page 18 of [2], i.e., $\lambda_1 > 0.238$, we get the following result.

**Theorem 1.2.** Let $X_\Gamma$ be the modular curve corresponding to a congruence subgroup $\Gamma$ of index $D_\Gamma := [\Gamma(1) : \pm \Gamma]$ and let $d_\Gamma$ be the gonality of $X_\Gamma$. Then

$$D_\Gamma < \frac{12000}{119} d_\Gamma.$$ 

In the following, we call the inequality in Theorem 1.2 Abramovich’s bound.

1.3. Clifford index

For a line bundle $L \in \text{Pic} X$, the *Clifford index* of $L$ is the integer

$$\text{Cliff}(L) := \deg(L) - 2(h^0(X, L) - 1)$$

and the *Clifford index* of $X$ itself is defined as

$$\text{Cliff}(X) := \min\{\text{Cliff}(L) \mid h^0(X, L) \geq 2, h^1(X, L) \geq 2\}.$$ 

It is well-known that $\text{Cliff}(X) + 2 \leq \text{Gon}(X) \leq \text{Cliff}(X) + 3$ [3].

1.4. Property $N_p$

If $X$ is a non-hyperelliptic curve, then the canonical line bundle $\omega_X$ defines an embedding $X \hookrightarrow \mathbb{P} H^0(X, \omega_X) = \mathbb{P}^{g-1}$. Consider the minimal free resolution

$$0 \to F_{g-2} \to \cdots \to F_2 \to F_1 \to S \to S_X \to 0$$

of the homogeneous coordinate ring $S_X = S/I_X$ as an $S$-module where $S = \mathbb{C}[X_0, X_1, \ldots, X_{g-1}]$ and $F_i = \bigoplus_{j \in \mathbb{Z}} S(-i-j)^{\delta_{i,j}}$. We call $\beta_{i,j}$ the graded Betti-numbers. Due to Green and Lazarsfeld[5], $X \hookrightarrow \mathbb{P}^{g-1}$ is said to satisfy property $N_p$ if the resolution is of the form

$$\cdots \to S^{\beta_{p,1}}(-p-1) \to \cdots \to S^{\beta_{2,1}}(-3) \to S^{\beta_{1,1}}(-2) \to S \to S_X \to 0.$$ 

Therefore property $N_1$ holds if and only if the homogeneous ideal is generated by quadrics, and property $N_p$ holds for $p \geq 2$ if and only if it has property $N_1$. 
and the $k^{th}$ syzygies among the quadrics are generated by linear syzygies for all $1 \leq k \leq p - 1$. Now we recall the following:

**Theorem 1.3** (M. Green and R. Lazarsfeld, Appendix in [4]). Let $X$ be a smooth non-hyperelliptic curve of genus $g \geq 3$. Then the canonical embedding $X \hookrightarrow \mathbb{P}^{g-1}$ fails to satisfy property $N_p$ for $p \geq \text{Cliff}(X)$.

Thus if the canonical embedding $X \hookrightarrow \mathbb{P}^{g-1}$ satisfies property $N_p$, then $\text{Cliff}(X) \geq p + 1$ and $\text{Gon}(X) \geq p + 3$.

2. Tetragonal curves

As we mentioned in Remark 0.2, all the rational, elliptic, hyperelliptic, bielliptic and bihyperelliptic curves are tetragonal. Using Theorem 1.1 one can get the following proposition.

**Proposition 2.1.** $X_1(M, N)$ has genus 0 or 1 if and only if $(M, N)$ is one of the 13 following ordered pairs:

- genus 0: $(1, N = 1 - 10, 12), (2, N = 2, 4, 6, 8), (3, N = 3, 6), (4, 4), (5, 5);$
- genus 1: $(1, N = 11, 14, 15), (2, N = 10, 12), (3, 9), (4, 8);$

Mestre prove that the only hyperelliptic modular curves $X_1(N)$ are $X_1(13), X_1(16), X_1(18)$. Ishii and Momose [7] asserted that there exist no hyperelliptic modular curves $X_\Delta(N)$ with $\{\pm 1\} \subset \Delta \subset (\mathbb{Z}/N\mathbb{Z})^*$. But the author and Kim [8] find one and only hyperelliptic modular curve $X_\Delta(21)$ with $\Delta = \{\pm 1, \pm 8\}$, and correct their result. Since $X_\Delta(21)$ is not birational to any $X_1(M, N)$, we can conclude that the only hyperelliptic modular curves $X_1(M, N)$ are $X_1(13), X_1(16), X_1(18)$.

The author and Kim determine all the bielliptic modular curves $X_1(M, N)$ as follows:

**Theorem 2.2.** The curve $X_1(M, N)$ is bielliptic only for the following $(M, N)$:

$(1, 13), (1, 16), (1, 17), (1, 18), (1, 20), (1, 21), (1, 24), (2, 14), (2, 16), (3, 12), (4, 12), (5, 10), (7, 7), (8, 8);$

Since there is a natural map $X_1(2, 18) \to X_1(18)$ of degree 2 and $X_1(18)$ is hyperelliptic, $X_1(2, 18)$ is bihyperelliptic.

