MIXED VECTOR $FQ$-IMPLICIT VARIATIONAL INEQUALITY WITH LOCAL NON-POSITIVITY

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Abstract. This paper introduces a local non-positivity of two set-valued mappings ($F, Q$) and considers the existences and properties of solutions for set-valued mixed vector $FQ$-implicit variational inequality problems and set-valued mixed vector $FQ$-complementarity problems in the neighborhood of a point belonging to an underlined domain $K$ of the set-valued mappings, where the neighborhood is contained in $K$.

This paper generalizes and extends many results in [1, 3-7].

1. Introduction

$F$-complementarity problem ($F$-CP): finding $x \in K$ such that
\[
\langle Tx, x \rangle + F(x) = 0 \text{ and } \langle Tx, y \rangle + F(y) \geq 0 \text{ for all } y \in K,
\]
and corresponding variational inequality problem;

finding $x \in K$ such that
\[
\langle Tx, y - x \rangle + F(y) - F(x) \geq 0 \text{ for all } y \in K,
\]
where $K$ is a nonempty closed and convex cone of a real Banach space $X$ with its dual $X^*$, $T : K \to X^*$ is a mapping and $F : K \to (-\infty, +\infty)$ is a positively homogeneous and convex function, were firstly considered in [7].

In 2003, Fang and Huang [1] considered a vector $F$-complementarity problem with demi-pseudomonotone mappings in Banach spaces by considering the solvability of the problems. Huang and Li [3] studied a scalar $F$-implicit variational inequality problem and another $F$-implicit complementarity problem in Banach spaces in 2004. Recently, the result of the scalar case in [3] was extended and generalized to the vector case by Li and Huang [6]. The equivalence between the $F$-implicit variational inequality problem and $F$-implicit complementarity problem was presented and some new existence theorems of solutions for $F$-implicit variational inequality problems were also proved.

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Recently, the following mixed vector $FQ$-implicit variational inequality problem (FQ-VI) and corresponding mixed vector $FQ$-implicit complementarity problems (FQ-CP) for set-valued mappings were considered in [4];

\begin{align*}
(FQ-VI): & \text{find } x \in K \text{ such that } p - s + w - z \in P(x) \text{ for any } p \in Q(x, g(y)), \\
& \hspace{1cm} s \in Q(x, h(x)), \ w \in F(g(y)), \text{ and } z \in F(h(x)), \text{ where } y \in K. \\
(FQ-CP): & \text{find } x \in K \text{ such that } \\
& \hspace{1cm} (a) \ p + w \in P(x) \text{ for any } p \in Q(x, g(y)) \text{ and } w \in F(g(y)), \text{ where } y \in K, \\
& \hspace{1cm} \text{and} \\
& \hspace{1cm} (b) \ s + z = 0 \text{ for any } s \in Q(x, h(x)) \text{ and } z \in F(h(x)), \text{ where } K \text{ is a nonempty closed convex cone of a real Banach space } X \text{ and } \{P(x) : x \in K\} \text{ is a family of nonempty pointed closed convex cones with the apex at the origin in a real Banach space } Y. \text{ Mappings } g, h : K \to K \text{ are single-valued, } F : K \to 2^Y \text{ and } Q : K \times K \to 2^Y \text{ are set-valued.}
\end{align*}

The following Theorem A and Theorem B in [4] show the equivalence between (FQ-VI) and (FQ-CP) and some existence theorems of solutions for them under some suitable assumptions without monotonicity, respectively.

**Theorem A.** Assume that a set-valued mapping $F : K \to 2^Y$ is positively homogeneous, a set-valued mapping $Q : K \times K \to 2^Y$ is also positively homogeneous in the second argument and $g : K \to K$ is surjective. Then (FQ-VI) is equivalent to (FQ-CP).

