EXTENDED CESÀRO OPERATORS FROM $F(p,q,s)$ SPACES TO BLOCH-TYPE SPACES IN THE UNIT BALL

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Abstract. In this paper, we characterize the boundedness and compactness of the extended Cesàro operators from general function spaces $F(p,q,s)$ to Bloch-type spaces $B_\mu$, where $\mu$ is normal function on $[0,1)$.

1. Introduction

Let $B$ be the open unit ball of $\mathbb{C}^n$, and let $\partial B$ be its boundary. $H(B)$ denotes the family of all holomorphic functions on $B$. For $a \in B$, let $h(z,a) = \log \frac{1}{|\varphi_a(z)|}$ be the Green’s function for $B$ with logarithmic singularity at $a$, where $\varphi_a$ is the Möbius transformation of $B$, satisfying $\varphi_a(0) = a$, $\varphi_a(a) = 0$ and $\varphi_a = \varphi_a^{-1}$. For $0 < p, s < \infty$, $-n - 1 < q < \infty$, we say $f \in F(p,q,s)$ provided that $f \in H(B)$ and

$$
\|f\|_{F(p,q,s)}^p = |f(0)|^p + \sup_{a \in \partial B} \int_{\partial B} \left| \Re f(z) \right|^p (1 - |z|^2)^q h^s(z, a) d\nu(z) < \infty.
$$

In one variable, the spaces $F(p,q,s)$ were first introduced by Zhao [12]. We call $F(p,q,s)$ general function space because we can get many function spaces, such as Hardy space, Bergman space, $Q_p$ space, BMOA space, Besov space and $\alpha$-Bloch space, if we take some special parameters of $p$, $q$ and $s$, see [7]. Notice that $F(p,q,s)$ is the space of constant functions if $q + s \leq -1$.

A positive continuous function $\mu$ on $[0, 1)$ is called normal if there are three constants $0 \leq \delta < 1$ and $0 < a < b$ such that

$$
(P_1) \quad \frac{\mu(r)}{(1 - r)^a} \text{ is decreasing on } [\delta, 1) \text{ and } \lim_{r \to 1} \frac{\mu(r)}{(1 - r)^a} = 0;
$$

$$
(P_2) \quad \frac{\mu(r)}{(1 - r)^b} \text{ is increasing on } [\delta, 1) \text{ and } \lim_{r \to 1} \frac{\mu(r)}{(1 - r)^b} = \infty.
$$
We extend it to $B$ by $\mu(z) = \mu(|z|)$. A function $f \in H(B)$ is said to belong to the Bloch-type space $B_\mu$ if
\[
\|f\|_{B_\mu} = \sup_{z \in B} \mu(z)|\nabla f(z)| < \infty,
\]
and it is said to belong to the little Bloch-type space $B_{\mu,0}$ if
\[
\lim_{|z| \to 1} \mu(z)|\nabla f(z)| = 0.
\]
Here $\nabla f(z) = \left( \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right)$ is the complex gradient of $f$. It is clear that both $B_\mu$ and $B_{\mu,0}$ are Banach spaces with the norm $\|f\|_{B_\mu} = |f(0)| + \|f\|_{\mu}$, and $B_{\mu,0}$ is a closed subspace of $B_\mu$. When $\mu(r) = 1 - r^2$, the induced space $B_\mu$ is the classic Bloch space.

In the unit ball, given $g \in H(B)$, we define the extended Cesàro operator to be
\[
T_g f(z) = \int_0^1 f(tz)\Re g(tz) \frac{dt}{t}, \quad z \in B,
\]
where $\Re f(z)$ is the radial derivative of $f$. Hu got the characterization on $g$ for which the operator $T_g$ is bounded or compact on the Bergman space in [2]. Stević [8] considered the boundedness of $T_g$ on $\alpha$-Bloch space. Xiao [10] obtained the property on $g$ such that $T_g$ is bounded or compact on $\alpha$-Bloch space and little $\alpha$-Bloch space. Recently, Li discussed the boundedness of $T_g$ from $F(p, q, s)$ to $\alpha$-Bloch spaces for some restricted $p, q, s$ and $\alpha$ in [9]. The purpose of this work is to obtain the boundedness and compactness of $T_g$ from $F(p, q, s)$ to $B_\mu$ (or $B_{\mu,0}$) for all $0 < p, s < \infty, -n - 1 < q < \infty$. Our work will generalize [3] and [8].

