SOME EISENSTEIN SERIES IDENTITIES RELATED TO MODULAR EQUATION OF THE FOURTH ORDER

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Abstract. We find some Eisenstein series related to modulus 4 using a theta function identity of McCullough and Shen and residue theorem for elliptic functions.

1. Introduction

The Eisenstein series $P(q)$, $Q(q)$ and $R(q)$ are defined for $|q| < 1$ by

\begin{align}
P(q) &:= 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n}, \\
Q(q) &:= 1 + 240 \sum_{n=1}^{\infty} \frac{n^3q^n}{1 - q^n}, \\
R(q) &:= 1 - 504 \sum_{n=1}^{\infty} \frac{n^5q^n}{1 - q^n}.
\end{align}

This is the notation used by Ramanujan in his lost notebook [8, pp. 136–162], but in his ordinary notebooks, $P$, $Q$ and $R$ are replaced by $L$, $M$ and $N$ respectively. We shall be using $L$, $M$ and $N$, respectively, for $P$, $Q$ and $R$. We studied the continued fraction of Ramanujan

\begin{equation}
C(q) = 1 + \frac{(1 + q) q^2 (q + q^3) q^4}{1 + \frac{1+\cdots}{1+\frac{1+\cdots}{1+\cdots}}},
\end{equation}

and called this continued fraction analogous to the celebrated Rogers-Ramanujan continued fraction $R(q)$

\begin{equation}
R(q) = q^{1/5} \frac{q}{1+1+1+\cdots} = q^{1/5} \frac{(q; q^5)_\infty (q^4; q^5)_\infty}{(q^2; q^5)_\infty (q^3; q^5)_\infty}.
\end{equation}

Received November 17, 2009.
2010 Mathematics Subject Classification. 11F20, 33D15.
Key words and phrases. Eisenstein series, theta functions, $q$-hypergeometric series.

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Since the continued fraction $C(q)$ sums to $q$-hypergeometric series on base 4, motivated me to study Eisenstein series identities related to modular equations of order 4. In this paper we will prove the following identities:

(1.6) \[ 2 \left( \frac{\theta_1'}{\theta_1} \right) \left( \frac{\pi}{4} q \right) + \left( \frac{\theta_1'}{\theta_1} \right) \left( \frac{\pi}{2} q \right) = \frac{1}{3} (L(\tau) - 4^2 L(4\tau)), \]

(1.7) \[ 2 \left( \frac{\theta_1'}{\theta_1} \right) \left( \frac{\pi}{4} q \right) + \left( \frac{\theta_1'}{\theta_1} \right) \left( \frac{2\pi}{4} q \right) = \frac{2}{15} (M(\tau) - 4^4 M(4\tau)), \]

(1.8) \[ 1 - 4 \sum_{n=0}^{\infty} \left( \frac{(4m+1)^2 q^{4m+1}}{1 - q^{4m+1}} - \frac{(4m+3)^2 q^{4m+3}}{1 - q^{4m+3}} \right) = \frac{\eta^4(\tau)\eta^4(2\tau)}{\eta(4\tau)^4}, \]

(1.9) \[ = 1 + 8 \left( \sum_{n=1}^{\infty} nq^n - \sum_{n=1}^{\infty} 4nq^n \right), \]

and

(1.10) \[ 160 + 128 \sum_{n=0}^{\infty} \left( \frac{(4n+1)^4 q^{4n+1}}{1 - q^{4n+1}} - \frac{(4n+3)^4 q^{4n+3}}{1 - q^{4n+3}} \right) \]

\[ = -2^5 \left[ 1 + 4 \sum_{n=0}^{\infty} \left( \frac{q^{4n+1}}{1 - q^{4n+1}} - \frac{q^{4n+3}}{1 - q^{4n+3}} \right) \right]^5 \]

