\textbf{\beta\text{-ALGEBRAS AND RELATED TOPICS}}

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\textbf{Abstract.} In this note we investigate some properties of \beta\text{-algebras and further relations with }B\text{-algebras. Especially, we show that if } (X, −, +, 0) \text{ is a } B^*\text{-algebra, then } (X, +) \text{ is a semigroup with identity } 0. \text{ We discuss some constructions of linear } \beta\text{-algebras in a field } K.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: \textit{BCK}\text{-algebras and }BCI\text{-algebras ([3, 4])}. We refer useful textbooks for \textit{BCK/BCI}\text{-algebra to [2, 6, 9]}. J. Neggers and H. S. Kim ([7]) introduced another class related to some of the previous ones, viz., \textit{B}\text{-algebras and studied some of its properties. They also introduced the notion of } \beta\text{-algebra ([8]) where two operations are coupled in such a way as to reflect the natural coupling which exists between the usual group operation and its associated }B\text{-algebra which is naturally defined by it.} \text{ P. J. Allen et al. ([1]) gave another proof of the close relationship of }B\text{-algebras with groups using the observation that the zero adjoint mapping is surjective. H. S. Kim and H. G. Park ([5]) showed that if } X \text{ is a 0-commutative } B\text{-algebra, then } (x * a) * (y * b) = (b * a) * (y * x). \text{ Using this property they showed that the class of } p\text{-semisimple } BCI\text{-algebras is equivalent to the class of 0-commutative }B\text{-algebras.}

In this note we investigate some properties of } \beta\text{-algebras and further relations with }B\text{-algebras. Especially, we show that if } (X, −, +, 0) \text{ is a } B^*\text{-algebra, then } (X, +) \text{ is a semigroup with identity } 0. \text{ Finally we discuss some constructions of linear } \beta\text{-algebras in a field } K.

2. Preliminaries

A \beta\text{-algebra ([8]) is a non-empty set } X \text{ with a constant } 0 \text{ and two binary operations } \text{"}+\text{" and }\text{"}−\text{" satisfying the following axioms: for any } x, y, z \in X,

\begin{align*}
(\text{I}) \quad & x - 0 = x, \\
(\text{II}) \quad & (0 - x) + x = 0,
\end{align*}

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(III) \((x - y) - x = x - (z + y)\).

**Example 2.1** ([8]). Let \(X := \{0, 1, 2, 3\}\) be a set with the following tables:

<table>
<thead>
<tr>
<th>+</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Then \((X, +, - , 0)\) is a \(\beta\)-algebra.

**Proposition 2.2** ([8]). Let \((G, \cdot, e)\) be a group. If we define \(x + y := x \cdot y, x - y := x \cdot y^{-1}, 0 := e\) for any \(x, y \in G\), then \((G, +, - , 0)\) is a \(\beta\)-algebra, called a group-derived \(\beta\)-algebra and denoted by \(A(G)\).

**Proposition 2.3** ([8]). Let \(S\) be a set. If we define \(x + y := x, x - y := x\) and \(0 \in S\), then \((S, +, - , 0)\) is a \(\beta\)-algebra, called a left \(\beta\)-algebra and denoted by \(A_S\).

It is known that the Cartesian product \(X \times Y\) of a group-derived \(\beta\)-algebra \(X\) and a left \(\beta\)-algebra \(Y\) is a \(\beta\)-algebra which is neither group-derived nor a left \(\beta\)-algebra, and denoted by \(A(G) \times A_S\).

We note that if a \(\beta\)-algebra is either \(A(G)\) or \(A_S\), then it is also the case that

(IV) \(x + y = x - (0 - y)\).

Hence the condition (IV) holds for \(\beta\)-algebras of the type \(A(G) \times A_S\) as well.

Group-derived and left \(\beta\)-algebras part ways via the following conditions:

\((V_a)\) \(x - x = 0\) (group derived),

\((V_b)\) \(x - x = x\) (left).

We list two classes of \(\beta\)-algebras of special interest. A \(\beta\)-algebra \(X\) is said to be a \(B^*\)-algebra if (IV) and \((V_a)\) hold. J. Neggers and H. S. Kim introduced the notion of \(B\)-algebra, and obtained various properties. An algebra \((X, - , 0)\) is said to be a \(B\)-algebra ([7]) if it satisfies (I), \((V_a)\) and

\((VI)\) \((x - y) - z = x - (z - (0 - y))\)

for any \(x, y, z \in X\).