In what follows we always suppose $0 < p, s < \infty, -n - 1 < q < \infty, q + s > -1$. $C$ will stand for positive constants whose value may change from line to line but not depend on the functions in $H(B)$. The expression $A \simeq B$ means $C^{-1}A \leq B \leq CA$.

2. Some preliminary results

**Lemma 2.1** ([11]). Suppose $f \in F(p, q, s)$. Then $f \in B_{\frac{n+1+s}{1-r^2}}$ and
\[
\|f\|_{B_{\frac{n+1+s}{1-r^2}}} \leq C\|f\|_{F(p, q, s)}.
\]

**Lemma 2.2** ([9]). Let $\mu$ be normal and $f \in H(B)$. Then
(i) $f \in B_\mu$ if and only if $\sup_{z \in B} \mu(z)|\Re f(z)| < \infty$. Moreover,
\[
\|f\|_{B_\mu} \simeq |f(0)| + \sup_{z \in B} \mu(z)|\Re f(z)|.
\]
(ii) $f \in B_{\mu,0}$ if and only if $\lim_{|z| \to 1} \mu(z)|\Re f(z)| = 0$. 
Lemma 2.3 ([8]). For $0 < \alpha < \infty$, if $f \in \mathcal{B}_{(1-r^2)^\alpha}$, then for any $z \in \mathcal{B}$,
\[
|f(z)| \leq \begin{cases} 
C \|f\|_{\mathcal{B}_{(1-r^2)^\alpha}}, & 0 < \alpha < 1; \\
C \|f\|_{\mathcal{B}_{(1-r^2)^\alpha}} \log \frac{1}{|z|}, & \alpha = 1; \\
C(1 - \|z\|^2)^{-\alpha} \|f\|_{\mathcal{B}_{(1-r^2)^\alpha}}, & \alpha > 1.
\end{cases}
\]

Lemma 2.4 ([5]). For $s > 1$, $r, t \geq 0$ and $r + t - s > n + 1$, then
\[
\int_{\mathcal{B}} \frac{(1 - \|z\|^2)^s}{|1 - w, a|^r |1 - w, z|^t} dv(z)
\leq \begin{cases} 
\frac{C}{|1 - w, a|^r \log \frac{1}{|z|}}, & \text{if } r - s, t - s < n + 1; \\
\frac{C}{(1 - |a|^2)^{r - s - 1} |1 - w, a|^t}, & \text{if } t - s < n + 1 < r - s.
\end{cases}
\]

Lemma 2.5. Let $p = n + 1 + q$. Suppose that for each $w \in \mathcal{B}$, $z$-variable functions $g_w$ satisfy $|g_w(z)| \leq \frac{C}{|1 - z, w|}$, then
\[
\int_{\mathcal{B}} |g_w(z)|^p (1 - \|z\|^2)^q h^s(z, a) dv(z) \leq C.
\]

Proof. If $0 < s < n + 1 + q$, Lemma 2.4 implies
\[
(1 - |a|^2)^{s} \int_{\mathcal{B}} (1 - \|z\|^2)^{q + s} \frac{dv(z)}{|1 - w, a|^r |1 - a, z|^t} \leq \frac{C(1 - |a|^2)^{s}}{|1 - w, a|^r \log \frac{1}{|z|}} \leq C.
\]

If $s > n + 1 + q$, we have
\[
(1 - |a|^2)^{s} \int_{\mathcal{B}} (1 - \|z\|^2)^{q + s} \frac{dv(z)}{|1 - w, a|^r |1 - a, z|^t} \leq \frac{C}{(1 - |a|^2)^{s - n - 1 - q} |1 - w, a|^r} \leq C.
\]

If $s = n + 1 + q$, choose $s_1 = \frac{n}{2}$, $s_2 = 2s$, $x = \frac{s_2 - s_1}{s_2 - s}$. By the fact $q + s > -1$, we know
\[
0 < s_1 < n + 1 + q < s_2, q + s_2 > q + s_1 > -1 \text{ and } x > 1.
\]

