ON INTUITIONISTIC FUZZY SUBSPACES

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Abstract. We introduce a new concept of intuitionistic fuzzy topological subspace, which coincides with the usual concept of intuitionistic fuzzy topological subspace due to Samanta and Mondal [18] in the case that 
\[ \mu = \chi_A \] for \( A \subseteq X \). Also, we introduce and study some concepts such as continuity, separation axioms, compactness and connectedness in this sense.

1. Introduction and preliminaries

ˇSostak [19], introduce the fundamental concept of a fuzzy topological structure as an extension of both crisp topology and Chang’s fuzzy topology [4], in the sense that not only the object were fuzzified, but also the axiomatics. In [20, 21] ˇSostak gave some rules and showed how such an extension can be realized. Chattopadhyay et al. [5, 6] have redefined the similar concept. In [16] Ramadan gave a similar definition namely “Smooth fuzzy topology” for lattice \( L = [0, 1] \), it has been developed in many direction [9-11]. As a generalization of fuzzy sets, the notion of intuitionistic fuzzy sets was introduced by Atanassov [1-3]. By using intuitionistic fuzzy sets, Çoker and his colleague [7, 8] introduced the topology of intuitionistic fuzzy sets. Samanta and Mondal [17, 18] introduced the notion of intuitionistic fuzzy topology which is a generalization of the concepts of fuzzy topology and the topology of intuitionistic fuzzy sets. Recently, much work has been done with this concept [12-14].

Throughout this paper, let \( X \) be a nonempty set \( I = [0, 1] \), \( I_0 = (0, 1] \), \( I_1 = [0, 1) \) and \( I^X \) denote the set of all fuzzy subsets of \( X \). For \( \mu \in I^X \), we call \( A_\mu = \{ \nu \in I^X : \nu \leq \mu \} \). IF stand for intuitionistic fuzzy. For \( \alpha \in I \), \( \alpha(x) = \alpha \) for all \( x \in X \). A fuzzy point \( x_t \) for \( t \in I_0 \) is an element of \( I^X \) such that, for \( y \in X \),

\[
x_t(y) = \begin{cases} t & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}
\]
The set of all fuzzy points in $X$ is denoted by $Pt(X)$. A fuzzy set $\lambda$ is quasi-coincident with a fuzzy set $\mu$, denoted by $\lambda\mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. Otherwise $\lambda \not\mu$ [15].

**Lemma 1.1** ([22]). If $f(x_1)q\lambda[f(\mu)], \text{then} \ x_1qf^{-1}(\lambda)[\mu], \text{where} \ x_1q\lambda[\mu]$ means $t + \lambda(x) > \mu(x)$.

**Definition 1.1** ([18]). An intuitionistic gradation of openness (IGO, for short) on $X$ is an ordered pair $(\tau, \tau^*)$ of mappings $\tau, \tau^*: I^X \rightarrow I$ satisfies the following conditions:

1. **(IGO1)** $\tau(\lambda) + \tau^*(\lambda) \leq 1$ for each $\lambda \in I^X$;
2. **(IGO2)** $\tau(\emptyset) = \tau(1) = 1$, $\tau^*(\emptyset) = \tau^*(1) = 0$;
3. **(IGO3)** $\tau(\lambda_1 \wedge \lambda_2) \geq \tau(\lambda_1) \wedge \tau(\lambda_2)$ and $\tau^*(\lambda_1 \wedge \lambda_2) \leq \tau^*(\lambda_1) \vee \tau^*(\lambda_2)$ for any $\lambda_1, \lambda_2 \in I^X$;
4. **(IGO4)** $\tau(\bigvee_{i \in I} \lambda_i) \leq \bigwedge_{i \in I} \tau(\lambda_i)$ and $\tau^*(\bigvee_{i \in I} \lambda_i) \leq \bigvee_{i \in I} \tau^*(\lambda_i)$ for any $\{\lambda_i : i \in I \} \subseteq I^X$.

The triplet $(X, \tau, \tau^*)$ is called an intuitionistic fuzzy topological space (briefly, IFTS). $\tau$ and $\tau^*$ may be interpreted as gradation of openness and gradation of nonopenness, respectively.

**Definition 1.2** ([18]). Let $f: (X, \tau, \tau^*) \rightarrow (Y, \sigma, \sigma^*)$ be a mapping from an IFTS $(X, \tau, \tau^*)$ to another IFTS $(Y, \sigma, \sigma^*)$. Then $f$ is said to be $IF$-continuous if for each $\nu \in I^Y$, 

$$\sigma(\nu) \leq \tau(f^{-1}(\nu)) \quad \text{and} \quad \sigma^*(\nu) \geq \tau^*(f^{-1}(\nu)).$$

## 2. Intuitionistic fuzzy subspaces

**Definition 2.1.** Let $(X, \tau, \tau^*)$ be an IFTS and $\mu \in I^X$. The pair of mappings $(\tau_{\mu}, \tau^*_{\mu}): A_\mu \rightarrow I$ defined by:

$$\tau_{\mu}(\nu) = \bigvee \{\tau(\lambda) : \lambda \in I^X, \lambda \wedge \mu = \nu\}$$

$$\tau^*_{\mu}(\nu) = \bigwedge \{\tau^*(\lambda) : \lambda \in I^X, \lambda \wedge \mu = \nu\}$$

is an intuitionistic fuzzy $\mu$-topology induced over $\mu$ by $(\tau, \tau^*)$. For any $\nu \in A_\mu$, the number $\tau_{\mu}(\nu)$ is called the $\mu$-openness degree of $\nu$, while $\tau^*_{\mu}(\nu)$ is called $\mu$-nonopenness degree of $\nu$.

**Remark 2.1.** If $A \subseteq X$ and $\mu = \chi_A$, we have just the usual concept of intuitionistic fuzzy subspace due to Samanta and Mondal [18]. Given $(\tau_{\mu}, \tau^*_{\mu})$ and $\nu \in A_\mu$, we can define $((\tau_{\mu})_\nu, (\tau^*_{\mu})_\nu)$, the intuitionistic fuzzy $\nu$-topology induced over $\nu$ by $(\tau_{\mu}, \tau^*_{\mu})$. We have trivially $\tau_{\nu} = (\tau_{\mu})_\nu$ and $\tau^*_{\nu} = (\tau^*_{\mu})_\nu$, that is, an intuitionistic fuzzy subspace of an intuitionistic fuzzy subspace is also an intuitionistic fuzzy subspace.

**Theorem 2.1.** Let $(X, \tau, \tau^*)$ be an IFTS and $\mu \in I^X$. Then $(\tau_{\mu}, \tau^*_{\mu})$ verifies the following properties:

1. **(μIGO1)** $\tau_{\mu}(\nu) + \tau^*_{\mu}(\nu) \leq 1$ for each $\nu \in A_\mu$. 

(µIGO2) τ_µ(0) = τ_µ(µ) = 1, τ_µ^*(0) = 0.
(µIGO3) τ_µ(ν_1 ∨ ν_2) ≥ τ_µ(ν_1) ∧ τ_µ(ν_2) and τ_µ^*(ν_1 ∨ ν_2) ≤ τ_µ^*(ν_1) ∨ τ_µ^*(ν_2) for each ν_1, ν_2 ∈ A_µ.
(µIGO4) τ_µ((\bigvee_{i \in J} ν_i) ≥ \bigwedge_{i \in J} τ_µ(ν_i) and τ_µ^*(\bigvee_{i \in J} ν_i) ≤ \bigvee_{i \in J} τ_µ^*(ν_i) for each \{ν_i : i \in J\} ⊆ A_µ.

Proof. (µIGO1) and (µIGO2) are clear.

(µIGO3) Suppose that there exist ν_1, ν_2 ∈ A_µ such that

τ_µ^*(ν_1 ∨ ν_2) ≤ τ_µ^*(ν_1) ∨ τ_µ^*(ν_2).

Then there exists s ∈ (0, 1) such that

τ_µ^*(ν_1 ∨ ν_2) ≥ s ≥ τ_µ^*(ν_1) ∨ τ_µ^*(ν_2).

Since τ_µ^*(ν_1) ≤ s and τ_µ^*(ν_2) ≤ s, there exist λ_1, λ_2 ∈ I^X with τ^*(λ_1) ≤ s and τ^*(λ_2) ≤ s such that ν_1 = λ_1 ∨ µ and ν_2 = λ_2 ∨ µ and hence ν_1 ∨ ν_2 = (λ_1 ∨ λ_2) ∨ µ.

Since τ^*(λ_1) ∨ τ^*(λ_2) ≤ s, τ_µ^*(ν_1 ∨ ν_2) ≤ s. It is a contradiction. Hence, τ_µ^*(ν_1 ∨ ν_2) ≥ τ_µ^*(ν_1) ∨ τ_µ^*(ν_2) for each ν_1, ν_2 ∈ A_µ. Similarly, we can show τ_µ(ν_1 ∨ ν_2) ≥ τ_µ(ν_1) ∧ τ_µ(ν_2) for each ν_1, ν_2 ∈ A_µ.

(µIGO4) Suppose that there exist a family \{ν_i : i \in J\} ⊆ A_µ such that

τ_µ^*(\bigvee_{i \in J} ν_i) ≤ \bigvee_{i \in J} τ_µ^*(ν_i).

Then there exists s ∈ (0, 1) such that

τ_µ^*(\bigvee_{i \in J} ν_i) > s ≥ \bigvee_{i \in J} τ_µ^*(ν_i).

Since τ_µ^*(ν_i) ≤ s for all i ∈ J, there exists λ_i ∈ I^X with τ^*(λ_i) ≤ s such that ν_i = λ_i ∨ µ. Thus \bigvee_{i \in J} ν_i = (\bigvee_{i \in J} λ_i) ∨ µ.

Since τ^*(\bigvee_{i \in J} λ_i) ≤ \bigvee_{i \in J} τ^*(λ_i) ≤ s, τ_µ^*(\bigvee_{i \in J} ν_i) ≤ s. It is a contradiction. Hence τ_µ^*(\bigvee_{i \in J} ν_i) ≥ \bigvee_{i \in J} τ_µ^*(ν_i) for each \{ν_i : i \in J\} ⊆ A_µ. Similarly we can show τ_µ(\bigvee_{i \in J} ν_i) ≥ \bigwedge_{i \in J} τ_µ(ν_i) for each \{ν_i : i \in J\} ⊆ A_µ.

