WEIERSTRASS POINTS ON $\Gamma_0(p)$ AND ITS APPLICATION

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Abstract. In this note, we study arithmetic properties for the exponents of modular forms on $\Gamma_0(p)$ for primes $p$. Our aim is to refine the result of [4] by using the geometric property of the modular curve of $\Gamma_0(p)$.

1. Introduction and results

Let $\theta := \frac{1}{2\pi i} \frac{d}{dz}$. This operator is called as the Ramanujan theta operator and plays important roles in number theory. Let $N$ be a positive integer. Suppose $f(z)$ are modular forms on $\Gamma_0(N)$. When $N = 1$, Bruinier, Kohnen, and Ono studied in [3] the images of modular forms under the Ramanujan theta operator. They also provided a relation between the infinite product expansion of a modular form and the values of a certain meromorphic modular function at points in the divisor of $f$. These results were extended to modular forms on the genus zero congruence subgroups in [1] and [5]. The author obtained in [4] analogues of these results for modular forms on $\Gamma_0(p)$, where $p$ is a prime. In this note, our aim is to refine the result of [4] by using the geometric property of the modular curve of $\Gamma_0(p)$.

Let $p$ be a prime and $g$ be the genus of $\Gamma_0(p)$. Let $\mathbb{H}$ denote the complex upper half plane. A modular curve $X_0(p)$ is defined by

$$X_0(p) := \Gamma_0(p) \setminus \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}. $$

Note that $\infty$ is not a Weierstrass point on $X_0(p)$ whose genus is larger than 1. This implies that for each integer $m \geq g + 1$ there exists a unique modular function $j_{p,m}(z) = q^{-m} + O(q^{-g})$ that has its only pole at $\infty$ and a zero at the cusp 0 (see Section 2 for details). Let $l_\tau$ be the order of isotropic subgroup of $\Gamma_0(p)$ at $\tau \in \mathbb{H}$. The order of zero or pole of $f$ at $\tau \in \mathbb{H}$ is denoted by $\nu^{(p)}_\tau(f)$ and has the form

$$\nu^{(p)}_\tau(f) = \frac{1}{l_\tau} \text{ord}_\tau(f),$$

where $\text{ord}_\tau(f)$ denotes the order of zero or pole of $f$ at $\tau$ as a complex function on $\mathbb{H}$. With these notations, we state our first theorem.

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Theorem 1. Suppose that \( f(z) := q h \prod_{n=1}^{\infty} (1 - q^n)^{c(n)} \) is a normalized modular form of weight \( k \) on \( \Gamma_0(p) \) and that the genus of \( \Gamma_0(p) \) is larger than 1. Then
\[
\sum_{m > g+1} \sum_{\tau \in \mathbb{H}_p} \nu_\tau^{(p)}(f(z)) j_{p,m}(\tau) q^m
\]
is a meromorphic modular form of weight 2. Moreover,
\[
f_\theta(z) := \theta f(z) - k \frac{E_2(z)}{2} + \sum_{m > g+1} \sum_{\tau \in \mathbb{H}_p} \nu_\tau^{(p)}(f(z)) j_{p,m}(\tau) q^m
\]
is a cusp form of weight 2 on \( \Gamma_0(p) \).

Remark 1.1. The main difference of our result from [4] is the definition of \( j_{p,m} \). In [4], \( j_{p,m} \) is defined by the sum of eta-quotients. Following the definition of \( j_{p,m} \) in [4], we have that
\[
\sum_{m > g+1} \sum_{\tau \in \mathbb{H}_p} \nu_\tau^{(p)}(f(z)) j_{p,m}(\tau) q^m
\]
is not a modular form in general.

Let \( K \) be the number field and \( \mathcal{O}_K \) denote the ring of integers in \( K \). Using Theorem 1, we have the following congruence for the exponents of modular forms.