In the next section, we prove that the modular curves $X_1(M, N)$ as mentioned above are the only tetragonal modular curves.

3. Non-tetragonal curves

Suppose $X_1(M, N)$ is tetragonal. Note that $X_1(M, N)$ is birational to $X_\Delta(MN)$. Applying Abramovich's bound to $X_\Delta(MN)$, we have the following:

**Lemma 3.1.** $X_1(M, N)$ is not tetragonal if $M > 10$ or $N > MN_M$ with $1 \leq M \leq 10$, where
\[ N_1 = 32, \; N_2 = 12, \; N_3 = 6, \; N_4 = 4, \; N_5 = 2, \; N_6 = 2, \; N_7 = 1, \]
\[ N_8 = 1, \; N_9 = 1, \; N_{10} = 1. \]

The author, Kim, and Park [9] prove that the modular curves \( X_1(N) \), \( X_1(2, 2N') \) are tetragonal only for \( N = 1 - 18, 20, 21, 22, 24 \) and \( N' = 1 - 9. \) Therefore we can cut out \( X_1(1, N), \; X_1(2, 2N') \) with \( N = 19, 23, 25 - 32 \) and \( N' = 10, 11, 12. \)

The remaining cases we don’t determine are \( X_1(M, N) \) for following \( (M, N) \):

\[ (3, 15), \; (3, 18), \; (4, 16), \; (6, 12), \; (9, 9), \; (10, 10). \]

To treat the above cases we need to compute the graded Betti numbers of the canonical embedding of \( X_\Delta(MN) \). We use the computer programs “Maple” and “Singular”. First we calculate the homogeneous ideal of the canonical embedding of \( X_\Delta(MN) \) by using Maple.

Note that, for such \( (M, N) \), \( X_\Delta(MN) \) is not hyperelliptic. Thus \( X_\Delta(MN) \) can be identified with the canonical curve which is the image of the canonical embedding

\[ X_\Delta(MN) \ni P \mapsto (f_1(P) : \cdots : f_g(P)) \in \mathbb{P}^{g-1}, \]

where \( \{f_1, \ldots, f_g\} \) is a basis of the space of cusp forms of weight 2 on \( X_\Delta(MN) \). One can get such a basis and their Fourier coefficients from [10]. Then to obtain the minimal generating system of the homogeneous ideal \( I(X_\Delta(MN)) \), we have only to compute the relations of the \( f_if_j \) (\( 1 \leq i, j \leq g \)) by Petri’s theorem. Since there are \( (g - 2)(g - 3)/2 \) linear relations among the \( f_if_j \), we get quadric generators \( Q_k(x_1, \ldots, x_g) \) with \( 1 \leq k \leq (g - 2)(g - 3)/2 \) by assigning \( x_i \) to \( f_i \) (for details see [6]).

Now we compute the Betti numbers by using Singular. In fact when the genus of \( X_\Delta(MN) \) is big then Singular doesn’t produce Betti numbers well. Note that since the canonical embedding is always projectively Cohen-Macaulay, the Betti numbers of the canonical curve are equal to those of the hyperplane section, which allows us to get Betti numbers easier.

We exhibit the so-called Betti-table of the canonical embedding for our cases in Table 1. All the cases satisfy property \( N_p \) for \( p \geq 2. \) Thus \( \text{Gon}(X_1(M, N)) \geq 5 \) by § 1.4 for \( (M, N) = (3, 15), (3, 18), (4, 16), (6, 12), (9, 9), (10, 10). \)

Table 1: The Graded Betti-numbers for the canonical embedding.

<table>
<thead>
<tr>
<th>genus</th>
<th>( X_1(M, N) )</th>
<th>( \beta_{1,2} )</th>
<th>( \beta_{2,2} )</th>
<th>( \beta_{3,2} )</th>
<th>genus</th>
<th>( X_1(M, N) )</th>
<th>( \beta_{1,2} )</th>
<th>( \beta_{2,2} )</th>
<th>( \beta_{3,2} )</th>
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<tr>
<td>9</td>
<td>( X_1(3, 15) )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( X_1(9, 9) )</td>
<td>0</td>
<td>20</td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td>21</td>
<td>64</td>
<td>70</td>
<td></td>
<td></td>
<td>28</td>
<td>105</td>
<td>162</td>
</tr>
<tr>
<td>13</td>
<td>( X_1(6, 12) )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( X_1(4, 16) )</td>
<td>0</td>
<td></td>
<td>0</td>
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<td>21</td>
<td>64</td>
<td>70</td>
<td></td>
<td></td>
<td>55</td>
<td>320</td>
<td>891</td>
</tr>
<tr>
<td>10</td>
<td>( X_1(3, 18) )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( X_1(10, 10) )</td>
<td>0</td>
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<td>0</td>
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References


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