**Theorem B.** Let $K$ be a nonempty closed convex subset of $X$ and $P : K \to 2^Y$ be upper semicontinuous on $K$. Assume that

\begin{enumerate}
\item[(a)] $g, h : K \to K$ are continuous, $F : K \to 2^Y$ is lower semicontinuous and $Q : K \times K \to 2^Y$ is lower semicontinuous in two arguments,
\item[(b)] there exists a single-valued mapping $T : K \times K \to Y$ satisfying
\begin{enumerate}
\item[(b1)] for $x \in K$, $T(x, x) \in P(x)$,
\item[(b2)] for $x, y \in K$,
$$a - b + c - d - T(x, y) \in P(x)$$
for any $a \in Q(x, g(y))$, $b \in Q(x, h(x))$, $c \in F(g(y))$ and $d \in F(h(x))$,
\item[(b3)] for $x \in K$ the set \{ $y \in K : T(x, y) \notin P(x)$ \} is convex,
\end{enumerate}
\end{enumerate}
(c) there exists a nonempty compact convex subset \( D \) of \( K \) such that for all \( x \in K \setminus D \) there exists a \( y \in D \) satisfying \( a - b + c - d \notin P(x) \) for some \( a \in Q(x, g(y)), b \in Q(x, h(x)), c \in F(g(y)) \) and \( d \in F(h(x)) \).

Then \((FQ-VI)\) has a solution. Furthermore, the solution set of \((FQ-VI)\) is closed.

This paper introduces a local non-positivity of set-valued mappings \((F, Q)\) and considers the existences and properties of solutions for \((FQ-VI)\) and \((FQ-CP)\) in the neighborhood of a point belonging to an underlined domain \( K \) of the set-valued mappings, where the neighborhood is contained in \( K \).

This paper generalizes and extends many results in [1, 3-7].

2. Preliminaries

Remark that \( P(x), x \in K \) is a closed set such that
(i) \( \lambda P(x) \subset P(x), \lambda > 0, x \in K \),
(ii) \( P(x) + P(x) \subset P(x), x \in K \),
(iii) \( P(x) \cap (-P(x)) = \{0\}, x \in K \).

An ordered Banach space \((Y, P(x))\) is a real Banach space with an ordering defined by a closed cone \( P(x) \subset Y \) as for any \( y, z \in Y \),
\[
y \geq z \quad \text{if and only if} \quad y - z \in P(x),
\]
\[
y \not\geq z \quad \text{if and only if} \quad y - z \notin P(x).
\]

Remark that
\[
z \leq 0 \quad \text{if and only if} \quad z \in -P(x),
\]
\[
z \not\leq 0 \quad \text{if and only if} \quad z \notin -P(x),
\]
\[
z \geq 0 \quad \text{if and only if} \quad z \in P(x),
\]
\[
z \not\geq 0 \quad \text{if and only if} \quad z \notin P(x).
\]

**Lemma 2.1** ([1]). Let \((Y, P)\) be an ordered Banach space induced by a pointed closed cone \( P \). Then \( x + y \in P \) for \( x, y \in P \).

**Definition 2.1** ([4]). Let \( X, Y \) be two vector spaces and \( K \) be a cone of \( X \). A set-valued mapping \( F : K \to 2^Y \) is said to be positively homogeneous if \( F(\alpha x) = \alpha F(x) \) for all \( x \in K \) and \( \alpha \geq 0 \). \( F \) is said to be linear if \( F(\alpha x + \beta y) = \alpha F(x) + \beta F(y) \) for \( x, y \in K \), \( \alpha + \beta = 1 \), \( \alpha, \beta \geq 0 \).

**Definition 2.2**. A set-valued mapping \( W : K \subset X \to 2^Y \) is upper semicontinuous at \( x_0 \in K \) if every open set \( V \) containing \( W(x_0) \) there exists an open set \( U \) containing \( x_0 \) such that \( W(U) \subset V \). \( W \) is lower semicontinuous at \( x_0 \in K \) if every open set \( V \) intersecting \( W(x_0) \) there exists an open set \( U \) containing \( x_0 \) such that \( W(x) \cap V \neq \emptyset \) for every \( x \in U \). \( W \) is upper semicontinuous (lower semicontinuous) on \( K \) if it is upper semicontinuous (lower semicontinuous) at every point of \( K \). \( W \) is continuous on \( K \) if it is both upper semicontinuous and lower semicontinuous on \( K \).
Lemma 2.2. Let $W : X \to 2^Y$ be a set-valued mapping and $x_0 \in X$.