\[ + 10 \left[ 1 + 4 \sum_{n=0}^{\infty} \left( \frac{q^{4n+1}}{1 - q^{4n+1}} - \frac{q^{4n+3}}{1 - q^{4n+3}} \right) \right] \left( \frac{16}{15} M(\tau) + \frac{2}{15} 4^4 M(4\tau) \right) \]

\[ + 10 \left[ 1 + 4 \sum_{n=0}^{\infty} \left( \frac{q^{4n+1}}{1 - q^{4n+1}} - \frac{q^{4n+3}}{1 - q^{4n+3}} \right) \right] \left( \frac{4}{3} L(\tau) - \frac{16}{3} L(4\tau) \right)^2. \]

The identity (1.10) is very interesting.

2. Preliminaries

Throughout the paper $q = e^{2\pi i \tau}$, $\text{Im}(\tau) > 0$ and the standard $q$-notations are used:

(2.1) \[ (a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - aq^k), \]
(2.2) \( (a; q)_n = \prod_{k=1}^{n} (1 - aq^{k-1}) \),

and

\( (a)_0 = (a; q)_0 = 1 \).

The Dedekind eta-function is defined by

(2.3) \( \eta(\tau) = q^{\frac{1}{24}} \eta(0) \).

Jacobi theta function is defined as follows, see [9, p. 464]

(2.4) \( \theta_1(z|q) = -iq^{\frac{1}{4}} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(n+1)}{2}} e^{(2n+1)iz} \)

(2.5) \( = 2q^{\frac{1}{8}} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \sin(2n+1)z \).

In terms of infinite products

(2.6) \( \theta_1(z|q) = 2q^{\frac{1}{8}} \sin z(q; q)_{\infty}(qe^{2iz}; q)_{\infty}(qe^{-2iz}; q)_{\infty} \)

(2.7) \( = iq^{\frac{1}{4}} e^{-iz}(q; q)_{\infty}(e^{2iz}; q)_{\infty}(qe^{-2iz}; q)_{\infty} \).

Differentiating partially with respect to \( z \) and then putting \( z = 0 \), we have the identity

(2.8) \( \theta_1'(q) = \theta_1'(0|q) = 2q^{\frac{1}{8}} (q; q)_{\infty}^{\frac{1}{2}} \).

From the definition of \( \theta_1(z|q) \),

(2.9) \( \theta_1(z + n\pi|q) = (-1)^n \theta_1(z|q) \),

and

(2.10) \( \theta_1(z + n\pi\tau|q) = (-1)^n q^{-\frac{n^2}{2}} e^{-2niz} \theta_1(z|q) \).

In this paper we shall be using the following residue theorem of elliptic functions:

**Theorem.** The sum of all the residues of an elliptic function in the period parallelogram is zero.

3. The proofs of (1.6) and (1.7)

By infinite product expansion for \( \theta_1(z|q) \) given in (2.7) and by simple computation, we have

(3.1) \( \theta_1(4z|q^4) = (q^4; q^4)_{\infty} \theta_1(z|q)^4 \theta_1(z - \pi\frac{1}{4}|q) \theta_1(z + \pi\frac{1}{4}|q) \theta_1(z - \pi\frac{1}{2}|q) \).

Taking logarithmic derivative of both the sides of (3.1), we obtain

(3.2) \( 4\frac{\theta_1'(4z|q^4)}{\theta_1(4z|q^4)} - \frac{\theta_1'(z|q)}{\theta_1(z|q)} = \frac{\theta_1'(z - \pi\frac{1}{4}|q)}{\theta_1(z - \pi\frac{1}{2}|q)} + \frac{\theta_1'(z + \pi\frac{1}{4}|q)}{\theta_1(z + \pi\frac{1}{2}|q)} + \frac{\theta_1'(z - \pi\frac{1}{2}|q)}{\theta_1(z - \pi\frac{1}{2}|q)} \).
We now use the following identity, [6, eq.(2.10), p. 109] to simplify the left hand side:

\[
\frac{\theta'_1(z|q)}{\theta_1(z|q)} = \frac{1}{z} - \frac{1}{3} L(\tau)z - \frac{1}{45} M(\tau)z^3 - \frac{2}{945} N(\tau)z^5
\]

\[
- \frac{1}{4725} \left( 1 + 480 \sum_{n=1}^{\infty} \frac{n^7 q^n}{1 - q^n} \right) z^7 + \cdots.
\]

