GENERALIZATION OF A TRANSFORMATION FORMULA FOUND BY BAILLON AND BRUCK

ARJUN K. RATHIE AND YONG SUP KIM

Abstract. We aim mainly at presenting a generalization of a transformation formula found by Baillon and Bruck. The result is derived with the help of the well-known quadratic transformation formula due to Gauss.

1. Introduction

There was an open problem posed by Baillon and Bruck [1, Eq.(9.10)] who needed to verify the following hypergeometric identity

\[ 2F1 \left[ \frac{1}{2}, -\frac{m}{2}; 4x(1-x) \right] = (m + 1)(1-x)x^{2m-1}2F1 \left[ -\frac{m}{2}, -\frac{m}{2}; \left( \frac{1-x}{x} \right)^{2} \right] \]

\[ + (2x-1)x^{2m-1}2F1 \left[ -\frac{m}{2}, -\frac{m}{2}; \left( \frac{1-x}{x} \right)^{2} \right] \]

in order to derive a quantitative form of the Ishikawa-Edelstein-O’Brian asymptotic regularity theorem. Using Zeilberger’s algorithm [4], Baillon and Bruck [1] gave a computer proof of this identity which is the key to the integral representation [1, Eq.(2.1)] of their main theorem. In 1995, Paule [2] gave the proof of (1.1) by using classical hypergeometric machinery by means of the following contiguous relations:

\[ \frac{abz}{c(c-1)x}2F1 \left[ a+1, b + 1; \frac{c+1}{z} \right] = 2F1 \left[ a, b; \frac{c+1}{z} \right] - 2F1 \left[ a, b; \frac{c}{z} \right] \]

\[ \left( \frac{c+1}{a} \right) 2F1 \left[ a+1, b; \frac{c+1}{z} \right] = \frac{a-c}{a} 2F1 \left[ a, b; \frac{c+1}{z} \right] + \frac{c}{a} 2F1 \left[ a, b; \frac{c}{z} \right] \]
and the following well-known quadratic transformation formula [3] due to Gauss

\[ \left. \begin{array}{c}
2F_1 \left[ \frac{a}{2b+1} : \frac{4z(1-z)}{(1+z)^2} \right] = (1+z)^{2a} 2F_1 \left[ \frac{a-b+\frac{1}{2}}{b+rac{1}{2}} : z^2 \right]. 
\end{array} \right. \]

The aim of this short paper is to provide a generalization of (1.1) by employing the transformation formula (1.4).

2. Main result

The following a generalization of the result (1.1) will be established:

\[ 2F_1 \left[ \frac{a}{2b+1} : 4z(1-z) \right] = \left. \begin{array}{c}
z^{-2a} \left\{ 2F_1 \left[ \frac{a}{2b+1} : \frac{a-b+\frac{1}{2}}{b+rac{1}{2}} \right] \left( \frac{1-z}{z} \right)^2 \right. \\
+ \frac{2a(1-z)}{(2b+1)z} 2F_1 \left[ \frac{a+1}{b+\frac{1}{2}} \left( \frac{1-z}{z} \right)^2 \right] \left( \frac{1-z}{z} \right)^2 \right\} 
\end{array} \right. \]

3. Proof of (2.1)

In order to prove the main result (2.1), we proceed as follows. From (1.2), we obtain the following relation:

\[ 2F_1 \left[ \frac{a}{2b+1} : x \right] = 2F_1 \left[ \frac{a}{2b} : x \right] - \frac{ax}{2(2b+1)} 2F_1 \left[ \frac{a+1}{2b+2} : x \right]. \]

Now, put \( x = -\frac{4y}{(1-y)^2} \), we get

\[ 2F_1 \left[ \frac{a}{2b+1} : -\frac{4y}{(1-y)^2} \right] = 2F_1 \left[ \frac{a}{2b} : -\frac{4y}{(1-y)^2} \right] + \frac{2ay}{(2b+1)(1-y)^2} 2F_1 \left[ \frac{a+1}{2b+2} : -\frac{4y}{(1-y)^2} \right]. \]

