A NOTE ON THE FIRST LAYERS OF $\mathbb{Z}_p$-EXTENSIONS

JANGHEON OH

ABSTRACT. In this paper we explicitly compute a Minkowski unit of a real abelian field and give a criterion when the first layer of anti-cyclotomic $\mathbb{Z}_3$-extension of an imaginary quadratic field is unramified everywhere.

1. Introduction

For each prime number $p$, a $\mathbb{Z}_p$-extension of a number field $k$ is an extension $k = k_0 \subset k_1 \subset \cdots \subset k_\infty$ with $Gal(k_\infty/k) \simeq \mathbb{Z}_p$, the additive group of $p$-adic integers. Let $k$ be an imaginary quadratic field, and $K$ an abelian extension of $k$. The number field $K$ is called an anti-cyclotomic extension of $k$ if it is Galois over $\mathbb{Q}$, and $Gal(k/\mathbb{Q})$ acts on $Gal(K/k)$ by $-1$. The explicit construction of the first layer $k_1^\alpha$ of the anti-cyclotomic $\mathbb{Z}_p$-extension of $k$ is given in [2, 3, 4].

In this paper, we prove two theorems (Theorem 1 and Theorem 2 in this paper) on questions raised from our previous paper [2, 3, 4]. First it is about the explicit construction of a Minkowski unit which plays a very important role in [4]. Let $L$ be a finite real Galois extension of $\mathbb{Q}$. It is well-known that there exists a unit in $L$ such that the set of units $\{c^\sigma|\sigma \neq 1, \sigma \in Gal(L/\mathbb{Q})\}$ is multiplicative independent and generates a subgroup of finite index in the full group of units. Such a unit is called a Minkowski unit. Theorem 1 gives an explicit construction of a Minkowski unit.

The first layer of anti-cyclotomic $\mathbb{Z}_p$-extension of an imaginary quadratic field $k$ may be or may not be contained in the Hilbert class field of $k$. Hence it is a natural question when the compositum $K$ of the $\mathbb{Z}_p$-extensions of a number field $k$ and Hilbert class field of $k$ are linearly disjoint over $k$. Theorem 2 gives an answer for this question when $k$ is an imaginary quadratic field and $p = 3$.

Let $n \equiv 2 \mod 4$, and let $n = \prod_{i=1}^s p_i^{\ell_i}$ be its prime factorization. Let $I$ run through all subsets of $\{1, \ldots, s\}$, except $\{1, \ldots, s\}$, and let $n_I = \prod_{i \in I} p_i^{\ell_i}$. For

Received January 30, 2008.
2000 Mathematics Subject Classification. 11R23.
Key words and phrases. Minkowski unit, anti-cyclotomic extension, $\mathbb{Z}_p$-extension.

This work was supported by the Korea Research Foundation Grant funded by the Korean Government (KRF-2008-314-C00004).

©2009 The Korean Mathematical Society
integer $a, (a, n) = 1$, define (see [5, Theorem 8.3])

$$\xi_a = \zeta_n^{d_a} \prod_l \frac{1 - \zeta_n^{an_l}}{1 - \zeta_n^{n_l}}, \quad d_a = \frac{1}{2} (1 - a) \sum_l n_l,$$

where $\zeta_n$ is a primitive $n$-th root of unity.

**Theorem 1.** Let $n \not\equiv 2 \mod 4$, and

$$(Z/n)^x/\{\pm 1\} = \langle t_1 \rangle \times \langle t_2 \rangle \times \cdots \times \langle t_m \rangle.$$

Then

$$\xi^{(n)} := \xi_{t_1} \xi_{t_2} \cdots \xi_{t_m}$$

is a Minkowski unit for $\mathbb{Q}(\zeta_n)^+$

Let $K$ be the compositum of all $\mathbb{Z}_3$-extensions of $k$, $H_k$ the 3-part of Hilbert class field of a number field $k$ and $A_k$ the 3-part of the ideal class group of $k$. Then we have the following theorem.

**Theorem 2.** Let $d \not\equiv 3 \mod 9$ be a positive integer and $k = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field. Then

$$H_k \cap K = k \iff \text{rank}_{\mathbb{Z}/3} A_{\mathbb{Q}(\sqrt{-d})} = \text{rank}_{\mathbb{Z}/3} A_{\mathbb{Q}(\sqrt{-d})}.$$

**Remark 1.** It is well-known that

$$\text{rank}_{\mathbb{Z}/3} A_{\mathbb{Q}(\sqrt{-d})} \leq \text{rank}_{\mathbb{Z}/3} A_{\mathbb{Q}(\sqrt{-d})} \leq \text{rank}_{\mathbb{Z}/3} A_{\mathbb{Q}(\sqrt{-d})} + 1.$$

2. **Proof of theorems**

First we prove Theorem 1. To prove Theorem 1 we need lemmas.

**Lemma 3.** For integers $a, b$ relatively prime to $n \not\equiv 2 \mod 4$, we have

$$\xi_a^{\sigma_b} \xi_b = \xi_{ab}, \quad \xi_a = \pm \xi_{-a},$$

where $\sigma_b$ is the Frobenius map.