So (3.2) can be written as

\[
\frac{\theta'_1(z - \pi/4|q)}{\theta_1(z - \pi/4|q)} + \frac{\theta'_1(z + \pi/4|q)}{\theta_1(z + \pi/4|q)} + \frac{\theta'_1(z - \pi/2|q)}{\theta_1(z - \pi/2|q)} = \frac{1}{3} \left( L(\tau) - 4^2 L(4\tau) \right) z + \frac{1}{45} \left( M(\tau) - 4^4 M(4\tau) \right) z^3 + O(z^5).
\]

Differentiate both side of (3.4) with respect to \( z \) and then put \( z = 0 \) to get

\[
2 \left( \frac{\theta'_1}{\theta_1} \right)^{\prime\prime\prime} \left( \frac{\pi}{4} | q \right) + \left( \frac{\theta'_1}{\theta_1} \right)^{\prime\prime\prime} \left( \frac{\pi}{2} | q \right) = \frac{1}{3} \left( L(\tau) - 4^2 L(4\tau) \right),
\]

which proves (1.6).

Differentiate thrice both side of (3.4) with respect to \( z \) and then put \( z = 0 \) to get

\[
2 \left( \frac{\theta'_1}{\theta_1} \right)^{\prime\prime\prime} \left( \frac{\pi}{4} | q \right) + \left( \frac{\theta'_1}{\theta_1} \right)^{\prime\prime\prime} \left( \frac{\pi}{2} | q \right) = \frac{2}{15} \left( M(\tau) - 4^4 M(4\tau) \right),
\]

which proves (1.7).

4. The proof of (1.8)

Recall the following identity [6, eq.(8.1), p. 117]

\[
\cot^2 y - \cot^2 x + 8 \sum_{n=1}^{\infty} \frac{ny^n}{1 - q^n} (\cos 2nx - \cos 2ny) = \theta'_1(0|q)^2 \theta_1(x - y|q) \theta_1(x + y|q) / \theta_1^2(x|q) \theta_1^2(y|q).
\]

There is a slight misprint which has been corrected.

Differentiate (4.1) partially with respect to \( x \) then putting \( y = x \) to obtain

\[
2 \cot x \csc^2 x - 16 \sum_{n=1}^{\infty} \frac{n^2 q^n}{1 - q^n} \sin 2nx = \frac{\theta'_1(0|q)^2 \theta_1(2|x|q)}{\theta_1^2(x|q)}.
\]

Putting \( x = \pi/4 \) in (4.2), we have

\[
4 - 16 \sum_{n=1}^{\infty} \frac{n^2 q^n}{1 - q^n} \sin \frac{n\pi}{2} = \theta'_1(0|q)^3 \left[ \frac{\theta_1(\pi/4|q)}{\theta_1^2(\pi/4|q)} \right].
\]
or
\[
4 - 16 \sum_{n=1}^{\infty} \frac{n^2 q^n}{1 - q^n} \sin \frac{n \pi}{2} = 4(q; q)_{\infty}^4 (q^2; q^2)_{\infty}^6
\]
so
\[
1 - 4 \sum_{m=0}^{\infty} \left( \frac{(4m + 1)^2 q^{4m+1}}{1 - q^{4m+1}} - \frac{(4m + 3)^2 q^{4m+3}}{1 - q^{4m+3}} \right) = \frac{(q; q)_{\infty}^4 (q^2; q^2)_{\infty}^6}{(q^4; q^4)_{\infty}^4}.
\]

Using the definition of Dedekind eta-function defined in (2.3), we have
\[
1 - 4 \sum_{m=0}^{\infty} \left( \frac{(4m + 1)^2 q^{4m+1}}{1 - q^{4m+1}} - \frac{(4m + 3)^2 q^{4m+3}}{1 - q^{4m+3}} \right) = \frac{\eta^4(\tau)\eta^6(2\tau)}{\eta(4\tau)\eta^3(4\tau)},
\]
which proves (1.8).