Theorem 2.2. Let (X, τ, τ^*) be an IFTS and µ ∈ I^X. Define the mappings F_{τ^*}, F_{τ^*}^*: A_µ → I by: F_{τ^*}(ν) = τ^*(µ - ν) and F_{τ^*}^*(ν) = τ^*(µ - ν) for each ν ∈ A_µ. Then (F_{τ^*}, F_{τ^*}^*) satisfies the following properties:

(µGIC1) F_{τ^*}(ν) + F_{τ^*}^*(ν) ≤ 1 for each ν ∈ A_µ.
(µGIC2) F_{τ^*}(0) = F_{τ^*}(µ) = 1, F_{τ^*}^*(0) = F_{τ^*}^*(µ) = 0.
(µGIC3) F_{τ^*}(ν_1 ∨ ν_2) ≥ F_{τ^*}(ν_1) ∧ F_{τ^*}(ν_2) and F_{τ^*}^*(ν_1 ∨ ν_2) ≤ F_{τ^*}^*(ν_1) ∨ F_{τ^*}^*(ν_2) for each ν_1, ν_2 ∈ A_µ.
(µGIC4) F_{τ^*}(\bigwedge_{i \in J} ν_i) ≥ \bigvee_{i \in J} F_{τ^*}(ν_i) and F_{τ^*}^*(\bigwedge_{i \in J} ν_i) ≤ \bigvee_{i \in J} F_{τ^*}^*(ν_i) for each \{ν_i : i ∈ J\} ⊆ A_µ.

Proof. It is clear.
Definition 2.2. Let \((X, \tau, \tau^*)\) be an IFTS, \(\mu \in I^X\) and \(x_t \in \mu\). Then for \(r \in I_0, s \in I_1\) with \(r + s \leq 1\). We say that \(\nu \in A_\mu\) is \((r, s)\)-\(IF\mu\)-\(q\)neighborhood of \(x_t\), if there is \(\eta \in A_\mu\) with \(\tau_\mu(\eta) \geq r\) and \(\tau^*_\mu(\eta) \leq s\) such that \(x_tq\eta[\mu]\) and \(\eta \leq \nu\). We denote the family of all \((r, s)\)-\(IF\mu\)-\(q\)neighborhoods of \(x_t\) by \(Q_{\tau_\mu, \tau^*_\mu}(x_t, r, s)\).

Theorem 2.3. Let \((X, \tau, \tau^*)\) be an IFTS, \(\mu \in I^X\) and \(x_t \in \mu\). Then for \(r \in I_0, s \in I_1\) with \(r + s \leq 1\),
\[Q_{\tau_\mu, \tau^*_\mu}(x_t, r, s) = \{\lambda \land \mu : \lambda \in Q_{\tau_\tau^*}(x_t, r, s)\}\]
Proof. Let \(\lambda \in Q_{\tau_\tau^*}(x_t, r, s)\). Then there is \(\xi \in I^X\) with \(\tau_\xi \geq r\) and \(\tau^*_\xi \leq s\) such that \(x_tq\xi\) and \(\xi \leq \lambda\). Then, \(\xi \land \mu \leq \lambda \land \mu\). Put \(\eta = \xi \land \mu\). Then \(\tau_\mu(\eta) \geq r\) and \(\tau^*_\mu(\eta) \leq s\). Since \(x_tq\xi, t + \xi(x) > 1\) which implies that \(t + (\xi \land \mu)(x) > \mu(x)\). Then \(x_tq\eta[\mu]\). Thus there is \(\eta = \xi \land \mu \in A_\mu\) with \(\tau_\mu(\eta) \geq r, \tau^*_\mu(\eta) \leq s, x_tq\eta[\mu]\) and \(\eta \leq \lambda \land \mu\). Hence \(\lambda \land \mu \in Q_{\tau_\mu, \tau^*_\mu}(x_t, r, s)\). □

Theorem 2.4. Let \((X, \tau, \tau^*)\) be an IFTS, \(\mu \in I^X\) and \(x_t \in \mu\). Then for \(r \in I_0, s \in I_1\) with \(r + s \leq 1\), \(Q_{\tau_\mu, \tau^*_\mu}(x_t, r, s)\) satisfies the following:
\begin{enumerate}
\item[(\(\mu Q1\))] If \(\nu \in Q_{\tau_\mu, \tau^*_\mu}(x_t, r, s)\), then \(x_tq\nu[\mu]\).
\item[(\(\mu Q2\))] If \(\nu_1, \nu_2 \in Q_{\tau_\mu, \tau^*_\mu}(x_t, r, s)\), then \(\nu_1 \land \nu_2 \in Q_{\tau_\mu, \tau^*_\mu}(x_t, r, s)\).
\item[(\(\mu Q3\))] If \(\nu \in Q_{\tau_\mu, \tau^*_\mu}(x_t, r, s)\) and \(\nu^* \in A_\mu\) such that \(\nu \leq \nu^*\), then \(\nu^* \in Q_{\tau_\mu, \tau^*_\mu}(x_t, r, s)\).
\item[(\(\mu Q4\))] If \(\nu \in Q_{\tau_\mu, \tau^*_\mu}(x_t, r, s)\), there is \(\nu^* \in Q_{\tau_\mu, \tau^*_\mu}(x_t, r, s)\) such that \(\nu \in Q_{\tau_\mu, \tau^*_\mu}(y_m, r, s)\) for each \(y_mq\nu[\mu]\).
\end{enumerate}
Proof. It is clear. □

Theorem 2.5. Let \((X, \tau, \tau^*)\) be an IFTS, \(\mu \in I^X\). Then for each \(\nu \in A_\mu\) and each \(r \in I_0, s \in I_1\) with \(r + s \leq 1\), we define the operator \(C_{\tau_\mu, \tau^*_\mu} : A_\mu \times I_0 \times I_1 \rightarrow A_\mu\) as follows:
\[C_{\tau_\mu, \tau^*_\mu}(\nu, r, s) = \bigwedge \{\eta \in A_\mu : \eta \geq \nu, \tau_\mu(\mu - \eta) \geq r, \tau^*_\mu(\mu - \eta) \leq s\}\]
For each \(\nu, \nu_1, \nu_2 \in A_\mu\) and \(r \in I_0, s \in I_1\) with \(r + s \leq 1\), the operator \(C_{\tau_\mu, \tau^*_\mu}\) satisfies the following:
\begin{enumerate}
\item[(\(\mu C1\))] \(C_{\tau_\mu, \tau^*_\mu}(0, r, s) = 0\).
\item[(\(\mu C2\))] \(\nu \leq C_{\tau_\mu, \tau^*_\mu}(\nu, r, s)\).
\item[(\(\mu C3\))] \(C_{\tau_\mu, \tau^*_\mu}(\nu_1 \lor \nu_2, r, s) = C_{\tau_\mu, \tau^*_\mu}(\nu_1, r, s) \lor C_{\tau_\mu, \tau^*_\mu}(\nu_2, r, s)\).
\item[(\(\mu C4\))] \(C_{\tau_\mu, \tau^*_\mu}(\nu, r, s) \leq C_{\tau_\mu, \tau^*_\mu}(\nu, m, n)\) if \(r \leq m, s \geq n\) and \(m + n \leq 1\).
\item[(\(\mu C5\))] \(C_{\tau_\mu, \tau^*_\mu}(\nu, r, s) = C_{\tau_\mu, \tau^*_\mu}(\nu, r, s)\).
\end{enumerate}
Proof. It is straightforward. □

Theorem 2.6. Let \((X, \tau, \tau^*)\) be an IFTS, \(\mu \in I^X\). Then for each \(\nu \in A_\mu\) and each \(r \in I_0, s \in I_1\) with \(r + s \leq 1\), we define the operator \(I_{\tau_\mu, \tau^*_\mu} : A_\mu \times I_0 \times I_1 \rightarrow A_\mu\) as follows:
$A_\mu$ as follows:

$$I_{\tau_\mu, \tau_\mu^*}(\nu, r, s) = \bigvee \{\eta \in A_\mu : \eta \leq \nu, \tau_\mu(\eta) \geq r, \tau_\mu^*(\eta) \leq s\}. \tag{\mu1}$$

For each $\nu, \nu_1, \nu_2 \in A_\mu$ and $r \in I_0$, $s \in I_1$ with $r + s \leq 1$, the operator $I_{\tau_\mu, \tau_\mu^*}$ satisfies the following:

(i) $I_{\tau_\mu, \tau_\mu^*}(\nu, r, s) = \mu$.

(ii) $I_{\tau_\mu, \tau_\mu^*}(\nu, r, s) \leq \nu$.

(iii) $I_{\tau_\mu, \tau_\mu^*}(\nu_1 \land \nu_2, r, s) = I_{\tau_\mu, \tau_\mu^*}(\nu_1, r, s) \land I_{\tau_\mu, \tau_\mu^*}(\nu_2, r, s)$.

(iv) $I_{\tau_\mu, \tau_\mu^*}(\nu, r, s) \geq I_{\tau_\mu, \tau_\mu^*}(\nu, m, n)$ if $r \leq m$, $s \geq n$ and $m + n \leq 1$.

(v) $I_{\tau_\mu, \tau_\mu^*}(I_{\tau_\mu, \tau_\mu^*}(\nu, r, s), r, s) = I_{\tau_\mu, \tau_\mu^*}(\nu, r, s)$.

Proof. It is straightforward. \qed

Theorem 2.7. Let $(X, \tau, \tau^*)$ be an IFTS and $\mu \in I^X$. For each $\nu \in A_\mu$ and each $r \in I_0$, $s \in I_1$ with $r + s \leq 1$, we have

(i) $I_{\tau_\mu, \tau_\mu^*}(\mu - \nu, r, s) = \mu - C_{\tau_\mu, \tau_\mu^*}(\nu, r, s)$.

(ii) $C_{\tau_\mu, \tau_\mu^*}(\mu - \nu, r, s) = \mu - I_{\tau_\mu, \tau_\mu^*}(\nu, r, s)$.

Proof. (i) For each $\nu \in A_\mu$ and each $r \in I_0$, $s \in I_1$ with $r + s \leq 1$, we have the following:

$$\mu - C_{\tau_\mu, \tau_\mu^*}(\nu, r, s) = \mu - \bigwedge \{\eta \in A_\mu : \nu \leq \eta, \tau_\mu(\eta) \geq r, \tau_\mu^*(\mu - \eta) \leq s\}
= \bigvee \{\mu - \eta : \mu - \eta \leq \mu - \nu, \tau_\mu(\mu - \eta) \geq r, \tau_\mu^*(\mu - \eta) \leq s\}
= I_{\tau_\mu, \tau_\mu^*}(\mu - \nu, r, s). \tag{\mu2}$$

(ii) It is similar to (i). \qed

Theorem 2.8. Let $(X, \tau, \tau^*)$ be an IFTS, $\mu \in I^X$ and $x_1 \in \mu$. For $\nu \in A_\mu$ and $r \in I_0$, $s \in I_1$ with $r + s \leq 1$, $x_1 \in C_{\tau_\mu, \tau_\mu^*}(\nu, r, s)$ if and only if for each $\eta \in A_\mu$ with $\tau_\mu(\eta) \geq r$, $\tau_\mu^*(\eta) \leq s$ and $x_1 \tau_\mu(\eta)$ we have $\nu \tau_\mu(\eta)$.