**Proof.** The proof comes directly from simple computations. \qed

**Lemma 4.** Let notations be as above and $\chi$ be a nontrivial even character mod $n$. Then

$$S_\chi := \sum_{(b, n) = 1}^{n} \chi^{-1}(b) \left( \sum_l \log |1 - \zeta_n^{bn_l}| \right) \neq 0.$$

**Proof.** In fact $S_\chi = \frac{1}{2} \tau(\chi^{-1}) L(1, \chi) \prod_{p_i \mid f_\chi} (\phi(p_\chi^{e_i}) + 1 - \chi^{-1}(p_i))$. See [5, Theorem 8.3] for details. \qed

Now we are ready to prove Theorem 1. It suffices to prove that

$$e_\chi \log |\xi^{(n)}| \neq 0,$$
for any nontrivial even character mod n. We may assume that the real number \( \xi_a \) is positive for any integer \( a \) relatively prime to \( n \). First let us compute

\[
e_\chi \log \xi_a = \frac{2}{\phi(n)} \sum_{(b, n) = 1, b \leq \frac{n}{2}} \chi^{-1}(b) \sigma_b \log \xi_a
\]

\[
= \frac{2}{\phi(n)} \sum_{(b, n) = 1, b \leq \frac{n}{2}} \chi^{-1}(b) \log \xi_a \sigma_b
\]

\[
= \frac{2}{\phi(n)} \sum_{(b, n) = 1, b \leq \frac{n}{2}} \chi^{-1}(b) \log \xi_a \sigma_b
\]

\[
= \frac{2}{\phi(n)} (\chi(a) - 1) \sum_{(b, n) = 1, b \leq \frac{n}{2}} \chi^{-1}(b) \log \xi_b
\]

\[
= \frac{1}{\phi(n)} (\chi(a) - 1) S_X.
\]

Therefore

\[
e_\chi \log |\xi_{(n)}| = \sum_{i=1}^{m} e_\chi \log \xi_{t_i}
\]

\[
= \sum_{i=1}^{m} \frac{1}{\phi(n)} (\chi(t_i) - 1) S_X = \frac{2}{\phi(n)} (\sum_{i=1}^{m} \chi(t_i) - m) S_X.
\]

Note that \( \left( \sum_{i=1}^{m} \chi(t_i) - m \right) \neq 0 \) since \( \chi \) is a nontrivial even character mod \( n \). By Lemma 4, \( S_X \neq 0 \). This completes the proof of Theorem 1.

**Corollary 5.** Let \( L \) be a real abelian field contained in \( \mathbb{Q}(\zeta_n + \zeta_n^{-1}) \). Then \( N_{\mathbb{Q}(\zeta_n + \zeta_n^{-1})/L}(\xi_{t_1} \cdots \xi_{t_m}) \) is a Minkowski unit of \( L \).

**Proof.** This directly comes from the finiteness of the index

\[
[ E_L : N_{\mathbb{Q}(\zeta_n + \zeta_n^{-1})/L}(E_{\mathbb{Q}(\zeta_n + \zeta_n^{-1})}) ]
\]

and Theorem 1. \( \square \)

Now we will prove Theorem 2. We assume that \( p = 3 \). For \( d \neq 3 \) mod 9, let \( k = \mathbb{Q}(\sqrt{-d}) \) be an imaginary quadratic field, \( F = \mathbb{Q}(\sqrt{-d}, \sqrt{-3}) \) a biquadratic field, \( M_F \) and \( M_k \) the maximal abelian \( p \)-extension of \( F \) and \( k \) unramified outside above \( p \), respectively.

Let \( X_F := \text{Gal}(M_F/F)/(p \text{Gal}(M_F/F)) \) and \( X_{F, \chi} \) be the \( \chi \)-component of \( X_F \) for the nontrivial character \( \chi \) of \( \text{Gal}(k/\mathbb{Q}) \). Let \( S \) be a subset of \( F^\times/(F^\times)^3 \) corresponding to the \( X_F \). Then, by Kummer theory, we have a perfect pairing \( S_{\chi \omega} \times X_{F, \chi} \to \mu_p \), where \( \omega \) is the nontrivial character of \( \text{Gal}(\mathbb{Q}(\sqrt{-3})/\mathbb{Q}) \) and \( S_{\chi \omega} \) is the \( \chi \omega \)-component of \( S \). Note that

\[
X_{F, \chi} \simeq X_{k, \chi}.
\]

By [1, Proposition 6.B], \( S \simeq E_F/E_F^p \times A_F/A_F^p \times (p)/(p)^p \), where \( E_F \) is the group of units of \( F \) and \( A_F \) is the \( p \)-part of the ideal class group of \( F \). Since the
$\omega$-component $E_{F,\omega}$ of the group of units $E_F$ is the group of the units of the real quadratic subfield $F^+ \left(= \mathbb{Q}(\sqrt{3d})\right)$ of $F$, the rank of $E_F/E_F^+$ is equal to 1. Therefore

$$H_k \cap K = k \iff \text{rank}_{\mathbb{Z}/p}[X_{k,\omega}] = 1 + \text{rank}_{\mathbb{Z}/p}[A_K]$$

$$\iff \text{rank}_{\mathbb{Z}/p}[X_{F,\omega}] = 1 + \text{rank}_{\mathbb{Z}/p}[A_K]$$

$$\iff \text{rank}_{\mathbb{Z}/p}[S_{\omega}] = 1 + \text{rank}_{\mathbb{Z}/p}[A_K]$$

$$\iff \text{rank}_{\mathbb{Z}/p}[A_F/A_F^+] = \text{rank}_{\mathbb{Z}/p}[A_K]$$

$$\iff \text{rank}_{\mathbb{Z}/p}[A_{F,\omega}] = \text{rank}_{\mathbb{Z}/p}[A_K]$$

$$\iff \text{rank}_{\mathbb{Z}/p}[A_{\mathbb{Q}(\sqrt{d})}] = \text{rank}_{\mathbb{Z}/p}[A_{\mathbb{Q}(\sqrt{-d})}]$$.

This completes the proof of Theorem 2.

References


DEPARTMENT OF APPLIED MATHEMATICS
SEOJONG UNIVERSITY
SEOUL 143-747, KOREA
E-mail address: oh@sejong.ac.kr