Take $t_1 = \frac{s_2 + s_1}{x}$, $t_2 = \frac{s_1}{x}$, $\frac{t_1}{s} = x = 1$. By (2.1), (2.2) and Hölder inequality,
\[
\int_{\mathcal{B}} \frac{(1 - |a|^2)^{s} (1 - \|z\|^2)^{q + s}}{|1 - w, a|^r |1 - a, z|^t} dv(z)
\leq \left\{ \int_{\mathcal{B}} \frac{(1 - |a|^2)^{s_1} (1 - \|z\|^2)^{q + s_1}}{|1 - w, a|^r |1 - a, z|^t} dv(z) \right\}^{\frac{1}{2}}
\times \left\{ \int_{\mathcal{B}} \frac{(1 - |a|^2)^{s_2} (1 - \|z\|^2)^{q + s_2}}{|1 - w, a|^r |1 - a, z|^t} dv(z) \right\}^{\frac{1}{2}}
\leq C \frac{(1 - |a|^2)^{s_1}}{|1 - w, a|^r} \frac{|1 - |a|^2|^{s_1}}{|1 - w, a|^r} \leq C.
\]
Given any \( a \in \mathcal{B} \), let \( x = 1 - |\varphi_a(z)|^2 \), we have

\[
h(z, a) = -\frac{1}{2} \log(1-x) \leq \frac{x}{2} \left[ \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \cdots \right] = 2x \quad \text{for} \quad \frac{1}{2} < |\varphi_a(z)| < 1.\]

Notice that

\[
1 - |\varphi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - < a, z >|^2}.
\]

Hence, (2.1), (2.2), and (2.3) yield, for \( p = n + 1 + q \),

\[
\int_{\frac{1}{2} < |\varphi_a(z)| < 1} |g_w(z)|^p (1 - |z|^2)^q h^s(z, a) dv(z)
\]

\[
\leq C \int_{\frac{1}{2} < |\varphi_a(z)| < 1} \frac{(1 - |a|^2)^s (1 - |z|^2)^q}{|1 - < z, w >|^{n+1+q}|1 - < a, z >|^{2s}} dv(z)
\]

\[
\leq C \int_{\mathcal{B}} \frac{(1 - |a|^2)^s (1 - |z|^2)^q}{|1 - < z, w >|^{n+1+q}|1 - < a, z >|^{2s}} dv(z) \leq C.
\]

At the same time,

\[
\int_{|\varphi_a(z)| \leq \frac{1}{2}} |g_w(z)|^p (1 - |z|^2)^q h^s(z, a) dv(z)
\]

\[
\leq C \int_{|\varphi_a(z)| \leq \frac{1}{2}} \frac{(1 - |z|^2)^q}{|1 - < z, w >|^{p}} h^s(z, a) dv(z)
\]

\[
= C \int_{|u| \leq \frac{1}{2}} \frac{(1 - |\varphi_a(u)|)^q}{|1 - < \varphi_a(u), w >|^{n+1+q}} \cdot \frac{(1 - |a|^2)^{n+1}}{|1 - < u, a >|^{2n+2}} \cdot \log^s \frac{1}{|u|} dv(u)
\]

\[
\leq C \int_{|u| \leq \frac{1}{2}} \frac{(1 - |a|^2)^{n+1+q}(1 - |u|^2)^q}{(1 - |\varphi_a(u)|)^{n+1+q}|1 - < u, a >|^{2n+2+2q}} \log^s \frac{1}{|u|} dv(u)
\]

\[
= C \int_{|u| \leq \frac{1}{2}} \frac{1}{(1 - |u|^2)^{n+1}} \log^s \frac{1}{|u|} dv(u)
\]

\[
\leq C \int_{|u| \leq \frac{1}{2}} \log^s \frac{1}{|u|} dv(u) = C \int_0^1 2nx^{2n-1} \log^s \frac{1}{r} dr \int_{\partial \mathcal{B}} d\sigma(\xi) \leq C,
\]

where \( u = \varphi_a(z) \). This, together with (2.4), means

\[
\int_{\mathcal{B}} |g_w(z)|^p (1 - |z|^2)^q h^s(z, a) dv(z)
\]

\[
= \left( \int_{\frac{1}{2} < |\varphi_a(z)| < 1} + \int_{|\varphi_a(z)| \leq \frac{1}{2}} \right) |g_w(z)|^p (1 - |z|^2)^q h^s(z, a) dv(z) \leq C.
\]