Multiplying both sides of (3.2) by \((1-y)^{-2a}\), we get

\[ (1-y)^{-2a} 2F_1 \left[ \frac{a}{2b+1} : -\frac{4y}{(1-y)^2} \right] = (1-y)^{-2a} 2F_1 \left[ \frac{a}{2b} : -\frac{4y}{(1-y)^2} \right] + \frac{2ay}{(2b+1)(1-y)^2} (1-y)^{-2(a+1)} 2F_1 \left[ \frac{a+1}{2b+2} : -\frac{4y}{(1-y)^2} \right]. \]

Now it is easy to see that the two \(2F_1\)’s on the right hand side of (3.3) can be evaluated with the help of the Gauss’ quadratic transformation formula (1.4),
we get

\[(1 - y)^{-2a} {}_2F_1 \left[ \begin{array}{c} a, b \\ 2b + 1 \end{array} ; -\frac{4y}{(1 - y)^2} \right] = {}_2F_1 \left[ \begin{array}{c} a, a - b + \frac{1}{2} \\ b + \frac{1}{2} \end{array} ; y^2 \right]
\]

\[(3.4)\]

\[+ \frac{2ay}{(2b + 1)} {}_2F_1 \left[ \begin{array}{c} a + 1, a - b + \frac{1}{2} \\ b + \frac{3}{2} \end{array} ; y^2 \right],
\]

which can be written as

\[2F_1 \left[ \begin{array}{c} a, b \\ 2b + 1 ; \frac{4y}{(1 - y)^2} \right] = (1 - y)^{2a} \left\{ 2F_1 \left[ \begin{array}{c} a, a - b + \frac{1}{2} \\ b + \frac{1}{2} ; y^2 \right] + \frac{2ay}{(2b + 1)} \right\}.
\]

\[(3.5)\]

Now, changing \( y \) to \( -y \), we get

\[2F_1 \left[ \begin{array}{c} a, b \\ 2b + 1 ; \frac{4y}{(1 + y)^2} \right] = (1 + y)^{2a} \left\{ 2F_1 \left[ \begin{array}{c} a, a - b + \frac{1}{2} \\ b + \frac{1}{2} ; y^2 \right] \right\}
\]

\[+ \frac{2ay}{(2b + 1)} 2F_1 \left[ \begin{array}{c} a + 1, a - b + \frac{1}{2} \\ b + \frac{3}{2} ; y^2 \right].
\]

\[(3.6)\]

Finally, taking \( y = \frac{1-z}{z} \) and we, after a little simplification, have

\[2F_1 \left[ \begin{array}{c} a, b \\ 2b + 1 ; 4z(1 - z) \right] = z^{-2a} \left\{ 2F_1 \left[ \begin{array}{c} a, a - b + \frac{1}{2} \\ b + \frac{1}{2} ; \left( \frac{1-z}{z} \right)^2 \right] + \frac{2a(1-z)}{(2b + 1)z} 2F_1 \left[ \begin{array}{c} a + 1, a - b + \frac{1}{2} \\ b + \frac{3}{2} ; \left( \frac{1-z}{z} \right)^2 \right] \right\}.
\]

\[(3.7)\]

This completes the proof of (2.1).

### 4. Special case

In our main transformation formula (2.1), if we take \( a = -m \) and \( b = \frac{1}{2} \), we obtain

\[2F_1 \left[ \begin{array}{c} \frac{1}{2}, -m \\ 2b + 1 ; 4z(1 - z) \right] = z^{2m} \left\{ 2F_1 \left[ \begin{array}{c} -m, -m + 1 \\ 1 ; \left( \frac{1-z}{z} \right)^2 \right] + m \left( \frac{1-z}{z} \right) 2F_1 \left[ \begin{array}{c} -m, -m + 1 \\ 2 ; \left( \frac{1-z}{z} \right)^2 \right] \right\}.
\]

\[(4.1)\]

Equation (4.1) is an alternate form of the result (1.1) due to Baillon and Bruck.

**Remark.** The result (1.1) in its exact form can be obtained from (4.1) by using (1.3) with \( a = b = -m \) and \( c = 1 \).
References


Arjun K. Rathie
Department of Mathematics
Vedant College of Engineering and Technology
Tulsi-323021, Dist. Bundi, Rajasthan State, India
E-mail address: akrathie@rediffmail.com

Yong Sup Kim
Department of Mathematics Education
Wonkwang University
Iksan 570-749, Korea
E-mail address: yspkim@wonkwang.ac.kr