Theorem 2. Let \( f(z) := q h \prod_{n=1}^{\infty} (1 - q^n)^{c(n)} \in \mathcal{O}_K[[q]] \) be a normalized modular form of weight \( k \) on \( \Gamma_0(p) \) and \( \beta \nmid (p-1) \) denote a prime ideal of \( \mathcal{O}_K \). Suppose that \( f_\theta \) is \( \beta \)-integral, and that \( s \) is a positive integer, and that the genus of \( \Gamma_0(p) \) is larger than 1. Then for almost all \( m \) coprime to \( p \)
\[
\sum_{d \mid m} d \cdot c(d) \equiv \sum_{\tau \in \mathbb{H}_p} \nu_\tau^{(p)}(f(z)) j_{p,m}(\tau) \pmod{\beta^s},
\]
where \( c(n) := \sum_{d \mid n} d^k \).

Remark 1.2. In Theorem 2, we mean “almost all” in the sense of density (i.e.,
\[
x \sim \{0 \leq m \leq x \mid \sum_{d \mid m} d \cdot c(d) \equiv \sum_{\tau \in \mathbb{H}_p} \nu_\tau^{(p)}(f(z)) j_{p,m}(\tau) \pmod{\beta^s}\}).
\]

Remark 1.3. Our method gives no information on the first \( g \) coefficients of \( \frac{d}{dz} f_\theta \). Thus, from the argument of this note we can not obtain an analogue for the recursive relations of the Fourier coefficients of modular forms in [3], [1] and [5].
2. Prerequisites

Suppose that $p$ is a prime. The group $\Gamma_0(p)$ is the congruence subgroup of $SL_2(\mathbb{Z})$ defined as

$$\Gamma_0(p) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod{p} \}.$$ 

Let $\Gamma$ denote $SL_2(\mathbb{Z})$ and $F_p$ be a fundamental domain for the action of $\Gamma_0(p)$ on $\mathbb{H}$. We denote the set of distinct cusps as $S_p$,

$$S_p = \{ 0, \infty \}.$$ 

From now on, we suppose that if $t$ is a cusp point, then $t$ is in $S_p$. The period of $q$-expansion at $t$ is denoted by $\nu_t^p$, where $\nu_t^p$ is given by the following way:

$$\nu_t^p = 1 \text{ if } t = \infty \text{ and } \nu_t^p = p \text{ if } t = 0.$$ 

Adjoining the cusps to $\Gamma_0(p) \backslash \mathbb{H}$, we obtain a compact Riemann surface $X_0(p)$. For $\tau \in \mathbb{H} \cup S_p$, let $Q_\tau$ be the image of $\tau$ by the canonical map from $\mathbb{H} \cup S_p$ to $X_0(p)$.

Suppose $G$ is a meromorphic modular form of weight 2 on $\Gamma_0(p)$. The residue of $G$ at $Q_\tau$ on $X_0(p)$, denoted by $\text{Res}_{Q_\tau} Gdz$, is well defined since we have the canonical correspondence between a meromorphic modular form of weight 2 on $\Gamma_0(p)$ and a meromorphic 1-form of $X_0(p)$. If $\text{Res}_\tau G$ denotes the residue of $G$ at $\tau$ on $\mathbb{H}$, then for $\tau \in \mathbb{H}$ we obtain

$$\text{Res}_{Q_\tau} Gdz = \frac{1}{l_\tau} \text{Res}_\tau G.$$ 

Here, $l_\tau$ is the order of isotropy group at $\tau$. Especially, if $f$ is a meromorphic modular form of weight $k$ on $\Gamma_0(p)$ and $G = \theta f$, then the residue of $G$ at $Q_\tau$ on $\tau \in \mathbb{H}$ is computed with the order of zero or pole of $f$ at $\tau \in \mathbb{H}$. The order of zero or pole of $f$ at $\tau \in \mathbb{H}$ is denoted by $\nu_{\tau}^p(f)$ and has the form

$$\nu_{\tau}^p(f) = \frac{1}{l_\tau} \text{ord}_\tau(f),$$ 

where $\text{ord}_\tau(f)$ denotes the order of zero or pole of $f$ at $\tau$ as a complex function on $\mathbb{H}$. Then we have

$$2\pi i \cdot \text{Res}_{Q_\tau} \frac{\theta f}{f} = \nu_{\tau}^p(f).$$

We introduce some notations to formulate $\text{Res}_{Q_\tau} Gdz$ at every cusp $t$. First, recall the usual slash operator $f(z)|_k \gamma$ given as