Proof. Let $x_1 \in C_{\tau_\mu, \tau_\mu^*}(\nu, r, s)$, $\eta \in A_\mu$, with $\tau_\mu(\eta) \geq r$, $\tau_\mu^*(\eta) \leq s$ and $x_1 \tau_\mu(\eta)$. Suppose that $\nu \not\tau_\mu(\eta)$, then $\nu \not\leq \mu - \eta$. Since $x_1 \tau_\mu(\eta)$, $\tau_\mu(\eta) > \mu(x)$. This implies $x_1 \not\in \mu - \eta$. Since $\nu \not\leq \mu - \eta$, $\tau_\mu(\mu - (\mu - \eta)) = \tau_\mu(\eta) \geq r$ and $\tau_\mu^*(\mu - (\mu - \eta)) = \tau_\mu^*(\eta) \leq s$, we have $x_1 \not\in C_{\tau_\mu, \tau_\mu^*}(\nu, r, s)$. It is a contradiction.

Conversely, let $\eta \in A_\mu$, with $\tau_\mu(\eta) \geq r$, $\tau_\mu^*(\eta) \leq s$, $x_1 \tau_\mu(\eta)$ and $\nu \tau_\mu(\eta)$. Suppose that $x_1 \not\in C_{\tau_\mu, \tau_\mu^*}(\nu, r, s)$, then there exists $\xi \in A_\mu$, with $\tau_\mu(\mu - \xi) \geq r$, $\tau_\mu^*(\mu - \xi) \leq s$, $\nu \not\leq \xi$ and $x_1 \not\in \xi$. Then $\xi(x) < t$ which implies $(\mu - \xi)(x) + t > \mu(x)$. Thus $x_1 \tau_\mu(\mu - \xi)$, then from our hypothesis $\nu \tau_\mu(\mu - \xi)$. Thus there is $y \in X$ such that $\nu(y) + (\mu - \xi)(y) > \mu(y)$. Thus $\nu(y) > \xi(y)$ which is a contradiction with $\nu \not\leq \xi$. Hence $x_1 \in C_{\tau_\mu, \tau_\mu^*}(\nu, r, s)$.
3. IFμ-continuity

**Definition 3.1.** Let \((X, \tau, \tau^\ast)\) and \((Y, \sigma, \sigma^\ast)\) be two IFTSs and \(\mu \in I^X\). Then the mapping \(f : (X, \tau, \tau^\ast) \rightarrow (Y, \sigma, \sigma^\ast)\) is called IFμ-continuous if \(\tau_\mu(f^{-1}(\nu) \land \mu) \geq \sigma_{f(\mu)}(\nu)\) and \(\tau^\ast_\mu(f^{-1}(\nu) \land \mu) \leq \sigma^\ast_{f(\mu)}(\nu)\) for each \(\nu \in A_f(\mu)\).

**Remark 3.1.** Every IF-continuous mapping is IFμ-continuous for all \(\mu\) but the converse is not true in general as the following example shows.

**Example 3.1.** Let \(X = I\). We define the IGO(\(\tau, \tau^\ast\)) and IGO(\(\sigma, \sigma^\ast\)) on \(X\) as follows: for each \(\lambda \in I^X\)

\[
\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = 0, 1 \\ 0.5, & \text{if } 0 < \lambda < 0.6 \\ 0, & \text{otherwise.} \end{cases} \quad \tau^\ast(\lambda) = \begin{cases} 0, & \text{if } \lambda = 0, 1 \\ 0.4, & \text{if } 0 < \lambda < 0.6 \\ 1, & \text{otherwise.} \end{cases}
\]

\[
\sigma(\lambda) = \begin{cases} 1, & \text{if } \lambda = 0, 1 \\ 0.2, & \text{if } 0 < \lambda < 0.6 \\ 0.4, & \text{if } \lambda = 0.7 \\ 0, & \text{otherwise.} \end{cases} \quad \sigma^\ast(\lambda) = \begin{cases} 0, & \text{if } \lambda = 0, 1 \\ 0.7, & \text{if } 0 < \lambda < 0.6 \\ 0.5, & \text{if } \lambda = 0.7 \\ 1, & \text{otherwise.} \end{cases}
\]

Let \(\mu = 0.5\), then

\[
\tau_\mu(\nu) = \begin{cases} 1, & \text{if } \nu = 0, \mu \\ 0.5, & \text{if } 0 < \nu < 0.5 \\ 0, & \text{otherwise.} \end{cases} \quad \tau^\ast_\mu(\nu) = \begin{cases} 0, & \text{if } \nu = 0, \mu \\ 0.4, & \text{if } 0 < \nu < 0.5 \\ 1, & \text{otherwise.} \end{cases}
\]

\[
\sigma_{f(\mu)}(\nu) = \begin{cases} 1, & \text{if } \nu = 0, f(\mu) \\ 0.2, & \text{if } 0 < \nu < 0.5 \\ 0, & \text{otherwise.} \end{cases} \quad \sigma^\ast_{f(\mu)}(\nu) = \begin{cases} 0, & \text{if } \nu = 0, f(\mu) \\ 0.7, & \text{if } 0 < \nu < 0.5 \\ 1, & \text{otherwise.} \end{cases}
\]

Then the identity mapping \(id_X : (X, \tau, \tau^\ast) \rightarrow (X, \sigma, \sigma^\ast)\) is an IFμ-continuous but not IF-continuous.

**Theorem 3.1.** Let \((X, \tau, \tau^\ast)\) and \((Y, \sigma, \sigma^\ast)\) be two IFTSs, \(f : X \rightarrow Y\) be a mapping and \(\{\mu_i : i \in J\} \subseteq I^X\) such that \(\bigcup_{i \in J} \mu_i = 1\). Then \(f\) is IFμ-continuous for each \(i \in J\) if and only if \(f\) is IF-continuous.

**Proof.** Due to Remark 3.1, it suffices to show that if \(f\) is IFμ-continuous for each \(i \in J\), then \(f\) is IF-continuous. Suppose there exists \(\lambda \in I^X\) such that

\[
\tau^\ast(f^{-1}(\lambda)) \not\leq \sigma^\ast(\lambda).
\]

Then there exists \(s \in (0, 1)\) such that

\[
\tau^\ast(f^{-1}(\lambda)) > s \geq \sigma^\ast(\lambda).
\]

Since \(\sigma^\ast(\lambda) \leq s\), \(\sigma^\ast_{f(\mu_i)}(\lambda \land f(\mu_i)) \leq s\) for each \(i \in J\). Since \(f\) is an IFμ\(_i\)-continuous for each \(i \in J\) we have

\[
\tau^\ast_{\mu_i}(f^{-1}(\lambda \land f(\mu_i)) \land \mu_i) \leq \sigma^\ast_{f(\mu_i)}(\lambda \land f(\mu_i)) \leq s
\]
but
\[ f^{-1}(\lambda \wedge f(\mu_i)) \wedge \mu_i = f^{-1}(\lambda) \wedge f^{-1}(f(\mu_i)) \wedge \mu_i = f^{-1}(\lambda) \wedge \mu_i. \]

Then \( \tau_J^*(f^{-1}(\lambda) \wedge \mu_i) \leq s \) for each \( i \in J \). Then for each \( i \in J \) there exists \( \nu_i \in I^X \) with \( \tau_J^*(\nu_i) \leq s \) such that \( f^{-1}(\lambda) \wedge \mu_i = \nu_i \wedge \mu_i \). This implies that \( \bigvee_{i \in J}(f^{-1}(\lambda) \wedge \mu_i) = \bigvee_{i \in J}(\nu_i \wedge \mu_i) \), thus
\[ f^{-1}(\lambda) \wedge (\bigvee_{i \in J} \mu_i) = (\bigvee_{i \in J} \nu_i) \wedge (\bigvee_{i \in J} \mu_i). \]

Since \( \bigvee_{i \in J} \mu_i = 1 \), \( f^{-1}(\lambda) = \bigvee_{i \in J} \nu_i \). Then
\[ \tau_J^*(f^{-1}(\lambda)) = \tau_J^*(\bigvee_{i \in J} \nu_i) \leq \bigvee_{i \in J} \tau_J^*(\nu_i) \leq s. \]

It is a contradiction. Thus \( \tau_J^*(f^{-1}(\lambda)) \leq \sigma^*(\lambda) \) for each \( \lambda \in I^Y \). Similarly, we can show \( \tau_J^*(f^{-1}(\lambda)) \leq \sigma^*(\lambda) \) for each \( \lambda \in I^Y \). Thus \( f \) is an IF-continuous. □

**Theorem 3.2.** Let \((X, \tau, \sigma^*)\) and \((Y, \sigma, \sigma^*)\) be two IFTSs, \( \mu \in I^X \) and \( f : X \to Y \) be an injective mapping. For \( r \in I_0, s \in I_1 \) with \( r + s \leq 1 \), the following statements are equivalent:

(i) \( f \) is IF-continuous.

(ii) \( \mathcal{F}_{\tau_J}(f^{-1}(\lambda) \wedge \mu) \geq \mathcal{F}_{\sigma_{f(\mu)}}(\lambda) \) and \( \mathcal{F}_{\tau_J}(f^{-1}(\lambda) \wedge \mu) \leq \mathcal{F}_{\sigma_{f(\mu)}}(\lambda) \) for each \( \lambda \in \mathcal{A}_{f(\mu)} \).

(iii) \( f(C_{\tau_J}(\mu, \nu, r, s)) \leq C_{\sigma_{f(\mu)}}(\nu, r, s) \) for each \( \nu \in \mathcal{A}_\mu \).

(iv) \( C_{\tau_J}(f^{-1}(\lambda) \wedge \mu, r, s) \leq f^{-1}(C_{\sigma_{f(\mu)}}(\lambda, r, s)) \wedge \mu \) for each \( \lambda \in \mathcal{A}_{f(\mu)} \).

(v) \( f^{-1}(I_{\sigma_{f(\mu)}}(\lambda, r, s)) \wedge \mu \leq I_{\tau_J}(f^{-1}(\lambda) \wedge \mu, r, s) \) for each \( \lambda \in \mathcal{A}_{f(\mu)} \).