### 3. \(\beta\)-algebras and related topics

Given a \(\beta\)-algebra \(X\), we denote \(x^* := 0 - x\) for any \(x \in X\).

**Proposition 3.1.** Let \((X, +, - , 0)\) be a \(\beta\)-algebra with condition (IV). Then the following holds: for any \(x, y, z \in X\),

1. \(x^* + y = x^* - y^*\),
2. \(x^* + x = 0\),
3. \(x^* - x^* = 0\),
4. \(x^* - y^* = (y^* - x^*)^*\),
(5) \( x + y = x - y^* \),
(6) \( x = (x - y) + y = (x - y) - y^* \),
(7) \( y - x = y' - x \) implies \( y = y' \).

Proof. (1) By (IV), \( x^* + y = (0 - x) + y = (0 - x) - (0 - y) = x^* - y^* \). (2) From (II) \( 0 = (0 - x) + x = x^* + x \). (3) If we let \( y := x \) in (1), then \( x^* - x^* = x^* + x = 0 \) by (2). (4) It follows from (III) and (1) that \( x^* - y^* = (0 - x) - (0 - y) = 0 - ((0 - y) + x) = 0 - (y^* + x) = 0 - (y^* - x^*) = (y^* - y^*)^* \). (5) It follows from (IV) immediately. (6) \( x = x - 0 = x - ((0 - y) + y) = (x - y) - y^* = (x - y) + y \).
(7) Suppose that \( y - x = y^* - x \). Then \( y = (y - x) + x = (y^* - x) + x = y^* \), proving the proposition. □

Let \((X, +, -, 0)\) be a \(\beta\)-algebra with condition (IV) and let \(x \in X\). We denote sum of \(x\) as follows:
\[
\begin{align*}
0x &= 0, \quad 1x = x, \\
2x &= x - (0 - x) = x + x, \quad 3x = 2x + x = (x + x) + x, \\
nx &= (n - 1)x + x \quad \text{where } n \text{ is a natural number } \geq 2.
\end{align*}
\]

**Proposition 3.2.** Let \((X, +, - , 0)\) be a \(\beta\)-algebra with condition (IV). Then
\[
(x - ny) + y = x - (n - 1)y
\]
for any \(x, y \in X\) where \(n\) is a natural number.

Proof. For any \(x, y \in X\), \((x - 2y) + y = (x - (y + y)) + y = ((x - y) - y) + y = x - y\) by Proposition 3.1(6).
\[
\begin{align*}
(x - 3y) + y &= (x - (2y + y)) + y \\
&= ((x - y) - 2y) + y \\
&= [(x - y) - (y + y)] + y \\
&= [(x - y) - y - y] + y \quad \text{[by (III)]]} \\
&= (x - y) - y \quad \text{[by Proposition 3.1(6)]} \\
&= x - 2y.
\end{align*}
\]
Using mathematical induction on \(n\), we obtain \((x - ny) + y = x - (n - 1)y\) for any natural number \(n\). □

**Proposition 3.3.** Let \((X, -, 0)\) be a \(\beta\)-algebra with (IV). Then \((X, -, +, 0)\) is a \(\beta\)-algebra.

Proof. (II) By applying (IV) and (V_\(a\)), we obtain \((0 - x) + x = (0 - x) - (0 - x) = 0\). (III) By applying (V_\(i\)) and (IV), we obtain \((x - y) - z = x - (z - (0 - y)) = x - (z + y)\). Hence \((X, -, +, 0)\) is a \(\beta\)-algebra. □

**Proposition 3.4.** Let \((X, -, +, 0)\) be a \(\beta\)-algebra with condition (IV). Then it satisfies the condition (VI).
Proof. Given \( x, y, z \in X \), we have
\[
x - (z - (0 - y)) = (x - (z + y)) \quad \text{[by (IV)]}
= (x - y) - z, \quad \text{[by (III)]}
\]
proving the proposition. \( \square \)

Lemma 3.5. Let \((X, -, +, 0)\) be a \(B^*\)-algebra. Then for any \( x \in X \), we have
\[ x = 0 - (0 - x). \]

Proof. For any \( x \in X \), we have
\[
x = x - 0 \quad \text{[by (I)]}
= x - [(0 - x) + x] \quad \text{[by (II)]}
= (x - x) - (0 - x) \quad \text{[by (III)]}
= 0 - (0 - x), \quad \text{[by (V\_a)]}
\]
proving the lemma. \( \square \)

Theorem 3.6. If \((X, -, +, 0)\) is a \(B^*\)-algebra, then \((X, +)\) is a semigroup with identity 0.