The proof is completed. \( \square \)

**Lemma 2.6.** Let \( \mu \) be normal and \( g \in H(\mathcal{B}) \). Suppose \( T_g : F(p, q, s) \to \mathcal{B}_\mu \) is bounded. Then \( T_g : F(p, q, s) \to \mathcal{B}_\mu \) is compact if and only if for any bounded
sequence \( \{f_j\} \subseteq F(p, q, s) \) which converges to 0 uniformly on any compact subset of \( B \), we have \( \lim_{j \to \infty} \|T_g f_j\|_{\mathcal{B}_\mu} = 0. \)

**Proof.** It can be proved by Lemma 2.1, Lemma 2.3 and the Montel Theorem. The details are omitted here. \( \square \)

To characterize the compactness of \( T_g \) from \( F(p, q, s) \) to \( \mathcal{B}_{\mu, 0} \), we give the following lemma, whose proof is similar to that of Lemma 1 in [4].

**Lemma 2.7.** Let \( \mu \) be a normal function. A closed subset \( E \) in \( \mathcal{B}_{\mu, 0} \) is compact if and only if it is bounded and satisfying

\[
\lim_{|z| \to 1} \sup_{f \in E} \mu(z)|\Re f(z)| = 0.
\]

### 3. Main results

**Theorem 3.1.** Let \( \mu \) be normal, \( g \in H(B) \), \( n + 1 + q \geq p \). Then \( T_g : F(p, q, s) \to \mathcal{B}_\mu \) is bounded if and only if

(i) for \( n + 1 + q > p \),

\[
\sup_{z \in B} \mu(z)|\Re g(z)|(1 - |z|^2)^{1 - \frac{n+1+q}{p}} < \infty.
\]

In this case,

\[
\|T_g\| \simeq \sup_{z \in B} \mu(z)|\Re g(z)|(1 - |z|^2)^{1 - \frac{n+1+q}{p}}.
\]

(ii) for \( n + 1 + q = p \),

\[
\sup_{z \in B} \mu(z)|\Re g(z)| \log \frac{2}{1 - |z|^2} < \infty.
\]

In this case,

\[
\|T_g\| \simeq \sup_{z \in B} \mu(z)|\Re g(z)| \log \frac{2}{1 - |z|^2}.
\]

**Proof.** (i) First, for \( f, g \in H(B) \), direct calculation shows

\[
\Re(T_g f)(z) = f(z)\Re g(z).
\]

Suppose \( n + 1 + q > p \), \( f \in F(p, q, s) \), by Lemmas 2.1, 2.2 and 2.3, we obtain

\[
\|T_g f\|_{\mathcal{B}_\mu} \simeq |T_g f(0)| + \sup_{z \in B} \mu(z)|f(z)||\Re g(z)|
\]

\[
\leq C\|f\|_{B_{(1-r)^{\frac{n+1+q}{p}}}} \sup_{z \in B} \mu(z)|\Re g(z)|(1 - |z|^2)^{1 - \frac{n+1+q}{p}}
\]

\[
\leq C\|f\|_{F(p, q, s)} \sup_{z \in B} \mu(z)|\Re g(z)|(1 - |z|^2)^{1 - \frac{n+1+q}{p}}.
\]

Hence, (3.1) implies that \( T_g : F(p, q, s) \to \mathcal{B}_\mu \) is bounded.

Conversely, suppose \( T_g : F(p, q, s) \to \mathcal{B}_\mu \) is bounded. For any \( w \in B \), set

\[
f_w(z) = \frac{1 - |w|^2}{(1 - z, w)^{\frac{n+1+q}{p}}}, \quad z \in B.
\]
Then \( \|f_w\|_{F(p,q,s)} \leq C \) by [11]. Hence,
\[
\mu(w)|\Re g(w)|(1 - |w|^2)^{1 - \frac{n+1+i}{p}} = \mu(w)|\Re g(w)||f_w(w)| \leq C\|T_g f_w\|_{\mathcal{B}_\mu} \leq C\|T_g\|.
\]
Therefore,
\[
(3.4) \quad \sup_{z \in \mathbb{B}} \mu(z)|\Re g(z)|(1 - |z|^2)^{1 - \frac{n+1+i}{p}} \leq C\|T_g\| < \infty.
\]