(vi) For each \( x_t \in \mu \) and \( \lambda \in \mathcal{A}_{f(\mu)} \) with \( \sigma_{f(\mu)}(\lambda) \geq r, \sigma_{f(\mu)}(\lambda) \leq s \) and \( f(x_t)q\lambda[f(\mu)] \), there is \( \nu \in \mathcal{A}_\mu \) with \( \tau_J(\nu) \geq r, \tau_J(\nu) \leq s \) such that \( x_tq\nu[f(\mu)] \) and \( f(\nu) \leq \lambda \).

(vii) For each \( x_t \in \mu \) and each \( \lambda \in Q_{\sigma_{f(\mu)}}(f(x_t), r, s) \), there exists \( \nu \in Q_{\sigma_{f(\mu)}}(f(x_t), r, s) \) such that \( f(\nu) \leq \lambda \).

(viii) For each \( x_t \in \mu \) and each \( \lambda \in Q_{\sigma_{f(\mu)}}(f(x_t), r, s) \),
\[ f^{-1}(\lambda) \in Q_{\tau_J}(x_t, r, s). \]

**Proof.** (i)⇒ (ii) For each \( \lambda \in \mathcal{A}_{f(\mu)} \), we have
\[ \mathcal{F}_{\sigma_{f(\mu)}}(\lambda) = \sigma_{f(\mu)}(f(\mu) - \lambda) \geq \tau_J^*(f^{-1}(f(\mu) - \lambda) \wedge \mu) \]
\[ = \tau_J^*(f^{-1}(f(\mu) - f^{-1}(\lambda)) \wedge \mu) \]
\[ = \tau_J^*(\mu - f^{-1}(\lambda) \wedge \mu) = \mathcal{F}_{\tau_J}(f^{-1}(\lambda) \wedge \mu). \]

Similarly, we can show \( \mathcal{F}_{\sigma_{f(\mu)}}(\lambda) \leq \mathcal{F}_{\tau_J}(f^{-1}(\lambda) \wedge \mu) \) for each \( \lambda \in \mathcal{A}_{f(\mu)} \).
(ii)⇒(iii) For each $\nu \in \mathcal{A}_1$ and $r \in I_0$, $s \in I_1$ with $r + s \leq 1$, we have
\[
\begin{align*}
&f^{-1}(C_{\sigma^*(\nu), \sigma^*(\mu)}(f(\nu), r, s)) \\
= &f^{-1}(\bigwedge \{ \lambda \in \mathcal{A}_1 : f(\nu) \leq \lambda, \sigma \lambda \leq r, \sigma^*(\mu) \leq s \}) \\
= &\bigwedge \{ f^{-1}(\lambda) : \nu \leq f^{-1}(\lambda), \mathcal{F}_\sigma(\lambda) \geq r, \mathcal{F}_{\sigma^*(\mu)}(\lambda) \leq s \} \\
\geq &\bigwedge \{ f^{-1}(\lambda) \land \mu \in \mathcal{A}_1 : \nu \leq f^{-1}(\lambda) \land \mu, \mathcal{F}_{\sigma^*(\mu)}(f^{-1}(\lambda) \land \mu) \geq r, \\
&\mathcal{F}_{\sigma^*(\mu)}(f^{-1}(\lambda) \land \mu) \leq s \} \\
= &C_{\tau^*_\mu}(\nu, r, s).
\end{align*}
\]

This implies that $f(C_{\tau^*_\mu}(\nu, r, s)) \leq C_{\sigma^*(\nu), \sigma^*(\mu)}(f(\nu), r, s)$.

(iii)⇒(iv) For each $\lambda \in \mathcal{A}_1$ and $r \in I_0$, $s \in I_1$ with $r + s \leq 1$, we have
\[
\begin{align*}
f(C_{\tau^*_\mu}(f^{-1}(\lambda) \land \mu, r, s)) &\leq C_{\sigma^*(\lambda), \sigma^*(\mu)}(f(f^{-1}(\lambda) \land \mu), r, s) \\
&\leq C_{\sigma^*(\lambda), \sigma^*(\mu)}(\lambda, r, s).
\end{align*}
\]

Thus
\[
C_{\tau^*_\mu}(f^{-1}(\lambda) \land \mu, r, s) \leq f^{-1}(C_{\sigma^*(\lambda), \sigma^*(\mu)}(\lambda, r, s)) \land \mu.
\]

(iv)⇒(v) It is clear from Theorem 2.7.

(v)⇒(i) Suppose that there exist $\lambda \in \mathcal{A}_1$ and $r_0 \in I_0$, $s_0 \in I_1$ with $r_0 + s_0 \leq 1$ such that
\[
\tau^*_\mu(f^{-1}(\lambda) \land \mu) < r_0 \leq \sigma^*(\lambda) \quad \text{and} \quad \tau^*_\mu(f^{-1}(\lambda) \land \mu) > s_0 \geq \sigma^*(\lambda).
\]

Since $\sigma^*(\lambda) \geq r_0$ and $\sigma^*(\lambda) \leq s_0$, $\lambda = I_{\sigma^*(\lambda), \sigma^*(\lambda)}(r_0, s_0)$. By (v) we have
\[
f^{-1}(\lambda) \land \mu = f^{-1}(I_{\sigma^*(\lambda), \sigma^*(\lambda)}(r_0, s_0)) \land \mu \leq I_{\tau^*_\mu, \tau^*_\mu}(f^{-1}(\lambda) \land \mu, r_0, s_0).
\]

Thus
\[
f^{-1}(\lambda) \land \mu = I_{\tau^*_\mu, \tau^*_\mu}(f^{-1}(\lambda) \land \mu, r_0, s_0).
\]

This meaning, $\tau^*_\mu(f^{-1}(\lambda) \land \mu) \geq r_0$ and $\tau^*_\mu(f^{-1}(\lambda) \land \mu) \leq s_0$. It is a contradiction. Then $\tau^*_\mu(f^{-1}(\lambda) \land \mu) \geq \sigma(\lambda)$ and $\tau^*_\mu(f^{-1}(\lambda) \land \mu) \leq \sigma^*(\lambda)$ for each $\lambda \in \mathcal{A}_1$. Hence $f$ is $IF\mu$-continuous.

(i)⇒(vi) Let $x_1 \in \mu$, $\lambda \in \mathcal{A}_1$ and $r \in I_0$, $s \in I_1$ with $r + s \leq 1$ such that $\sigma(\lambda) \geq r$, $\sigma^*(\lambda) \leq s$ and $f(x_1) \notin \sigma(\mu)$. Since $f$ is $IF\mu$-continuous we have
\[
\tau^*_\mu(f^{-1}(\lambda) \land \mu) \geq \sigma(\lambda) \geq r
\]
and
\[
\tau^*_\mu(f^{-1}(\lambda) \land \mu) \leq \sigma^*(\lambda) \leq s.
\]
Since $f(x_t)q\lambda[f(\mu)]$ and by Lemma 1.1, we have $x_tqf^{-1}(\lambda)[\mu]$. Since $f$ is injective, $f^{-1}(\lambda) \leq \mu$ and hence $x_tqf^{-1}(\lambda) \land \mu[\mu]$. Then there exists $\nu = f^{-1}(\lambda) \land \mu \in A_\mu$ with $\tau_\mu(\nu) \geq r$, $\tau_\mu^*(\nu) \leq s$ and $x_tq\nu[\mu]$. Also,

$$f(\nu) = f(f^{-1}(\lambda) \land \mu) \leq f(f^{-1}(\lambda)) \land f(\mu) \leq \lambda \land f(\mu) = \lambda.$$  

(vi)$\Rightarrow$(iii) Let $\nu \in A_\mu$, $x_t \in C_{\tau_\mu,\tau_\mu^*}(\nu, r, s)$ and $\lambda \in A_{f(\mu)}$ with $\sigma_{f(\mu)}(\lambda) \geq r$ and $\sigma_{f(\mu)}^*(\lambda) \leq s$ such that $f(x_t)q\lambda[f(\mu)]$. By (vi), there exists $\eta \in A_\mu$ with $\tau_\mu(\eta) \geq r$, $\tau_\mu^*(\eta) \leq s$ such that $x_tq\eta[\mu]$ and $f(\eta) \leq \lambda$. Since $x_t \in C_{\tau_\mu,\tau_\mu^*}(\nu, r, s)$, $\tau_\mu(\eta) \geq r$, $\tau_\mu^*(\eta) \leq s$ and $x_tq\eta[\mu]$, then by using Theorem 2.8, we have $\nu\eta[\mu]$ which implies that $f(\nu)q\eta[f(\mu)]$ and hence $f(\nu)q\lambda[f(\mu)]$. Thus $f(x_t) \in C_{\sigma_{f(\mu)},\sigma_{f(\mu)}^*}(f(\nu), r, s)$. Then

$$C_{\tau_\mu,\tau_\mu^*}(\nu, r, s) \leq f^{-1}(C_{\sigma_{f(\mu)},\sigma_{f(\mu)}^*}(f(\nu), r, s)).$$

Hence

$$f(C_{\tau_\mu,\tau_\mu^*}(\nu, r, s)) \leq C_{\sigma_{f(\mu)},\sigma_{f(\mu)}^*}(f(\nu), r, s).$$

(vii)$\Rightarrow$(viii) Let $x_t \in \mu$ and $\lambda \in Q_{\sigma_{f(\mu)},\sigma_{f(\mu)}^*}(f(x_t), r, s)$. Then there exists $\eta \in A_\mu$ with $\sigma_{f(\mu)}(\eta) \geq r$, $\sigma_{f(\mu)}^*(\eta) \leq s$ such that $f(x_t)q\eta[f(\mu)]$ and $\eta \leq \lambda$. By (vii), there exists $\nu \in A_\mu$ with $\tau_\mu(\nu) \geq r$, $\tau_\mu^*(\nu) \leq s$ such that $x_tq\nu[\mu]$ and $f(\nu) \leq \eta \leq \lambda$. Hence $\nu \in Q_{\tau_\mu,\tau_\mu^*}(x_t, r, s)$ and $f(\nu) \leq \lambda$. 