5. The proof of (1.9)

For proving (1.9) we first construct the following elliptic function:
\[
f(z) = \frac{\theta_2^2(z + \frac{\pi}{2} | q) \theta_1(z + \frac{\pi}{2} | q)}{\theta_1(z | q)}
\]
and use residue theorem of elliptic functions.

It is easy to see that \( f(z) \) is an elliptic function of periods \( \pi \) and \( \pi \tau \), and has a pole of order 3 at \( z = 0 \). We now compute residue of \( f(z) \) at \( z = 0 \).

Now
\[
\text{res}(f; 0) = \frac{1}{2} \left[ \frac{d^2}{dz^2} (z^3 f(z)) \right]_{z=0}.
\]

Let
\[
F(z) = z^3 f(z) \quad \text{and} \quad \varphi(z) = \frac{F'(z)}{F(z)}.
\]

By logarithmic differentiation
\[
\text{res}(f; 0) = \frac{1}{2} \left[ \frac{d^2}{dz^2} (z^3 f(z)) \right]_{z=0} = \frac{1}{2} F(0) \left[ \varphi(0)^2 + \varphi'(0) \right].
\]

Now
\[
\varphi(z) = \frac{3z^2 f(z) + z^3 f'(z)}{z^3 f(z)} = \frac{3}{z} + \frac{f'(z)}{f(z)}
\]
\[= \frac{3}{z} - 3 \frac{\theta_2'(z | q) + 2 \theta_1' \theta_1(z + \frac{\pi}{4} | q) + \theta_1' \theta_1(z + \frac{\pi}{2} | q)}{\theta_1(z + \frac{\pi}{4} | q) + \theta_1(z + \frac{\pi}{2} | q) + O(z^3)}.
\]
Now
\[
\frac{\theta'_1(\pi/4|q)}{\theta_1(\pi/4|q)} = \cot \frac{\pi}{4} + 4 \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n} \sin \frac{n\pi}{2}
\]
(5.6)
\[
= 1 + 4 \sum_{n=0}^{\infty} \left( \frac{q^{4n+1}}{1 - q^{4n+1}} - \frac{q^{4n+3}}{1 - q^{4n+3}} \right)
\]
and
\[
\frac{\theta'_1(\pi|2|q)}{\theta_1(\pi|2|q)} = 4 \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n} \sin n\pi = 0.
\]
(5.7)
So
\[
\frac{2\theta'_1(\pi|4|q) + \theta'_1(\pi|2|q)}{\theta_1(\pi|4|q) + \theta_1(\pi|2|q)} = 2 \left[ 1 + 4 \sum_{n=0}^{\infty} \left( \frac{q^{4n+1}}{1 - q^{4n+1}} - \frac{q^{4n+3}}{1 - q^{4n+3}} \right) \right].
\]
(5.8)
Putting \(z = 0\) in (5.5) and using (5.8), we have
\[
\varphi(0) = 2 \frac{\theta'_1(\pi|4|q) + \theta'_1(\pi|2|q)}{\theta_1(\pi|4|q) + \theta_1(\pi|2|q)}
\]
(5.9)
Differentiating (5.5) with respect to \(z\) and then putting \(z = 0\) and using (3.5), we have
\[
\varphi'(0) = L(\tau) + 2 \left( \frac{\theta'_1(\pi|4|q)}{\theta_1(\pi|4|q)} \right) + \left( \frac{\theta'_1(\pi|2|q)}{\theta_1(\pi|2|q)} \right)
\]
(5.10)
\[
= \frac{1}{3} \left( 4L(\tau) - 16L(4\tau) \right)
\]
\[
= \frac{1}{3} \left[ 4 \left( 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} \right) - 16 \left( 1 - 24 \sum_{n=1}^{\infty} \frac{nq^{4n}}{1 - q^{4n}} \right) \right]
\]
\[
= -4 - 32 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} + 128 \sum_{n=1}^{\infty} \frac{nq^{4n}}{1 - q^{4n}}.
\]
Also
\[
F(0) = \frac{\theta^3_1(\pi/4|q)\theta_1(\pi/2|q)}{\theta^3_1(0|q)^3} \neq 0.
\]
Now by the residue theorem and using (5.4), we obtain
\[
\varphi(0)^2 = -\varphi'(0).
\]
(5.11)
Hence from (5.9) and (5.10)