Moreover, (3.3) and (3.4) yield
\[
\|T_g\| \simeq \sup_{z \in \mathbb{B}} \mu(z)|\Re g(z)|(1 - |z|^2)^{1 - \frac{n+1+i}{p}}.
\]

(ii) If \( n + 1 + q = p \), by Lemma 2.1, \( F(p,q,s) \subseteq B_{1-r^2} \). For \( f \in F(p,q,s) \), combining Lemma 2.2 and Lemma 2.3, we get
\[
\|T_g f\|_{\mathcal{B}_\mu} \simeq |T_g f(0)| + \sup_{z \in \mathbb{B}} |\mu(z)||f(z)||\Re g(z)|
\]
\[
\leq C\|f\|_{B_{1-r^2}} \sup_{z \in \mathbb{B}} \mu(z)|\Re g(z)| \log \frac{2}{1 - |z|^2}
\]
\[
\leq C\|f\|_{F(p,q,s)} \sup_{z \in \mathbb{B}} \mu(z)|\Re g(z)| \log \frac{2}{1 - |z|^2}.
\]
Thus, (3.2) yields that \( T_g : F(p,q,s) \to B_{\mu} \) is bounded.

Conversely, suppose \( T_g : F(p,q,s) \to B_{\mu} \) is bounded. Given any \( w \in \mathbb{B} \), set
\[
f_w(z) = \log \frac{2}{1 - \langle z, w \rangle}, \quad z \in \mathbb{B}.
\]
Then \( |\Re f_w(z)| \leq \frac{C}{1 - |z|^2} \), by Lemma 2.5
\[
\|f_w\|_{F(p,q,s)} \leq C.
\]
By the boundedness of \( T_g \), we have
\[
\mu(w)|\Re g(w)| \log \frac{2}{1 - |w|^2} = \mu(w)|\Re g(w)||f_w(w)| \leq C\|T_g f_w\|_{\mathcal{B}_\mu} \leq C\|T_g\|.
\]
This means
\[
(3.6) \quad \sup_{z \in \mathbb{B}} \mu(z)|\Re g(z)| \log \frac{2}{1 - |z|^2} \leq C\|T_g\| < \infty.
\]

Furthermore, (3.5) and (3.6) imply
\[
\|T_g\| \simeq \sup_{z \in \mathbb{B}} \mu(z)|\Re g(z)| \log \frac{2}{1 - |z|^2}.
\]
The proof is completed.

Remark. Set \( \mu(z) = (1 - |z|^2)^\alpha \), when \( n + 1 + q \leq p\alpha \) in (i) and \( \alpha \geq 1 \), \( s > n \) in (ii), respectively. Theorem 3.1 is just the main results in [3], which are Theorem 2.4 and Theorem 2.10.
Theorem 3.2. Let μ be normal, g ∈ H(B), n + 1 + q ≥ p. Then the following statements are equivalent:

(A) $T_g : F(p, q, s) \to B_\mu$ is compact;

(B) $T_g : F(p, q, s) \to B_{\mu, 0}$ is compact;

(C) (i) for $n + 1 + q > p$,

$$\lim_{|z| \to 1} \mu(z)|\Re g(z)|(1 - |z|^2)^{1 - \frac{n+1+q}{p}} = 0;$$

(ii) for $n + 1 + q = p$,

$$\lim_{|z| \to 1} \mu(z)|\Re g(z)| \log \frac{2}{1 - |z|^2} = 0.$$

Proof. The implication (B)⇒(A) is trivial.