(viii)$\Rightarrow$(vi) Let $x_t \in \mu$ and $\lambda \in A_{f(\mu)}$ with $\sigma_{f(\mu)}(\lambda) \geq r$, $\sigma_{f(\mu)}^*(\lambda) \leq s$ and $f(x_t)q\lambda[f(\mu)]$. Then $\lambda \in Q_{\sigma_{f(\mu)},\sigma_{f(\mu)}^*}(f(x_t), r, s)$. By (viii), we have $f^{-1}(\lambda) \in Q_{\tau_\mu,\tau_\mu^*}(x_t, r, s)$ and hence there is $\nu \in A_\mu$ with $\tau_\mu(\nu) \geq r$, $\tau_\mu^*(\nu) \leq s$ such that $x_tq\nu[\mu]$ and $\nu \leq f^{-1}(\lambda)$, so $f(\nu) \leq \lambda$. 

**Theorem 3.3.** Let $(X, \tau, \tau^*)$, $(Y, \sigma, \sigma^*)$ and $(Z, \delta, \delta^*)$ be IFTSs, $\mu \in I^X$, $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. If $f$ is IF-continuous and $g$ is IF-continuous, then $g \circ f$ is IF-continuous. 

**Proof.** For each $\lambda \in A_{(g \circ f)(\mu)}$ we have

$$\delta_{g(f(\mu))}^*(\lambda) \leq \tau_\mu((g \circ f)^{-1}(\lambda) \land \mu).$$

Similarly, $\delta_{g(f(\mu))}^*(\lambda) \leq \tau_\mu((g \circ f)^{-1}(\lambda) \land \mu)$. Thus $g \circ f$ is IF-continuous. 

**Definition 3.2.** Let $(X, \tau, \tau^*)$ be an IFTS and $\mu \in I^X$. For $r \in I_0$, $s \in I_1$ with $r + s \leq 1$, $\nu \in A_\mu$ is called:
(i) \( (r, s) \)-IF\( \mu \)-regular open set if \( I_{\tau, \tau^*}(C_{\tau^*, \tau^*}(\nu, r, s), r, s) = \nu \).
(ii) \( (r, s) \)-IF\( \mu \)-regular closed set if \( C_{\tau^*, \tau^*}(I_{\tau, \tau^*}(\nu, r, s), r, s) = \nu \).

**Definition 3.3.** Let \((X, \tau, \tau^*)\), \((Y, \sigma, \sigma^*)\) be two IFTSs and \( \mu \in I^X \). Then the mapping \( f : X \rightarrow Y \) is called IF\( \mu \)-almost continuous if \( \tau^*_\mu(f^{-1}(\nu) \land \mu) \geq r \) and \( \tau^*_\mu(f^{-1}(\nu) \land \mu) \leq s \) for each \((r, s)\)-IF\( \mu \)-regular open set \( \nu \) in \( A_f(\mu) \).

**Remark 3.2.** Every IF\( \mu \)-continuous mapping is also IF\( \mu \)-almost continuous but the converse is not true in general, as the following example shows.

**Example 3.2.** Consider Example 3.1, and put

\[
\sigma(\lambda) = \begin{cases} 
1, & \text{if } \lambda = 0.1 \\
0.5, & \text{if } \lambda = 0.5 \\
0.6, & \text{if } \lambda = 2.5 \\
0, & \text{otherwise.}
\end{cases}
\]

\[
\sigma^*(\lambda) = \begin{cases} 
0, & \text{if } \lambda = 0.1 \\
0.5, & \text{if } \lambda = 0.5 \\
2.2, & \text{if } \lambda = 2.5 \\
1, & \text{otherwise.}
\end{cases}
\]

Since \( \mu = 0.5 \), we have

\[
\sigma_f(\mu)(\nu) = \begin{cases} 
1, & \text{if } \nu = 0, f(\mu) \\
0.6, & \text{if } \nu = 2.5 \\
0, & \text{otherwise.}
\end{cases}
\]

\[
\sigma^*_f(\mu)(\nu) = \begin{cases} 
0, & \text{if } \nu = 0, f(\mu) \\
0.2, & \text{if } \nu = 2.5 \\
1, & \text{otherwise.}
\end{cases}
\]

Then, the identity mapping \( id_X : (X, \tau, \tau^*) \rightarrow (X, \sigma, \sigma^*) \) is IF\( \mu \)-almost continuous but not IF\( \mu \)-continuous.

**Theorem 3.4.** Let \((X, \tau, \tau^*)\) and \((Y, \sigma, \sigma^*)\) be two IFTSs, \( \mu \in I^X \) and \( f : X \rightarrow Y \) be an injective mapping. For \( r \in I_0, s \in I_1 \) such that \( r + s \leq 1 \) the following statements are equivalent:

(i) \( f \) is IF\( \mu \)-almost continuous.
(ii) \( F_{\tau^*}(f^{-1}(\lambda) \land \mu) \geq r \) and \( F_{\tau^*}^*(f^{-1}(\lambda) \land \mu) \leq s \) for each \((r, s)\)-IF\( \mu \)-regular closed set \( \lambda \) in \( A_f(\mu) \).
(iii) \( f^{-1}(\lambda) \land \mu \leq I_{\tau, \tau^*}(f^{-1}(C_{\sigma(\mu), \sigma^*}(\lambda, r, s) \land \mu), r, s) \) for each \( \lambda \in A_f(\mu) \) with \( \sigma_f(\mu)(\lambda) \geq r \) and \( \sigma^*_f(\mu)(\lambda) \leq s \).
(iv) \( f^{-1}(\lambda) \land \mu \geq C_{\tau^*, \tau^*}(f^{-1}(C_{\sigma(\mu), \sigma^*}(I_{\sigma_f(\mu), \sigma^*}(\lambda, r, s) \land \mu), r, s)) \) for each \( \lambda \in A_f(\mu) \) with \( \sigma_f(\mu)(f(\mu) - \lambda) \geq r \) and \( \sigma^*_f(\mu)(f(\mu) - \lambda) \leq s \).
(v) For each \( x_t \in \mu \) and \( \lambda \in A_f(\mu) \) with \( \sigma_f(\mu)(\lambda) \geq r \), \( \sigma^*_f(\mu)(\lambda) \leq s \) and \( f(x_t) \in \lambda \), there is \( \nu \in A_\mu \) with \( \tau_\mu(\nu) \geq r, \tau^*_\mu(\nu) \leq s \) such that \( f(\nu) \leq I_{\sigma_f(\mu), \sigma^*_f(\mu)}(C_{\sigma(\mu), \sigma^*}(\lambda, r, s), r, s) \).
(vi) For each \( x_t \in \mu \) and \( \lambda \in Q_{\sigma_f(\mu), \sigma^*}(f(\nu), r, s), \) there exists \( \nu \in Q_{\tau^*, \tau^*}(x_t, r, s) \) such that \( f(\nu) \leq I_{\sigma_f(\mu), \sigma^*_f(\mu)}(C_{\sigma(\mu), \sigma^*}(\lambda, r, s), r, s) \).
(vii) For each \( x_t \in \mu \) and each \( \lambda \in Q_{\sigma_f(\mu), \sigma^*}(f(\nu), r, s) \) we have

\[
f^{-1}(I_{\sigma_f(\mu), \sigma^*_f(\mu)}(C_{\sigma(\mu), \sigma^*}(\lambda, r, s), r, s)) \in Q_{\tau^*, \tau^*}(x_t, r, s).
\]
Proof. (i)⇒ (ii) Let \( \lambda \) be \((r, s)\)-IF\( \mu \)-regular closed set in \( A_{f(\mu)} \). Then \( f(\mu) - \lambda \) is \((r, s)\)-IF\( \mu \)-regular open set. Since \( f \) is IF\( \mu \)-almost continuous we have
\[
\tau_\mu(f^{-1}(f(\mu) - \lambda) \land \mu) \geq r \quad \text{and} \quad \tau_\mu^*(f^{-1}(f(\mu) - \lambda) \land \mu) \leq s.
\]
Since \( f \) is injective, \( \tau_\mu((\mu - f^{-1}(\lambda)) \land \mu) \geq r \) and \( \tau_\mu^*((\mu - f^{-1}(\lambda)) \land \mu) \leq s \). Let \( \nu = \mu - (f^{-1}(\lambda) \land \mu) \). Then
\[
\nu(x) = \mu(x) - (f^{-1}(\lambda) \land \mu)(x) = \mu(x) - \min\{f^{-1}(\lambda)(x), \mu(x)\}.
\]
If \( \mu(x) \leq f^{-1}(\lambda)(x) \), then \( \nu(x) = \mu(x) - \mu(x) = 0 \). Then
\[
F_\nu(f^{-1}(\lambda) \land \mu) = \tau_\mu((\mu - (f^{-1}(\lambda) \land \mu)) = \tau_\mu(\nu) = \tau_\mu(1) = 1 \geq r.
\]
and
\[
F_\nu^*(f^{-1}(\lambda) \land \mu) = \tau_\mu^*((\mu - (f^{-1}(\lambda) \land \mu)) = \tau_\mu^*(\nu) = \tau_\mu^*(0) = 0 \leq s.
\]
If \( \mu(x) > f^{-1}(\lambda)(x) \), then
\[
\nu = \mu - f^{-1}(\lambda) = (\mu - f^{-1}(\lambda)) \land \mu.
\]
Then
\[
\tau_\mu(\nu) = \tau_\mu((\mu - f^{-1}(\lambda)) \land \mu) \geq r \quad \text{and} \quad \tau_\mu^*(\nu) = \tau_\mu^*((\mu - f^{-1}(\lambda)) \land \mu) \leq s.
\]
Thus
\[
F_\nu(f^{-1}(\lambda) \land \mu) = \tau_\mu(\nu) \geq r \quad \text{and} \quad F_\nu^*(f^{-1}(\lambda) \land \mu) = \tau_\mu^*(\nu) \leq s.
\]
(iii)⇒ (i) It is clear.