\[
\left[ 1 + 4 \sum_{n=0}^{\infty} \left( \frac{q^{4n+1}}{1 - q^{4n+1}} - \frac{q^{4n+3}}{1 - q^{4n+3}} \right) \right]^2 \\
= 1 + 8 \sum_{n=1}^{\infty} \frac{\eta q^n}{1 - q^n} - \sum_{n=1}^{\infty} \frac{4nq^n}{1 - q^n},
\]

which is (1.9).

6. The proof of (1.10)

For proving (1.10) we construct the following elliptic function and use the residue theorem of elliptic functions.

Let

\[
f(z) = \frac{\theta_1(2z|q)\theta_1^2(z + \frac{z}{4}q)\theta_1(z + \frac{z}{4}q)}{\theta_1^4(z|q)}.
\]

By (2.9) it is easily seen that \(f(z)\) is an elliptic function of periods \(\pi\) and \(\pi r\) with only one pole at \(z = 0\) of order 6.

Now

\[
\text{res}(f; 0) = \frac{1}{120} \left[ \frac{d^5}{dz^5} (z^6 f(z)) \right]_{z=0}.
\]

To compute \(\text{res}(f; 0)\), let

\[
F(z) = z^6 f(z) \quad \text{and} \quad \varphi(z) = \frac{F'(z)}{F(z)}.
\]

By logarithmic differentiation and elementary computation, we get

\[
\frac{d^5}{dz^5} F(z) = F(z) \left[ \varphi(z)^5 + 10 \varphi(z)^3 \varphi'(z) + 5 \varphi(z) \varphi''(z) + 10 \varphi(z)^2 \varphi'''(z) + 5 \varphi(z) \varphi'(z)^2 + 10 \varphi'(z) \varphi''(z) + \varphi^{(4)}(z) \right].
\]

It is obvious that

\[
F(0) = \lim_{z \to 0} z^6 f(z) = \frac{\theta_1^2(z|q)\theta_1(z|q)}{\theta_1^4(0|q)^6} \neq 0.
\]

We now calculate \(\varphi(0)\).
From (6.3)

$$
\varphi(z) = \frac{6z^5f(z) + z^6f'(z)}{z^n f(z)} = \frac{6}{z} + \frac{f'(z)}{f(z)}
$$

$$
= \frac{6}{z} + 2\frac{\theta_1'(2z|q)}{\theta_1} - 7\frac{\theta_1'(z|q)}{\theta_1} + 2\frac{\theta_1'(z + \frac{\pi}{4}|q)}{\theta_1} + \frac{\theta_1'(z + \frac{\pi}{2}|q)}{\theta_1}
$$

$$
= L(\tau)z - \frac{3}{5}M(\tau) + 2\frac{\theta_1'(z + \frac{\pi}{4}|q)}{\theta_1} + \frac{\theta_1'(z + \frac{\pi}{2}|q)}{\theta_1} + O(z^5) \text{ by (3.3)}.
$$