(i) $W$ is upper semicontinuous at $x_0$ if and only if for any net $\{x_{\alpha}\} \subset X$ with $x_{\alpha} \to x_0$ and for any net $\{y_{\alpha}\}$ in $Y$ with $y_{\alpha} \in W(x_{\alpha})$ such that $y_{\alpha} \to y_0$ in $Y$, we have $y_0 \in W(x_0)$.

(ii) $W$ is lower semicontinuous at $x_0$ if and only if for any net $\{x_{\alpha}\} \subset X$ with $x_{\alpha} \to x_0$, and for any $y_0 \in W(x_0)$, there exists a net $\{y_{\alpha}\}$ such that $y_{\alpha} \in W(x_{\alpha})$ and $y_{\alpha} \to y_0$.

Lemma 2.3 ([2]). Let $W : X \to 2^Y$ be a set-valued mapping. If for any $x \in X$, $W(x)$ is compact, then $W$ is upper semicontinuous at $x_0$ if and only if for any net $\{x_{\alpha}\} \subset X$ such that $x_{\alpha} \to x_0$ and for every $y_0 \in W(x_0)$, there exists $y_0 \in W(x_\alpha)$ and a subnet $\{y_{\alpha,\beta}\}$ of $\{y_{\alpha}\}$ such that $y_{\alpha,\beta} \to y_0$.

3. Main results

Unless otherwise specified, we assume that $K$ is a nonempty closed convex cone of a real Banach space $X$ and $\{P(x) : x \in K\}$ is a family of nonempty pointed closed convex cones with the apex at the origin in a real Banach space $Y$.

Definition 3.1. Let $g, h : K \to K$ be single-valued mappings and $F : K \to 2^Y$, $Q : K \times K \to 2^Y$ set-valued mappings. Let $P : K \to 2^Y$ be a set-valued mapping with nonempty pointed closed convex cones with the apex at the origin in $Y$. $(F,Q)$ is said to be locally non-positive at $x_0 \in K$ with respect to $(g,h)$ if there exist a neighborhood $N(x_0)$ of $x_0$ and $z_0 \in K \cap \text{Int}N(x_0)$ such that $a-b+c-d \not\in P(x)$ for any $a \in Q(x,g(z_0))$, $b \in Q(x,h(x))$, $c \in F(g(z_0))$ and $d \in F(h(x))$ for $x \in K \cap \partial N(x_0)$, the boundary of $N(x_0)$.

Example 3.1. Let $X = Y = \mathbb{R}$, $K = [0, \infty)$ and $P(x) = [0, \infty)$ for all $x \in K$. Define mappings $g, h : K \to K$ by $g(x) = 2x$ and $h(x) = 2x$, set-valued mappings $F : K \to 2^\mathbb{R}$ by $F(x) = \left[\frac{1}{2}x, x\right]$, $Q : K \times K \to 2^\mathbb{R}$ by $Q(x,y) = \left[\frac{1}{2}(x+y), x+y\right]$, then $(F,Q)$ is locally non-positive at $x_0 = 0 \in K$ with respect to $(g,h)$. If we take a neighborhood $N(0) = (-\frac{1}{4}, \frac{1}{4})$ of $x_0 = 0$ and $z_0 = \frac{1}{4} \in K \cap \text{Int}N(0) = [0, \frac{1}{2})$, then for the unique element $x = \frac{1}{2}$ of $K \cap \partial N(0) = \left\{\frac{1}{2}\right\}$, we have for any $a \in Q\left(\frac{1}{2}, g\left(\frac{1}{4}\right)\right)$, $b \in Q\left(\frac{1}{2}, h\left(\frac{1}{4}\right)\right)$, $c \in F\left(g\left(\frac{1}{4}\right)\right)$ and $d \in F\left(h\left(\frac{1}{4}\right)\right)$,