(C)⇒(B) Suppose (3.7) holds for the case of $n + 1 + q > p$. For $f \in F(p, q, s)$, by Lemmas 2.1 and 2.3, we obtain

$$\mu(z)|f(z)||\Re g(z)| \leq C\|f\|_{F(p, q, s)} \mu(z)|\Re g(z)|(1 - |z|^2)^{1 - \frac{n+1+q}{p}} \leq C\|f\|_{F(p, q, s)} \mu(z)|\Re g(z)|(1 - |z|^2)^{1 - \frac{n+1+q}{p}}.$$

Thus, (3.7) shows

$$\lim_{|z| \to 1} \sup_{\|f\|_{F(p, q, s)} \leq 1} \mu(z)|\Re(T_g f)(z)| = 0.$$

Similarly, we can obtain

$$\lim_{|z| \to 1} \sup_{\|f\|_{F(p, q, s)} \leq 1} \mu(z)|\Re(T_g f)(z)| = 0$$

for the case of $n + 1 + q = p$ by (3.8). Therefore, $T_g : F(p, q, s) \to B_{\mu, 0}$ is compact by Lemma 2.7.

(A)⇒(C) First, we deal with the case of $n + 1 + q > p$. Suppose (3.7) did not hold. Then there would be some $\varepsilon_0 > 0$ and some sequence $\{z^j\} \subseteq B$ satisfying $\lim_{j \to \infty} |z^j| = 1$, but for each $j$,

$$\mu(z^j)|\Re g(z^j)|(1 - |z^j|^2)^{1 - \frac{n+1+q}{p}} \geq \varepsilon_0.$$ 

Set

$$f_j(z) = \frac{1 - |z|^2}{(1 - <z, z^j>|^\frac{n+1+q}{p}}, \quad z \in B.$$

Then $\|f_j\|_{F(p, q, s)} \leq C$, and $\{f_j\}$ converges to 0 uniformly on any compact subset of $B$. By Lemma 2.6 and (A),

$$\|T_g f_j\|_{B_\mu} \to 0 \quad (j \to \infty).$$
On the other hand, (3.9) implies
\[
\|Tg f_j\|_{\mathcal{B}_n} \sim |Tg f_j(0)| + \sup_{z \in \mathcal{B}} |f_j(z)\Re g(z)| \\
\geq \mu(z^j)|f_j(z^j)\Re g(z^j)| \\
= \mu(z^j)|\Re g(z^j)|(1 - |z^j|^2)^{1 - n + q + n \over p} \geq \varepsilon_0.
\]
This is a contradiction to (3.11). If \(n + 1 + q = p\), suppose (3.8) did not hold. Then there would be some \(\varepsilon_0 > 0\) and some sequence \(\{z^j\} \subseteq \mathcal{B}\) satisfying \(\lim_{j \to \infty} |z^j| = 1\), but for each \(j\),
\[
(3.12) \quad \mu(z^j)|\Re g(z^j)| \log \frac{2}{1 - |z^j|^2} \geq \varepsilon_0.
\]
Take the test function
\[
f_j(z) = \left(\frac{\log \frac{1 - |z|}{1 - |z^j|^2}}{\log \frac{1 - |z|}{1 - |z^j|^2}}\right)^2, \quad z \in \mathcal{B}.
\]
Then
\[
|\Re f_j(z)| &= \left| \frac{2 < z, z^j > \log \frac{2}{1 - |z^j|^2}}{(1 - < z, z^j >) \log \frac{2}{1 - |z^j|^2}} \right| \\
&\leq 2 \left| \frac{\log \frac{2}{1 - |z|}}{\log \frac{2}{1 - |z^j|^2}} \right| \frac{1}{|1 - < z, z^j >|} \\
&\leq 2 \pi + \log \frac{1 - |z|}{1 - |z^j|^2} \cdot \frac{1}{|1 - < z, z^j >|} \leq C.
\]
Then \(\|f_j\|_{F(p,q,s)} \leq C\) by Lemma 2.5, and \(\{f_j\}\) converges to 0 uniformly on any compact subset of \(\mathcal{B}\). By Lemma 2.6 and (A), we have
\[
(3.13) \quad \|Tg f_j\|_{\mathcal{B}_n} \to 0 \quad \text{as} \quad j \to \infty.
\]
However, (3.12) yields
\[
\|Tg f_j\|_{\mathcal{B}_n} \sim |Tg f_j(0)| + \sup_{z \in \mathcal{B}} |f_j(z)\Re g(z)| \\
\geq \mu(z^j)|f_j(z^j)\Re g(z^j)| \\
= \mu(z^j)|\Re g(z^j)| \log \frac{2}{1 - |z^j|^2} \geq \varepsilon_0.
\]
This is a contradiction to (3.13). The proof is completed. \(\square\)

**Theorem 3.3.** Let \(\mu\) be normal, \(g \in H(\mathcal{B})\), \(n + 1 + q < p\). Then the following statements are equivalent:

(A) \(Tg: F(p,q,s) \to \mathcal{B}_\mu\) is bounded;

(B) \(Tg: F(p,q,s) \to \mathcal{B}_\mu\) is compact;

(C) \(g \in \mathcal{B}_\mu\).