Putting \( z = 0 \), we have

$$
\varphi(0) = 2\frac{\theta_1'(\frac{\pi}{4}|q)}{\theta_1} + \frac{\theta_1'(\frac{\pi}{2}|q)}{\theta_1}.
$$

Hence by (5.8)

$$
\varphi(0) = 2\left[ 1 + 4\sum_{n=0}^{\infty} \left( \frac{q^{4n+1}}{1-q^{4n+1}} - \frac{q^{4n+3}}{1-q^{4n+3}} \right) \right].
$$

Differentiating (6.5) with respect to \( z \) and then setting \( z = 0 \) and using (1.6), we have

$$
\varphi'(0) = L(\tau) + 2\left( \frac{\theta_1'}{\theta_1} \right)'\left( \frac{\pi}{4}|q \right) + \left( \frac{\theta_1'}{\theta_1} \right)'\left( \frac{\pi}{2}|q \right)
$$

$$
= \frac{4}{3}L(\tau) - \frac{16}{3}L(4\tau)
$$

$$
= -4 - 32\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} + 128\sum_{n=1}^{\infty} \frac{nq^{4n}}{1-q^{4n}} \text{ by (1.1)}.
$$

Differentiating (6.5) twice with respect to \( z \) and setting \( z = 0 \), we have

$$
\varphi''(0) = 2\left( \frac{\theta_1'}{\theta_1} \right)''\left( \frac{\pi}{4}|q \right) + \left( \frac{\theta_1'}{\theta_1} \right)''\left( \frac{\pi}{2}|q \right).
$$

Now for determining the right hand side of (6.9) we use the following identity [9, p. 489]

$$
\frac{\theta_1'(z|\tau)}{\theta_1} = \cot z + 4\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} \sin 2nz.
$$

Differentiating (6.10) twice with respect to \( z \), we get

$$
\left( \frac{\theta_1'}{\theta_1} \right)''(z|\tau) = 2\cosec^2 z \cot z - 16\sum_{n=1}^{\infty} \frac{n^2q^n}{1-q^n} \sin 2nz.
$$
Setting \( z = \frac{\pi}{4} \) and \( z = \frac{\pi}{2} \), respectively in (6.11), we have

\[
\left( \frac{\theta_1'}{\theta_1} \right)'' \left( \frac{\pi}{4} \mid \tau \right) = 4 - 16 \sum_{n=1}^{\infty} \frac{n^2 q^n}{1 - q^n} \sin \frac{n\pi}{2}
\]

\[= 4 - 16 \sum_{n=0}^{\infty} \left[ \frac{(4n + 1)^2 q^{4n+1}}{1 - q^{4n+1}} - \frac{(4n + 3)^2 q^{4n+3}}{1 - q^{4n+3}} \right]
\]

and

\[
\left( \frac{\theta_1'}{\theta_1} \right)'' \left( \frac{\pi}{2} \mid \tau \right) = 0.
\]

Hence

\[
(6.12) \quad \varphi''(0) = 8 \left[ 1 - 4 \sum_{n=0}^{\infty} \left( \frac{(4n + 1)^2 q^{4n+1}}{1 - q^{4n+1}} - \frac{(4n + 3)^2 q^{4n+3}}{1 - q^{4n+3}} \right) \right].
\]

Differentiating (6.5) thrice with respect to \( z \) and then putting \( z = 0 \) and using (1.7), we have

\[
(6.13) \quad \varphi''(0) = -\frac{6}{5} M(\tau) + 2 \left( \frac{\theta_1'}{\theta_1} \right)'' \left( \frac{\pi}{4} \mid q \right) + \left( \frac{\theta_1'}{\theta_1} \right)'' \left( \frac{\pi}{2} \mid q \right)
\]

\[= -\frac{16}{15} M(\tau) - \frac{2}{15} 4^4 M(4\tau) \text{ by (1.7)}
\]

and

\[
\varphi^{(4)}(0) = 2 \left( \frac{\theta_1'}{\theta_1} \right)^{(4)} \left( \frac{\pi}{4} \mid q \right) + \left( \frac{\theta_1'}{\theta_1} \right)^{(4)} \left( \frac{\pi}{2} \mid q \right).
\]