$$a-b+c-d \not\in -K.$$ 

In fact, $Q\left(\frac{1}{2}, g\left(\frac{1}{4}\right)\right) = Q\left(\frac{1}{2}, \frac{1}{2}\right) = \left[\frac{1}{2}, 1\right]$, $Q\left(\frac{1}{2}, h\left(\frac{1}{4}\right)\right) = Q\left(\frac{1}{2}, 1\right) = [1, \frac{3}{2}]$, 

$$F\left(g\left(\frac{1}{4}\right)\right) = F\left(\frac{1}{4}\right) = \left[\frac{1}{4}, \frac{1}{2}\right], F\left(h\left(\frac{1}{4}\right)\right) = F(1) = \left[\frac{1}{2}, 1\right],$$

thus

$$1 - 1 + \frac{1}{2} - \frac{1}{2} = 0 \in -K.$$ 

Theorem 3.1. Let $K$ be a nonempty closed and convex subset of $X$. Let $P : K \to 2^Y$ be a set-valued mapping with nonempty pointed closed convex cones with the apex at the origin in $Y$. Assume that
(a) single-valued mappings $g, h : K \to K$ are continuous and set-valued mappings $F : K \to 2^V$, $Q : K \times K \to 2^Y$ are continuous and $P$ is upper semicontinuous,

(b) a single-valued mapping $T : K \times K \to Y$ satisfies

(b1) for $x \in K$, $T(x, x) \in P(x)$,

(b2) for $x, y \in K$,

$$a - b + c - d - T(x, y) \in P(x)$$

for any $a \in Q(x, g(y))$, $b \in Q(x, h(x))$, $c \in F(g(y))$ and $d \in F(h(x))$,

(b3) for $x \in K$ the set $\{y \in K : T(x, y) \notin P(x)\}$ is convex,

(c) $(F, Q)$ is locally non-positive at $x_0 \in K$ with respect to $(g, h)$ and there exists a nonempty compact convex subset $D$ of $K \cap N(x_0)$ such that for all $x \in (K \cap N(x_0)) \setminus D$ there exists $y \in D$ satisfying

$$a - b + c - d \notin P(x)$$

for any $a \in Q(x, g(y))$, $b \in Q(x, h(x))$, $c \in F(g(y))$ and $d \in F(h(x))$,

(d) $g, h$ and $F$ are linear and $Q$ is linear in the second argument.

Then $(FQ\text{-VI})$ has a solution in the neighborhood of $x_0$, that is, there exists $x^* \in K \cap N(x_0)$ such that, for $y \in K$,

$$a^* - b^* + c - d^* \notin P(x^*)$$

for any $a^* \in Q(x^*, g(y))$, $b^* \in Q(x^*, h(x^*))$, $c \in F(g(y))$ and $d^* \in F(h(x^*))$.

Proof. Since $(F, Q)$ is locally non-positive at $x_0 \in K$ with respect to $(g, h)$, we can assume that $N(x_0)$ is a closed and convex set without loss of generality. Since $K \cap N(x_0)$ is also closed and convex, from Theorem B, $(FQ\text{-VI})$ has a solution $x^* \in K \cap N(x_0)$ such that, for $y \in K \cap N(x_0)$

\begin{equation}
(3.1)
 a^* - b^* + c - d^* \notin P(x^*)
\end{equation}

for any $a^* \in Q(x^*, g(y))$, $b^* \in Q(x^*, h(x^*))$, $c \in F(g(y))$ and $d^* \in F(h(x^*))$.

Now we show that for $y \in K$, (3.1) also holds.

(i) If $x^* \in K \cap \text{Int}N(x_0)$, then $N(x_0) \setminus \{x^*\}$ is a neighborhood of the origin and so it is absorbing. For any $y \in K$, there exists $t \in (0, 1)$ such that $t(y - x^*) \in N(x_0) \setminus \{x^*\}$ and so $y_t := ty + (1 - t)x^* \in K \cap N(x_0)$. Hence

\begin{equation}
(3.2)
 a^*_t - b^*_t + c_t - d^*_t \notin P(x^*)
\end{equation}

for any $a^*_t \in Q(x^*, g(y_t))$, $b^*_t \in Q(x^*, h(x^*))$, $c \in F(g(y_t))$ and $d^*_t \in F(h(x^*))$.