$$f(z)|_k \gamma = \text{det}(\gamma)^\frac{k}{2} (cz + d)^{-k} f(\gamma z),$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$ and $\gamma z$ denotes $\frac{az + b}{cz + d}$. From now on, $q$ denotes $e^{2\pi i z}$. We define a matrix $\gamma_t^p$ as the following way:

$$\gamma_t^p := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \quad \text{if } t = 0,$$

$$\gamma_t^p := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{if } t = \infty.$$
If $G$ has the Fourier expansion of the form at each cusps

$$G(z) \mid_{2 \gamma_t^{(p)}} = \sum_{n=m_t}^{\infty} a_t(n)q^n \text{ at } \infty,$$

then we have

$$(2.2) \quad \text{Res}_{Q_t} Gdz = \frac{a_t(0)}{2\pi i} \text{ for } t \in S_p.$$  

Now, we recall the definition of Weierstrass point. Let $X$ be a compact Riemann surface with the genus $g$. At a given point $P$ of a Riemann surface $X$, genus $g$, we say that $m$ is a gap if no function exists with a pole of order $m$ at $P$ and regular elsewhere on $X$. It is known that there are just $g$ gaps at $P$ and that these satisfy $1 \leq m \leq 2g - 1$; moreover except for finitely many $P$, the gaps are just the integers $1$ to $g$. Those exceptional $P$ for which this is not so are called Weierstrass points of $X$. It is known that the point $\infty$ on $X_0(p)$ is not a Weierstrass point (see [8]). So, for each integer $m \geq g + 1$ there exists a modular function on $\Gamma_0(p)$ such that $\text{ord}_{\infty}(F_m(z)) = -m$ and that $F_m(z)$ is holomorphic elsewhere on $X_0(p)$. Using $F_j(z)$ for $g + 1 \leq j \leq m$, we can construct a modular function $G_m(z)$ on $X_0(p)$ satisfying the followings:

- $G_m(z) = q^{-m} + O(q^{-g})$,
- $\text{ord}_{0}(G_m(z)) \geq 1$,
- $G_m(z)$ is holomorphic on $X_0(p)$ except $\infty$.

Moreover, $G_m(z)$ is uniquely determined by its properties.

3. Proofs

We begin by stating a lemma which was proved by Eholzer and Skoruppa in [6].

**Lemma 3.1.** Suppose that $f = \sum_{n=h}^{\infty} a(n)q^n$ is a meromorphic modular function in a neighborhood of $q = 0$ and that $a(h) = 1$. Then there are uniquely determined complex number $c(n)$ such that

$$f = q^h \prod_{n=1}^{\infty} (1 - q^n)^{c(n)},$$

where the product converges in a small neighborhood of $q = 0$. Moreover, the following identity is true