By (6.10)

\[
\varphi^{(4)}(0) = 2 \cot^{(4)} \left( \frac{\pi}{4} \right) + \cot^{(4)} \left( \frac{\pi}{2} \right) + 128 \sum_{n=1}^{\infty} \frac{n^4 q^n}{1 - q^n} \sin \frac{n\pi}{2}
\]

\[= 160 + 128 \sum_{n=1}^{\infty} \frac{n^4 q^n}{1 - q^n} \sin \frac{n\pi}{2}
\]

\[= 160 + 128 \sum_{n=0}^{\infty} \left[ \frac{(4n + 1)^4 q^{4n+1}}{1 - q^{4n+1}} - \frac{(4n + 3)^4 q^{4n+3}}{1 - q^{4n+3}} \right].
\]

Now by (6.2) and (6.4) the res\((f; 0)\) is

\[
(6.15) \quad \text{res}(f; 0) = \frac{1}{120} F(0) \left[ \varphi(0)^5 + 10\varphi(0)^3 \varphi'(0) + 5\varphi(0)\varphi''(0) + 10\varphi(0)^2 \varphi''(0)
\right.
\]

\[+ 5\varphi(0)\varphi'(0)^2 + 10\varphi'(0)\varphi''(0) + \varphi^{(4)}(0) \right] .
\]
We see that \( \varphi(0) \) calculated in (5.9) is the same as in (6.7) and \( \varphi'(0) \) calculated in (5.10) is the same as in (6.8). Hence by (5.11)
\[
  \varphi(0)^2 = -\varphi'(0).
\]
Putting this in (6.15), \( \text{res}(f; 0) \) simplifies to
\[
  \text{res}(f; 0) = \frac{1}{120} F(0) \left[ \varphi(0)^5 + 5\varphi(0)\varphi''(0) - 5\varphi(0)\varphi'(0)^2 + \varphi^{(4)}(0) \right].
\]
Since \( \text{res}(f; 0) = 0 \) by the residue theorem, we have
\[
  \varphi^{(4)}(0) = - \left[ \varphi(0)^5 + 5\varphi(0)\varphi''(0) - 5\varphi(0)\varphi'(0)^2 \right].
\]
Using (6.7), (6.8), (6.13) and (6.14) the identity in (6.16) simplifies to
\[
  160 + 128 \sum_{n=0}^{\infty} \left( \frac{(4n+1)^4 q^{4n+1}}{1 - q^{4n+1}} - \frac{(4n+3)^4 q^{4n+3}}{1 - q^{4n+3}} \right)
  = -2^5 \left[ 1 + 4 \sum_{n=0}^{\infty} \left( \frac{q^{4n+1}}{1 - q^{4n+1}} - \frac{q^{4n+3}}{1 - q^{4n+3}} \right) \right]^5
  + 10 \left[ 1 + 4 \sum_{n=0}^{\infty} \left( \frac{q^{4n+1}}{1 - q^{4n+1}} - \frac{q^{4n+3}}{1 - q^{4n+3}} \right) \right] \left( \frac{16}{15} M(\tau) + \frac{2}{15} 4^4 M(4\tau) \right)
  + 10 \left[ 1 + 4 \sum_{n=0}^{\infty} \left( \frac{q^{4n+1}}{1 - q^{4n+1}} - \frac{q^{4n+3}}{1 - q^{4n+3}} \right) \right] \left( \frac{4}{3} L(\tau) - \frac{16}{3} L(4\tau) \right)^2,
\]
which is (1.10).

References