(iii)⇒ (ii) Let \( \lambda \in A_{f(\mu)} \) and \( r \in I_0, s \in I_1 \) with \( r + s \leq 1 \) such that \( \sigma_{f(\mu)}(\lambda) \geq r, \sigma_{f(\mu)}^*(\lambda) \leq s \). Then \( \lambda \leq I_{\sigma_{f(\mu)}, \sigma_{f(\mu)}^*}(C_{\sigma_{f(\mu)}, \sigma_{f(\mu)}^*}((\lambda, r, s), r, s)) \) is \((r, s)\)-IF\( \mu \)-regular open and \( f \) is IF\( \mu \)-almost continuous,
\[
\tau_\mu(f^{-1}(I_{\sigma_{f(\mu)}, \sigma_{f(\mu)}^*}(C_{\sigma_{f(\mu)}, \sigma_{f(\mu)}^*}((\lambda, r, s), r, s)) \land \mu) \geq r
\]
and
\[
\tau_\mu^*(f^{-1}(I_{\sigma_{f(\mu)}, \sigma_{f(\mu)}^*}(C_{\sigma_{f(\mu)}, \sigma_{f(\mu)}^*}((\lambda, r, s), r, s)) \land \mu, r, s) \leq s.
\]
So,
\[
f^{-1}(\lambda) \land \mu \leq f^{-1}(I_{\sigma_{f(\mu)}, \sigma_{f(\mu)}^*}(C_{\sigma_{f(\mu)}, \sigma_{f(\mu)}^*}((\lambda, r, s), r, s)) \land \mu
\]
\[
= I_{\tau_\mu, \tau_\mu^*}(f^{-1}(I_{\sigma_{f(\mu)}, \sigma_{f(\mu)}^*}(C_{\sigma_{f(\mu)}, \sigma_{f(\mu)}^*}((\lambda, r, s), r, s)) \land \mu, r, s).
\]
(iv)⇒ (i) It follows from Theorem 2.7, (iv)⇒ (ii) Let \( \lambda \in A_{f(\mu)} \) and \( r \in I_0, s \in I_1 \) such that \( \sigma_{f(\mu)}(\lambda) \geq r, \sigma_{f(\mu)}^*(\lambda) \leq s \). By (iv) we have,
\[
f^{-1}(\lambda) \land \mu \geq C_{\tau_\mu, \tau_\mu^*}(f^{-1}(C_{\sigma_{f(\mu)}, \sigma_{f(\mu)}^*}(I_{\sigma_{f(\mu)}, \sigma_{f(\mu)}^*}((\lambda, r, s), r, s) \land \mu), r, s)
\]
\[
= C_{\tau_\mu, \tau_\mu^*}(f^{-1}(\lambda) \land \mu, r, s).
\]
Thus $f^{-1}(\lambda) \wedge \mu = C_{\tau, \tau}(f^{-1}(\lambda) \wedge \mu, r, s)$. By Theorem 2.5, we have

$$F_{\tau}(f^{-1}(\lambda) \wedge \mu) = \tau_{\mu}(\mu - f^{-1}(\lambda) \wedge \mu) \geq r$$

and

$$F_{\tau}(f^{-1}(\lambda) \wedge \mu) = \tau_{\mu}(\mu - f^{-1}(\lambda) \wedge \mu) \leq s.$$  

(i)$\Rightarrow$(vi) and (vi)$\Rightarrow$(vii) are similar to that of Theorem 3.2. \qed

4. $IF\mu$-separation axioms

**Definition 4.1.** Let $(X, \tau, \tau^*)$ be an IFTS, $\mu \in I^X$ and $r \in I_0$, $s \in I_1$ with $r + s \leq 1$. $\mu$ is said to be (r, s)-$IF\mu T_2$-space if for each $x_\alpha, y_\beta(x \neq y) \in \mu$, there are $\nu_1, \nu_2 \in A_\mu$ with $\tau_{\mu}(\nu_1) \geq r$, $\tau_{\mu}(\nu_2) \geq r$, $\tau_{\mu}^*(\nu_1) \leq s$ and $\tau_{\mu}^*(\nu_2) \leq s$ such that $x_\alpha \in \nu_1$, $y_\beta \in \nu_2$ and $\nu_1 \not\in \nu_2[\mu]$.

**Theorem 4.1.** Let $(X, \tau, \tau^*)$ be an IFTS, $\mu \in I^X$ and $r \in I_0$, $s \in I_1$ with $r + s \leq 1$. $\mu$ is (r, s)-$IF\mu T_2$-space if and only if for each $x_\alpha, y_\beta(x \neq y) \in \mu$, we have

$$y_\beta \not\in \{C_{\tau, \tau}(r, r, s) : \tau_{\mu}(\nu) \geq r, \tau_{\mu}^*(\nu) \leq s, x_\alpha \in \nu\}.$$  

Proof. Let $x_\alpha, y_\beta(x \neq y) \in \mu$ and $m = \mu(y) - \beta$. Then $x_\alpha, y_m(x \neq y) \in \mu$. Since $\mu$ is (r, s)-$IF\mu T_2$-space, there are $\nu_1, \nu_2 \in A_\mu$ with $\tau_{\mu}(\nu_1) \geq r$, $\tau_{\mu}(\nu_2) \geq r$, $\tau_{\mu}^*(\nu_1) \leq s$ and $\tau_{\mu}^*(\nu_2) \leq s$ such that $x_\alpha \in \nu_1$, $y_m \in \nu_2$ and $\nu_1 \not\in \nu_2[\mu]$. Thus $\nu_1 \leq \mu - \nu_2$, $\tau(\mu - (\mu - \nu_2)) \geq r$ and $\tau^*(\mu - (\mu - \nu_2)) \leq s$ which implies, $C_{\tau, \tau}(\nu_1, r, s) \leq \mu - \nu_2$. Since $y_m \in \nu_2$,

$$\beta = \mu(y) - m > \mu(y) - \nu_2(y) \geq (C_{\tau, \tau}(\nu_1, r, s))(y)$$

and hence $y_\beta \not\in \{C_{\tau, \tau}(r, r, s) : \tau_{\mu}(\nu) \geq r, \tau_{\mu}^*(\nu) \leq s, x_\alpha \in \nu\}$.

Conversely, let $x_\alpha, y_\beta(x \neq y) \in \mu$. Then, $x_\alpha, y_{m(y) - \beta}(x \neq y) \in \mu$. By hypothesis, there is $\nu_1 \in A_\mu$ with $\tau_{\mu}(\nu_1) \geq r$, $\tau_{\mu}^*(\nu_1) \leq s$ such that $x_\alpha \in \nu_1$ and $y_{\mu(y) - \beta} \not\in C_{\tau, \tau}(\nu_1, r, s)$ and hence, $\mu(y) - \beta > (C_{\tau, \tau}(\nu_1, r, s))(y)$ which implies $y_\beta \in \mu - C_{\tau, \tau}(\nu_1, r, s) = \nu_2$ and $\tau_{\mu}(\nu_2) \geq r$, $\tau_{\mu}^*(\nu_2) \leq s$. Since $\nu_2 = \mu - C_{\tau, \tau}(\nu_1, r, s) \leq \mu - \nu_1$, $\nu_1 \not\in \nu_2[\mu]$. Hence $\mu$ is (r, s)-$IF\mu T_2$-space. \qed

**Definition 4.2.** Let $(X, \tau, \tau^*)$ be an IFTS, $\mu \in I^X$ and $r \in I_0$, $s \in I_1$ with $r + s \leq 1$. $\mu$ is said to be (r, s)-$IF\mu$-regular space if for each $\xi \in A_\mu$ with $\tau_{\mu}(\mu - \xi) \geq r$, $\tau_{\mu}^*(\mu - \xi) \leq s$ and for each fuzzy point $x_\mu \in \mu$ with $x_\mu \not\in \mu[\mu]$, there are $\nu, \eta \in A_\mu$ with $\tau_{\mu}(\nu) \geq r$, $\tau_{\mu}(\eta) \geq r$, $\tau_{\mu}(\nu) \leq s$ and $\tau_{\mu}^*(\eta) \leq s$ such that $x_\mu \in \nu, \xi \leq \eta$ and $\nu \not\in \eta[\mu]$.

**Theorem 4.2.** Let $(X, \tau, \tau^*)$ be an IFTS, $\mu \in I^X$ and $r \in I_0$, $s \in I_1$ with $r + s \leq 1$. Then the following are equivalent:

(i) $\mu$ is (r, s)-$IF\mu$-regular.
(ii) For each fuzzy point \(x_t \in \mu\) and each \(\nu \in A_\mu\) with \(x_t \in \nu, \tau_\mu(\nu) \geq r, \tau_\nu^*(\nu) \leq s\) there is a \(\eta \in A_\mu\) with \(\tau_\mu(\eta) \geq r, \tau_\nu^*(\eta) \leq s\) such that \(x_t \in \eta \leq C_{\tau_\nu^*}(r, r, s) \leq \nu\).

(iii) For each fuzzy point \(x_t \in \mu\) and each \(\xi \in A_\mu\) with \(\tau_\mu(\mu - \xi) \geq r\) and \(\tau_\nu^*(\mu - \xi) \leq s\) there is \(\nu, \eta \in A_\mu\) with \(\tau_\mu(\nu) \geq r, \tau_\nu^*(\nu) \leq s\) and \(\tau_\eta^*(\eta) \leq s\) such that \(x_t \in \nu, \xi \leq \eta \) and \(C_{\tau_\nu^*}(r, r, s) \leq \nu\).

Proof. (i)\(\Rightarrow\) (ii) Let \(x_t \in \mu\) be a fuzzy point, \(r \in I_0, s \in I_1\) with \(r + s \leq 1\) and \(\nu \in A_\mu\) with \(\tau_\mu(\nu) \geq r, \tau_\nu^*(\nu) \leq s\) and \(x_t \in \nu\). Then \(\tau_\mu(\mu - \nu) \geq r, \tau_\nu^*(\mu - \nu) \leq s\) and \(x_t \in \eta \leq \eta \leq \nu\). By (i), there are \(\eta, \nu \in A_\mu\) with \(\tau_\mu(\eta) \geq r, \tau_\nu^*(\eta) \leq s\) and \(\tau_\eta^*(\eta) \leq s\) such that \(x_t \in \eta, \mu - \nu \leq \nu\) and \(\eta \leq \eta \leq \nu\). Since \(\eta \leq \eta \leq \nu\), \(\eta \leq \mu - \nu \leq \nu\) and hence \(C_{\tau_\nu^*}(r, r, s) \leq \nu\). Thus \(x_t \in \eta \leq \nu \leq C_{\tau_\nu^*}(r, r, s) \leq \nu\).

(ii)\(\Rightarrow\) (iii) Let \(x_t \in \mu\) be a fuzzy point, \(r \in I_0, s \in I_1\) with \(r + s \leq 1\) and \(\xi \in A_\mu\) with \(\tau_\mu(\mu - \xi) \geq r, \tau_\nu^*(\mu - \xi) \leq s\) and \(x_t \in \nu\). Then \(x_t \in \mu - \xi\). By (ii), there is \(\eta \in A_\mu\) with \(\tau_\mu(\eta) \geq r, \tau_\nu^*(\eta) \leq s\) such that \(x_t \leq \eta \leq \eta \leq \nu\). Then \(x_t \in \eta \leq \eta \leq \nu\). By (ii) again, there is \(\nu \in A_\mu\) with \(\tau_\nu(\nu) \geq r\) and \(\tau_\nu^*(\nu) \leq s\) such that

\[
x_t \in \nu \leq C_{\tau_\nu^*}(r, r, s) \leq \eta \leq C_{\tau_\nu^*}(r, r, s) \leq \mu - \xi.
\]

Put \(\eta = \mu - C_{\tau_\nu^*}(r, r, s)\). Hence there are \(\eta, \nu \in A_\mu\) with \(\tau_\mu(\nu) \geq r, \tau_\mu(\eta) \geq r, \tau_\nu^*(\nu) \leq s\) and \(\tau_\nu^*(\eta) \leq s\) such that \(x_t \in \nu, \xi \leq \eta \) and \(C_{\tau_\nu^*}(r, r, s) \leq \nu\).