On the other hand, the following set

$$A = \{y \in K : a - b + c - d \in P(x) \text{ for any } a \in Q(x, g(y)), b \in Q(x, h(x))$$

$$c \in F(g(y)) \text{ and } d \in F(h(x))\},$$

is convex for all $x \in K$. In fact, if $y_1, y_2 \in A$, then for $x \in K$,

$$a_1 - b + c_1 - d \in P(x)$$
for any \( a_1 \in Q(x, g(y_1)), b \in Q(x, h(x)), c_1 \in F(g(y_1)) \) and \( d \in F(h(x)) \) and
\[
a_2 - b + c_2 - d \in P(x)
\]
for any \( a_2 \in Q(x, g(y_2)), b \in Q(x, h(x)), c_2 \in F(g(y_2)) \) and \( d \in F(h(x)) \).
Hence for \( t \in (0, 1) \), from the condition (d), we have, for \( x \in K \)
\[
ta_1 + (1 - t)a_2 - b + tc_1 + (1 - t)c_2 - d \in P(x)
\]
for any \( ta_1 + (1 - t)a_2 \in tQ(x, g(y_1)) + (1 - t)Q(x, g(y_2)) = Q(x, g(ty_1 + (1 - t)y_2)), \)
\[
b \in Q(x, h(x)),
\]
\[
tc_1 + (1 - t)c_2 \in tF(g(y_1)) + (1 - t)F(g(y_2)) = F(g(ty_1 + (1 - t)y_2)), \)
\[
d \in F(h(x)).
\]
Hence \( ty_1 + (1 - t)y_2 \in A \), which shows that \( A \) is convex. Thus by the continuities of \( g, h, F \) and \( Q \) from (3.2) we have for \( y \in K \)
\[
a^* - b^* + c - d^* \in P(x^*)
\]
for any \( a^* \in Q(x^*, g(y)), b^* \in Q(x^*, h(x^*)), c \in F(g(y)) \) and \( d^* \in F(h(x^*)) \).
(ii) Since \((F, Q)\) is locally non-positive at \( x_0 \in K \) with respect to \((g, h)\), for \( x^* \in K \cap \partial N(x_0) \) there exists \( z_0 \in K \cap \text{Int}N(x_0) \) such that
\[
a_0 - b^* + c_0 - d^* \in -P(x^*)
\]
for any \( a_0 \in Q(x^*, g(z_0)), b^* \in Q(x^*, h(x^*)), c_0 \in F(g(z_0)) \) and \( d^* \in F(h(x^*)) \).
By a similar method, for any \( y \in K \), there exists a \( t \in (0, 1) \) such that
\( t(y - z_0) \in N(x_0) \setminus \{z_0\} \), so \( z_t := ty + (1 - t)z_0 \in K \cap N(x_0) \). Hence it follows from (3.1)
\[
a_t - b^* + c_t - d^* \in P(x^*)
\]
for any \( a_t \in Q(x^*, g(z_t)), b^* \in Q(x^*, h(x^*)), c_t \in F(g(z_t)) \) and \( d^* \in F(h(x^*)) \).
Letting \( t \to 0 \) in (3.4), we obtain
\[
a_0 - b^* + c_0 - d^* \in P(x^*)
\]
for any \( a_0 \in Q(x^*, g(z_0)), b^* \in Q(x^*, h(x^*)), c_0 \in F(g(z_0)) \) and \( d^* \in F(h(x^*)) \).
Thus by (3.3) and (3.5),
\[
a_0 - b^* + c_0 - d^* = 0
\]
for any \( a_0 \in Q(x^*, g(z_0)), b^* \in Q(x^*, h(x^*)), c_0 \in F(g(z_0)) \) and \( d^* \in F(h(x^*)) \). 
Thus by (3.4) and (3.6), we have
\[
ta^*_0 + (1 - t)b^* - a_0 + tc_t + (1 - t)d^* - c_0 \in P(x^*)
\]
for any \( a^*_0 \in Q(x^*, g(z_0)), b^* \in Q(x^*, h(x^*)), a_0 \in Q(x^*, g(z_0)), c_t \in F(g(z_t)), d^* \in F(h(x^*)) \) and \( c_0 \in F(g(z_0)) \).
Hence by (3.6) and (3.7)
\[
a^*_0 - b^* + c_t - d^* \in P(x^*)
\]
for any \( a \in Q(x^*, g(z)) \), \( b \in Q(x^*, h(x^*)) \), \( c \in F(g(z)) \), and \( d \in F(h(x^*)) \).