$$\frac{\theta f}{f} = h - \sum_{n \geq 1} \sum_{d|n} c(d) dq^n.$$

**Proof of Theorem 1.** Let

$$F(z) = \frac{\theta f(z)}{f(z)} = \frac{k}{12} E_2(z) + \left( \frac{k}{12} - h \right) \frac{1}{p - 1} (pE_2(pz) - E_2(z)).$$
Here, \( E_2(z) \) is the usual normalized Eisenstein series of weight 2 defined by
\[
E_2(z) = 1 - 24 \sum_{n \geq 1} \sigma_1(n)q^n.
\]
The function \( F(z) \) is a meromorphic modular form of weight 2 on \( \Gamma_0(p) \) and has the \( q \)-expansion of the form
\[
F(z) = \sum_{n=1}^{\infty} a_F(n)q^n.
\]
Since \( \infty \) is not a Weierstrass point on \( X_0(p) \) for each integer \( v \), \( 1 \leq v \leq g \), there exits a cusp form \( \psi(z) \) of weight 2 such that \( \text{ord}_\infty(\psi) = v \) (see [7] or [2]).
So, we can choose a cusp form \( g(z) := \sum_{n=1}^{\infty} a_g(n) \) of weight 2 such that \( a_g(n) = a_F(n) \) for \( 1 \leq n \leq g \).
Let \( F'(z) = F(z) - g(z) \). Its Fourier expansion at \( t \in S_N \) is given by
\[
F'(z) = \frac{\theta(f(z))}{f(z)} \gamma_t^{(p)} + \frac{p(k - 12h)}{12(p - 1)} E_2(pz) \gamma_t^{(p)} - \frac{p - 12h}{12(p - 1)} E_2(z) \gamma_t^{(p)} - \frac{p(k - 12h)}{12(p - 1)} E_2(pz) \gamma_t^{(p)} - \frac{p - 12h}{12(p - 1)} E_2(z) \gamma_t^{(p)} - h(z) \gamma_t^{(p)}.
\]
Since \( F'(z)j_{p,m}(z)dz \) is a meromorphic 1-form on \( X_0(p) \), we obtain from (2.2) that
\[
2\pi i \text{Res}_{Q_v} F'(z)j_{p,m}(z)dz = -a_g(m) \left( \sum_{d|m} \varphi(d) \right) + \frac{2pk - 24h}{p - 1} \sigma_1(m) + \frac{24h - 2k}{p - 1} p \sigma_1(m/p),
\]
and that \( 2\pi i \text{Res}_{Q_v} F'(z)j_{p,m}(z)dz = 0 \) since \( \text{ord}_0(j_{p,m}(z)) \geq 1 \) and \( F'(z) \) is holomorphic at 0. Next we compute \( \text{Res}_{Q_v} F'(z)j_{p,m}(z)dz \) for \( \tau \in \mathbb{H} \). For each \( \tau \in \mathbb{H} \), we obtain that from (2.1)
\[
2\pi i \text{Res}_{Q_v} F'(z)j_{p,m}(z)dz = 2\pi i \text{Res}_{\tau} \frac{\theta(f(z))}{f(z)} j_{p,m}(z) = \nu_{\tau}^{(N)}(f) j_{p,m}(z)
\]
since \( E_2(z) \) and \( j_{p,m}(z) \) are holomorphic on \( \mathbb{H} \).
The residue theorem implies that
\[
2\pi i \sum_{Q_v \in X_0(N)} \text{Res}_{Q_v} F'(z)j_{p,m}(z)dz = 0
\]
since $X_0(N)$ is a compact Riemann surface. Thus, we have
\[
- \sum_{m \geq g+1} \sum_{\tau \in \mathcal{H}} \nu^{(p)}_\tau(f(z)) j_{p,m}(\tau) q^m
\]
\[
= F'(z) = F(z) - g(z)
\]
\[
= \frac{\theta f(z)}{f(z)} - g(z) + \frac{-kp + 12h}{12(p-1)} E_2(z) + \frac{-12h + k}{12(p-1)} pE_2(pz).
\]
Therefore, this completes the proof. \hfill \Box

To prove Theorem 2 we need the following proposition.

**Proposition 3.2** (Serre [9], Corollaire du Théorème 1). Let
\[
f(z) = \sum_{n=0}^{\infty} c_f(n) q^{n/M}, \ M \geq 1
\]
be a modular form of integral weight $k \geq 1$ on a congruence subgroup of $SL_2(\mathbb{Z})$, and suppose that the coefficients $c_f(n)$ lie in the ring of integers of an algebraic number field $K$. Then for any integer $m \geq 1$,
\[
c_f(n) \equiv 0 \pmod{m}
\]
for almost all $n$.

**Proof of Theorem 2.** Theorem 1 implies that
\[
g(z) := \sum_{m \geq g+1} \sum_{\tau \in \mathcal{H}} \nu^{(p)}_\tau(f(z)) j_{p,m}(\tau) q^m + \frac{\theta f(z)}{f(z)}
\]
\[
+ \frac{-kp + 12h}{12(p-1)} E_2(z) + \frac{-12h + k}{12(p-1)} pE_2(pz)
\]
is a cusp form. From the assumption the coefficients of $g(z)$ are $\beta$-integral. Applying Proposition 3.2 to $g(z)$, we complete the proof. \hfill \Box

**References**


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