(iii)\(\Rightarrow\) (i) It is clear. \(\square\)

5. \(IF\mu\)-compactness

**Definition 5.1.** Let \((X, \tau, \tau^*)\) be an IFTS, \(\mu \in I^X\) and \(r \in I_0, s \in I_1\) with \(r + s \leq 1\). \(\mu\) is said to be \((r, s)\)-\(IF\mu\)-compact if for every family \(\{\nu_i : i \in I\}\) in \(\{\nu : \nu \in A_\mu, \tau_\mu(\nu) \geq r, \tau_\nu^*(\nu) \leq s\}\) such that \((\bigwedge_{i \in I} \nu_i)(x) = \mu(x)\) for each \(x \in X\), there exists a finite subset \(J_0\) of \(J\) such that \((\bigvee_{i \in J_0} \nu_i)(x) = \mu(x)\).

**Definition 5.2.** Let \(X\) be a non-empty set and \(\mu \in I^X\). A collection \(\beta \subseteq A_\mu\) is said to form a fuzzy \(\mu\)-filterbasis if for each finite subcollection \(\{\nu_1, \nu_2, \ldots, \nu_n\}\) of \(\beta\), \((\bigwedge_{i=1}^n \nu_i)(x) > 0\) for some \(x \in X\).

**Theorem 5.1.** Let \((X, \tau, \tau^*)\) be an IFTS, \(\mu \in I^X\) and \(r \in I_0, s \in I_1\) with \(r + s \leq 1\). \(\mu\) is \((r, s)\)-\(IF\mu\)-compact if and only if for each fuzzy \(\mu\)-filterbasis \(\beta \subseteq A_\mu\), \((\bigwedge_{\nu \in \beta} C_{\tau_\nu^*}(r, r, s))(x) > 0\) for some \(x \in X\).

**Proof.** Let \(\{\nu_i : i \in I\}\) be a family in \(\Lambda = \{\nu : \nu \in A_\mu, \tau_\mu(\nu) \geq r, \tau_\nu^*(\nu) \leq s\}\) such that \((\bigwedge_{i \in I} \nu_i)(x) = \mu(x)\). Suppose that there is no finite subset \(J_0\) of \(J\) such that \((\bigvee_{i \in J_0} \nu_i)(x) = \mu(x)\). Then for each finite subcollection \(\{\nu_1, \nu_2, \ldots, \nu_n\}\) of \(\Lambda\), there exists \(x \in X\) such that \(\nu_i(x) < \mu(x)\) for each \(i = 1, 2, \ldots, n\). Then \(\mu(x) - \nu_i(x) \geq 0\) for each \(i = 1, 2, \ldots, n\). So, \(\bigwedge_{i=1}^n (\mu - \nu_i)(x) > 0\) and hence \(\beta = \{\mu - \nu_i : \nu_i \in \Lambda, i \in I\}\) forms a fuzzy \(\mu\)-filterbasis.
Since \((\bigvee_{i \in J} \nu_i)(x) = \mu(x)\) for each \(x \in X\) and \(\tau^*_\mu(\nu_i) \geq r, \tau^*_\mu(\nu_i) \leq s\) for each \(i \in J\) we have
\[
(\bigwedge_{i \in J} C_{\tau^*_\mu}(\mu - \nu_i, r, s))(x) = (\bigwedge_{i \in J} (\mu - \nu_i))(x) = (\mu - \bigvee_{i \in J} \nu_i)(x) = 0
\]
for each \(x \in X\). It is a contradiction. Thus there exists a finite subset \(J_0\) of \(J\) such that \((\bigvee_{i \in J_0} \nu_i)(x) = \mu(x)\) for all \(x \in X\). Thus \(\mu\) is \((r, s)\)-IF\(\mu\)-compact.

Conversely, Suppose that there exists fuzzy \(\mu\)-filterbasis \(\beta\) such that
\[
(\bigwedge_{\nu \in \beta} C_{\tau^*_\mu}(\nu, r, s))(x) = 0
\]
for each \(x \in X\). Then
\[
(\bigvee_{\nu \in \beta} (\mu - C_{\tau^*_\mu}(\nu, r, s)))(x) = \mu(x) \quad \text{for each} \quad x \in X.
\]
Since \(\tau^*_\mu(\mu - C_{\tau^*_\mu}(\nu, r, s)) \geq r\) and \(\tau^*_\mu(\mu - C_{\tau^*_\mu}(\nu, r, s)) \leq s\) and \(\mu\) is \((r, s)\)-IF\(\mu\)-compact, there exists a finite subcollection \(\{\mu - C_{\tau^*_\mu}(\nu_i, r, s) : i = 1, 2, \ldots, n\}\) of \(\{\mu - C_{\tau^*_\mu}(\nu_i, r, s) : \nu \in \beta\}\) such that
\[
(\bigvee_{i=1}^{n} (\mu - C_{\tau^*_\mu}(\nu_i, r, s)))(x) = \mu(x) \quad \text{for each} \quad x \in X.
\]
Since \(\nu_i(x) \leq C_{\tau^*_\mu}(\nu_i, r, s)(x)\) for each \(x \in X\) we have
\[
(\bigvee_{i=1}^{n} (\mu - \nu_i))(x) \geq (\bigvee_{i=1}^{n} (\mu - C_{\tau^*_\mu}(\nu_i, r, s)))(x) = \mu(x) \quad \text{for each} \quad x \in X.
\]
Then \((\bigvee_{i=1}^{n} (\mu - \nu_i))(x) = \mu(x)\) for each \(x \in X\). Thus \((\bigwedge_{\nu \in \beta} C_{\tau^*_\mu}(\nu, r, s))(x) > 0\) for some \(x \in X\). It is a contradiction. Hence \((\bigwedge_{\nu \in \beta} C_{\tau^*_\mu}(\nu, r, s))(x) > 0\) for some \(x \in X\).

\(\square\)

**Theorem 5.2.** Let \((X, r, \tau^*)\), \((Y, \sigma, \sigma^*)\) be two IFTSs, \(f : X \to Y\) an IF\(\mu\)-continuous bijective mapping. For \(r \in I_0, s \in I_1\) with \(r + s \leq 1\), if \(\mu\) is \((r, s)\)-IF\(\mu\)-compact, then \(f(\mu)\) is \((r, s)\)-IF\(f(\mu)\)-compact.

**Proof.** Let \((\lambda_i : i \in J)\) be a family in \(\{\lambda : \lambda \in A_{f(\mu)}, \sigma_{f(\mu)}(\lambda) \geq r, \sigma^*_{f(\mu)}(\lambda) \leq s\}\) such that \((\bigvee_{i \in J} \lambda_i)(y) = (f(\mu))(y)\) for all \(y \in Y\). Since \(f\) is IF\(\mu\)-continuous for each \(i \in J\) we have,
\[
\tau_\mu(f^{-1}(\lambda_i) \wedge \mu) \geq \sigma_{f(\mu)}(\lambda_i) \geq r,
\]
\[
\tau^*_\mu(f^{-1}(\lambda_i) \wedge \mu) \leq \sigma^*_{f(\mu)}(\lambda_i) \leq s.
\]
Since \(f\) is injective,
\[
\bigvee_{i \in J} (f^{-1}(\lambda_i) \wedge \mu) = f^{-1}(\bigvee_{i \in J} \lambda_i) \wedge \mu = f^{-1}(f(\mu)) \wedge \mu = \mu.
\]
Since \(\mu\) is \((r, s)\)-IF\(\mu\)-compact, there exists a finite subset \(J_0\) of \(J\) such that \((\bigvee_{i \in J_0} f^{-1}(\lambda_i))(x) = \mu(x)\) for all \(x \in X\). This implies that
Let \( \iota \) since \(-\), all \( y \) Then \( \eta \eta \) \((\ast)\). Since \( \iota \) let \( (\ast)\). Then \( \eta \eta \) \((\ast)\).

Thus \( (\ast)\) for all \( y \in Y \). Hence \( f(\mu) \) is \((r, s)\)-IFT-compact.

6. IF\(\mu\)-connected sets

**Definition 6.1.** Let \((X, \tau, \tau^*)\) be an IFTS, \(\mu \in I^X\) and \(r \in I_0\), \(s \in I_1\) with \(r + s \leq 1\). Then \(\nu, \eta \in \mathcal{A}_\mu\) are said to be \((r, s)\)-IFT\(\mu\)-separated if \(\nu \not{\mathcal{A}} r, s (\ast)\eta \not{\mathcal{A}} (\ast)\nu \not{\mathcal{A}} \mu\).

**Theorem 6.1.** Let \((X, \tau, \tau^*)\) be an IFTS, \(\mu \in I^X\) and \(r \in I_0\), \(s \in I_1\) with \(r + s \leq 1\). Then for \(\nu, \eta \in \mathcal{A}_\mu\),

(i) If \(\nu, \eta \in \mathcal{A}_\mu\) such that \(\nu \neq \eta \leq \mu\), then \(\nu \not{\mathcal{A}} \eta \not{\mathcal{A}} \mu\) are \((r, s)\)-IFT\(\mu\)-separated.

(ii) If \(\nu \not{\mathcal{A}} \eta \not{\mathcal{A}} \mu\) and either \(\tau_\mu(\nu) \geq r\), \(\tau_\mu(\eta) \geq r\), \(\tau_\mu(\nu) \leq s\) or \(\tau_\mu(\mu - \nu) \geq r\), \(\tau_\mu(\mu - \eta) \geq r\), \(\tau_\mu(\mu - \nu) \leq s\) and \(\tau_\mu(\mu - \eta) \leq s\), then \(\nu \not{\mathcal{A}} \eta \not{\mathcal{A}} \mu\) are \((r, s)\)-IFT\(\mu\)-separated.