Letting \( t \to 1 \) in (3.8), by the condition (d) we have

\[
a^\ast - b^\ast + c - d^\ast \in P(x^*)
\]

for any \( a^\ast \in Q(x^*, g(y)) \), \( b^\ast \in Q(x^*, h(x^*)) \), \( c \in F(g(y)) \) and \( d \in F(h(x^*)) \).

Hence by (i) and (ii), the proof is completed. \( \square \)

Letting \( D = K \) in the condition (c) of Theorem 3.1, we have the following result as a corollary.

**Theorem 3.2.** Let \( K \) be a nonempty compact and convex subset of a real Banach space \( X \), and assume that the condition (a), (b) and (d) of Theorem 3.1 hold with the following condition (c)' instead of (c) of Theorem 3.1:

(c)' the mappings \( (F, Q) \) is locally non-positive at \( x_0 \in K \) with respect to \( (g, h) \).

Then (FQ-VI) has a solution in the neighborhood of \( x_0 \), that is, there exists \( x^* \in K \cap N(x_0) \) such that, for \( y \in K \)

\[
a^\ast - b^\ast + c - d^\ast \in P(x^*)
\]

for any \( a^\ast \in Q(x^*, g(y)) \), \( b^\ast \in Q(x^*, h(x^*)) \), \( c \in F(g(y)) \) and \( d \in F(h(x^*)) \).

**Theorem 3.3.** Assume that

(a) \( g, h : K \to K \) are continuous and surjective, set-valued mappings

\( F : K \to 2^Y \) and \( Q : K \times K \to 2^Y \) are continuous and \( P \) is upper semicontinuous,

(b) a single-valued mapping \( T : K \times K \to Y \) satisfies

(b1) for \( x \in K \), \( T(x, x) \in P(x) \),

(b2) for \( x, y \in K \),

\[
a - b + c - d - T(x, y) \in P(x)
\]

for any \( a \in Q(x, g(y)) \), \( b \in Q(x, h(x)) \), \( c \in F(g(y)) \) and \( d \in F(h(x)) \),

(b3) for \( x \in K \) the set \( \{ y \in K : T(x, y) \notin P(x) \} \) is convex,

(c) \( (F, Q) \) is locally non-positive at \( x_0 \in K \) with respect to \( (g, h) \), and there exists a nonempty compact and convex subset \( D \) of \( K \cap N(x_0) \) such that for all \( x \in K \cap N(x_0) \setminus D \) there exists \( y \in D \) satisfying

\[
a - b + c - d \notin P(x)
\]

for any \( a \in Q(x, g(y)) \), \( b \in Q(x, h(x)) \), \( c \in F(g(y)) \) and \( d \in F(h(x)) \).

(d) \( g \) and \( F \) are linear and \( Q \) is linear in the second argument.

Then (FQ-CP) has a solution in the neighborhood of \( x_0 \), that is, there exists \( x^* \in K \cap N(x_0) \) such that,

\[
a^\ast + b^\ast = 0 \text{ for any } a^\ast \in Q(x^*, h(x^*)) \text{ and } b^\ast \in F(h(x^*))
\]
and for $y \in K$,

$$a^*_y + c \in P(x^*) \quad \text{for any } a^*_y \in Q(x^*, g(y)) \text{ and } c \in F(g(y)).$$

**Proof.** The conclusion follows directly from Theorem A and Theorem 3.1. □

**Remark 3.1.** Though Theorem A is used to prove Theorem 3.3 and Theorem B is used to prove Theorem 3.1 and Theorem 3.2, Theorem 3.1, 3.2 and 3.3 extend and generalize Theorems A and B.

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