In this case,
\[
\|Tg\| \simeq \|g - g(0)\|_{\mathcal{B}_n}.
\]
Proof. The implication (B)⇒(A) is trivial.

(A)⇒(C) Suppose $T_g: F(p, q, s) \rightarrow \mathcal{B}_\mu$ is bounded. By the fact that $g(z) = g(0) + T_g(1)(z)$, we know $g \in \mathcal{B}_\mu$. Moreover,

$$\|g - g(0)\|_{\mathcal{B}_\mu} = \|T_g(1)\|_{\mathcal{B}_\mu} \leq C\|T_g\| < \infty.$$  \hfill (3.14)

(C)⇒(B) Suppose $\{f_j\} \subseteq F(p, q, s)$ is any bounded sequence converging to 0 uniformly on any compact subset of $\mathcal{B}$. By Lemma 2.1 and [9, Lemma 4.2],

$$\lim_{j \to \infty} \sup_{z \in \mathcal{B}} |f_j(z)| = 0.$$

Hence,

$$\|T_g f_j\|_{\mathcal{B}_\mu} \simeq |T_g f_j(0)| + \sup_{z \in \mathcal{B}} \mu(z)|f_j(z)\Re g(z)| \leq C\|g\|_{\mathcal{B}_\mu} \sup_{z \in \mathcal{B}} |f_j(z)| \to 0 \quad (j \to \infty).$$

This means $T_g: F(p, q, s) \rightarrow \mathcal{B}_\mu$ is compact.

Furthermore, for any $f \in F(p, q, s)$, Lemmas 2.1 and 2.3 yield

$$\|T_g f\|_{\mathcal{B}_\mu} \simeq |T_g f(0)| + \sup_{z \in \mathcal{B}} \mu(z)|f(z)\Re g(z)| \leq C\|g - g(0)\|_{\mathcal{B}_\mu} \|f\|_{(1-r^2)^{\frac{n+1+q}{p}}(1-r^2)} \leq C\|g - g(0)\|_{\mathcal{B}_\mu} \|f\|_{F(p, q, s)}.$$

This, combining with (3.14), shows

$$\|T_g\| \simeq \|g - g(0)\|_{\mathcal{B}_\mu}.$$

The proof is completed. \hfill \Box

Theorem 3.4. Let $\mu$ be normal, $g \in H(\mathcal{B})$, $n + 1 + q < p$. Then the following statements are equivalent:

(A) $T_g: F(p, q, s) \rightarrow \mathcal{B}_{\mu,0}$ is bounded;

(B) $T_g: F(p, q, s) \rightarrow \mathcal{B}_{\mu,0}$ is compact;

(C) $g \in \mathcal{B}_{\mu,0}$.

Proof. The implication (B)⇒(A) is trivial.

(A)⇒(C) It is trivial from the fact that $g(z) = g(0) + T_g(1)(z)$.

(C)⇒(B) By Theorem 3.3, the condition (C) implies that $T_g$ is compact from the $F(p, q, s)$ space to Bloch-type space $\mathcal{B}_{\mu}$. We claim that $T_g(F(p, q, s)) \subseteq \mathcal{B}_{\mu,0}$. In fact, for any $f \in F(p, q, s) \subseteq \mathcal{B}_{(1-r^2)^{\frac{n+1+q}{p}}(1-r^2)}$, Lemmas 2.2 and 2.3 imply

$$0 \leq \mu(z)|\Re g(z)||f(z)| \leq C\|f\|_{\mathcal{B}_{(1-r^2)^{\frac{n+1+q}{p}}(1-r^2)}} \mu(z)|\Re g(z)| \to 0 \quad \text{as} \quad |z| \to 1.$$

The proof is completed. \hfill \Box

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