(iii) If \(\tau_\mu(\nu) \geq r\), \(\tau_\mu(\eta) \geq r\), \(\tau_\mu(\nu) \leq s\) or \(\tau_\mu(\mu - \nu) \geq r\), \(\tau_\mu(\mu - \eta) \geq r\), \(\tau_\mu(\mu - \nu) \leq s\) and \(\tau_\mu(\mu - \eta) \leq s\), then \(\nu \not{\mathcal{A}} \eta \not{\mathcal{A}} \mu\) are \((r, s)\)-IFT\(\mu\)-separated.

**Proof.** (i) Since \(\nu_1 \leq \nu\), \(C_{\tau_\mu, \tau_\mu^*}(\nu_1, r, s) \leq C_{\tau_\mu, \tau_\mu^*}(\nu, r, s)\). Since \(\nu, \eta \in \mathcal{A}_\mu\), \(\nu \not{\mathcal{A}} \eta \not{\mathcal{A}} (\ast)\mu\) are \((r, s)\)-IFT\(\mu\)-separated. Then \(\eta \not{\mathcal{A}} C_{\tau_\mu, \tau_\mu^*}(\nu_1, r, s)[\mu]\). Thus \(\eta \leq \mu \leq \mu - C_{\tau_\mu, \tau_\mu^*}(\nu, r, s) \leq \mu - C_{\tau_\mu, \tau_\mu^*}(\nu_1, r, s)\).

Then \(\eta \not{\mathcal{A}} C_{\tau_\mu, \tau_\mu^*}(\nu_1, r, s)[\mu]\). Similarly, \(\nu_1 \not{\mathcal{A}} C_{\tau_\mu, \tau_\mu^*}(\eta_1, r, s)[\mu]\). Hence \(\nu_1\) and \(\eta_1\) are \((r, s)\)-IFT\(\mu\)-separated.

(ii) Let \(\nu \not{\mathcal{A}} \eta \not{\mathcal{A}} \mu\), \(\tau_\mu(\nu) \geq r\), \(\tau_\mu(\eta) \geq r\), \(\tau_\mu(\nu) \leq s\) and \(\tau_\mu(\eta) \leq s\). Since \(\nu \not{\mathcal{A}} \eta \not{\mathcal{A}} \mu\), \(\nu \leq \mu - \eta\). Thus

\(C_{\tau_\mu, \tau_\mu^*}(\nu, r, s) \leq C_{\tau_\mu, \tau_\mu^*}(\mu - \eta, r, s) = \mu - \eta\).

Then \(C_{\tau_\mu, \tau_\mu^*}(\nu, r, s) \not{\mathcal{A}} \eta \not{\mathcal{A}} \mu\). Similarly, \(C_{\tau_\mu, \tau_\mu^*}(\eta, r, s) \not{\mathcal{A}} \nu \not{\mathcal{A}} \mu\). Then \(\nu, \eta \in \mathcal{A}_\mu\) are \((r, s)\)-IFT\(\mu\)-separated. Let \(\tau_\mu(\mu - \nu) \geq r\), \(\tau_\mu(\mu - \eta) \geq r\), \(\tau_\mu(\mu - \nu) \leq s\) and \(\tau_\mu(\mu - \eta) \leq s\). By Theorem 2.5, \(\nu \not{\mathcal{A}} \eta \not{\mathcal{A}} \mu\) and \(\eta \not{\mathcal{A}} \nu \not{\mathcal{A}} \mu\). Since \(\nu \not{\mathcal{A}} \eta \not{\mathcal{A}} \mu\) we have, \(C_{\tau_\mu, \tau_\mu^*}(\nu, r, s) \not{\mathcal{A}} \eta \not{\mathcal{A}} \mu\) and \(C_{\tau_\mu, \tau_\mu^*}(\eta, r, s) \not{\mathcal{A}} \nu \not{\mathcal{A}} \mu\). Then \(\nu, \eta \in \mathcal{A}_\mu\) are \((r, s)\)-IFT\(\mu\)-separated.

(iii) Let \(\tau_\mu(\nu) \geq r\), \(\tau_\mu(\eta) \geq r\), \(\tau_\mu(\nu) \leq s\) and \(\tau_\mu(\eta) \leq s\). Since \(\nu \not{\mathcal{A}} \eta \not{\mathcal{A}} \mu\), \(\tau_\mu(\nu \wedge (\mu - \eta), r, s) \leq C_{\tau_\mu, \tau_\mu^*}(\nu \wedge (\mu - \eta), r, s) = \mu - \eta\).

Then

\(\eta \leq \mu - C_{\tau_\mu, \tau_\mu^*}(\nu \wedge (\mu - \eta), r, s)\).

Since \(\eta \not{\mathcal{A}} \mu - \nu \not{\mathcal{A}} \eta\),

\(\eta \wedge (\mu - \nu) \leq \mu - C_{\tau_\mu, \tau_\mu^*}(\nu \wedge (\mu - \eta), r, s)\).
Let \( \nu \), \( \tau \), and \( \rho \) be \( (r, s)\)-\( IFTS \) with \( r + s \leq 1 \). Then \( \nu \) and \( \rho \) are \((r, s)\)-\( IFTS \)-separated if and only if there exist \( \nu_1, \rho_1 \in \mathcal{A}_\mu \) with \( \tau_\mu(\nu_1) \geq r \), \( \tau_\mu(\rho_1) \geq r \), \( \tau_\mu^*(\nu_1) \leq s \) and \( \tau_\mu^*(\rho_1) \leq s \) such that \( \nu \leq \nu_1 \), \( \rho \leq \rho_1 \), \( \nu \not\sim \rho \mu \), \( \rho \not\sim \nu \mu \), and \( \nu \not\sim \rho \mu \). Hence, \( \nu \) and \( \rho \) are \((r, s)\)-\( IFTS \)-separated.

**Theorem 6.2.** Let \((X, \tau, \tau^*)\) be an IFTS, \( \mu \in \mathcal{I}_X \) and \( r, s \in I_0 \), \( s \in I_1 \) with \( r + s \leq 1 \). Then \( \nu \), \( \eta \in \mathcal{A}_\mu \) are \((r, s)\)-\( IFTS \)-separated if and only if there exist \( \nu_1, \eta_1 \in \mathcal{A}_\mu \) with \( \tau_\mu(\nu_1) \geq r \), \( \tau_\mu(\eta_1) \geq r \), \( \tau_\mu^*(\nu_1) \leq s \) and \( \tau_\mu^*(\eta_1) \leq s \) such that \( \nu \leq \nu_1 \), \( \eta \leq \eta_1 \), \( \nu \not\sim \eta \mu \), \( \eta \not\sim \nu \mu \), and \( \nu \not\sim \eta \mu \). Hence, \( \nu \) and \( \eta \) are \((r, s)\)-\( IFTS \)-separated.

**Definition 6.2.** Let \((X, \tau, \tau^*)\) be an IFTS, \( \mu \in \mathcal{I}_X \) and \( r, s \in I_0 \), \( s \in I_1 \) with \( r + s \leq 1 \). Then \( \nu \in \mathcal{A}_\mu \) is said to be \((r, s)\)-\( IFTS \)-connected if it can’t be expressed as the union of two \((r, s)\)-\( IFTS \)-separated sets.

**Theorem 6.3.** Let \((X, \tau, \tau^*)\) and \((Y, \sigma, \sigma^*)\) be IFTS, \( \mu \in \mathcal{I}_X \) and \( f : X \to Y \) be \( IFTS \)-continuous injective mapping. For \( r, s \in I_0 \), \( s \in I_1 \) with \( r + s \leq 1 \), if \( \lambda \) is \((r, s)\)-\( IFTS \)-connected, then \( f(\lambda) \) is \((r, s)\)-\( IFTS \)-connected.

**Proof.** Suppose that \( f(\lambda) \) is not \((r, s)\)-\( IFTS \)-connected. Then there exist two \((r, s)\)-\( IFTS \)-separated sets \( \nu, \eta \in \mathcal{A}_f(\mu) \) such that \( \nu \not\sim \eta \mu \). By Theorem 6.2, there exist \( \nu_1, \eta_1 \in \mathcal{A}_f(\mu) \) with \( \sigma_f(\nu_1) \geq r \), \( \sigma_f(\eta_1) \geq r \), \( \sigma_f^*(\nu_1) \leq s \) and \( \sigma_f^*(\eta_1) \leq s \) such that \( \nu \leq \nu_1 \), \( \eta \leq \eta_1 \), \( \nu \not\sim \eta_1 \mu \), \( \eta \not\sim \nu_1 \mu \), and \( \nu \not\sim \eta_1 \mu \).
η \not\subseteq \mu_1[f(\mu)]. Since \(f\) is injective and \(\nu_1 \leq f(\mu)\), \(f^{-1}(\nu_1) \leq f^{-1}(f(\mu)) = \mu\) and hence \(f^{-1}(\nu) \leq f^{-1}(\nu_1) \wedge \mu\). Similarly, \(f^{-1}(\eta) \leq f^{-1}(\eta_1) \wedge \mu\). Since \(f\) is \(IF\mu\)-continuous, we have

\[
\tau_\mu(f^{-1}(\nu_1) \wedge \mu) \geq \sigma_{f(\mu)}(\nu_1) \geq r \quad \text{and} \quad \tau_\mu(f^{-1}(\eta_1) \wedge \mu) \leq \sigma_{f(\mu)}(\eta_1) \leq s.
\]

Similarly, \(\tau_\mu(f^{-1}(\eta_1) \wedge \mu) \geq r\), \(\tau_\mu(f^{-1}(\nu_1) \wedge \mu) \leq s\). Since \(f\) is injective,

\[
\begin{align*}
 f^{-1}(\nu)(x) + (f^{-1}(\eta_1) \wedge \mu)(x) &= f^{-1}(\nu)(x) + f^{-1}(\eta_1)(x) \\
 &= \nu(f(x)) + \eta_1(f(x)) \\
 &= \nu(y) + \eta_1(y) \leq f(\mu)(y) = \mu(x)
\end{align*}
\]

and hence \(f^{-1}(\nu) \not\subseteq (f^{-1}(\eta_1) \wedge \mu)[\mu]\). Similarly, \(f^{-1}(\eta) \not\subseteq (f^{-1}(\nu_1) \wedge \mu)[\mu]\). Then by Theorem 6.2, \(f^{-1}(\nu)\) and \(f^{-1}(\eta)\) are \((r, s)-IF\mu\)-separated. Since \(f\) is injective,

\[
\lambda = f^{-1}(f(\lambda)) = f^{-1}(\nu \lor \eta) = f^{-1}(\nu) \lor f^{-1}(\eta).
\]

It is a contradiction with \(\lambda\) is \((r, s)-IF\mu\)-connected. Hence \(f(\lambda) = (r, s)-IF\mu\)-connected. \(\square\)

References

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