Proof. We claim that \((0 - z) + (0 - y) = 0 - (y + z)\). By applying (IV), Lemma 3.5 and (III), we obtain
\[
(0 - z) + (0 - y) = (0 - z) - (0 - (0 - y)) = (0 - z) - y = 0 - (y + z).
\]
For any \( x, y, z \in X \), we have
\[
(x + y) + z = (x + y) - (0 - z)
\]
\[
= (x - (0 - y)) - (0 - z)
\]
\[
= x - [(0 - z) + (0 - y)]
\]
\[
= x - [0 - (y + z)] \quad \text{[by claim]}
\]
\[
= x + (y + z).
\]
Hence \((X, +)\) is a semigroup. Since \( x + 0 = x - (0 - 0) = x - 0 = x \) and \( 0 + x = 0 - (0 - x) = x \), 0 acts as an identity. \( \square \)

Corollary 3.7. Let \((X, -, +, 0)\) be a \(B^*\)-algebra. If \(0 - x = 0 - y\), then \(x + y = 0\).

Proof. Suppose that \(0 - x = 0 - y\). Then \(0 = (0 - x) + x = (0 - y) + x = (0 - y) - (0 - x) = 0 - (x + y)\) by applying the claim in the proof of Theorem 3.6. Since \(0 - 0 = 0\), by applying Proposition 3.1(7), we obtain \(x + y = 0\). \( \square \)

4. Linear \(\beta\)-algebras

Let \((K, +, \cdot, e)\) be a field (sufficiently large) and let \(x, y \in K\). Define two binary operations “\(\ominus, \oplus\)” on \(K\) as follows:
\[
x \ominus y := \alpha + \beta x + \gamma y,
\]
\[
x \oplus y := A + Bx + Cy,
\]
where $\alpha, \beta, \gamma, A, B, C \in K$ (fixed). Assume that $(K, \ominus, \oplus, e)$ is a $\beta$-algebra.

It is necessary to find proper solutions for two equations. Since $x = x \ominus e = \alpha + \beta x + \gamma e$, we obtain $(\beta - 1)x + (\alpha + \gamma e) = 0$, and hence $\beta = 1$ and $\alpha = -\gamma e$. It follows that

$$(1)\quad x \ominus y = x + \gamma(y - e).$$

Since $(e \ominus x) \oplus x = e$, we obtain

$$(2)\quad [A - Be(1 - \gamma) - e] + (B\gamma + C)x = 0, \forall x \in K.$$

It follows from (2) that $C = -B\gamma, A = [1 + B(1 - \gamma)]e$. Hence we have

$$(3)\quad x \oplus y = [1 + B(1 - \gamma)]e + B(x - \gamma y).$$

Using (1) we obtain

$$(4)\quad (x \ominus y) \ominus z = x + \gamma(y + z - 2e)$$

and

$$(5)\quad x \ominus (z \ominus y) = x + \gamma(z \ominus y - e).$$

To satisfy condition (III), if $\gamma \neq 0$, then

$$z \ominus y - e = y + z - 2e,$$

i.e., $z \ominus y = z + y - e$. Hence $x \ominus y = x + y - e$ and $B = C = 1$. Since $C = -B\gamma, \gamma = -1$, and hence $x \ominus y = x - y + e$. In the case of $\gamma = 0$, we obtain from (1) and (3) that $x \ominus y = x$ and $x \ominus y = (1 + B)e + Bx$, which leads to a contradiction, since $(e \ominus x) \oplus x = 1 + 2Be \neq e$. We summarize:

**Theorem 4.1.** Let $(K, +, \cdot, e)$ be a field (sufficiently large) and let $x, y \in K$. Then $(K, \ominus, \oplus, e)$ is a $\beta$-algebra, where $x \ominus y = x - y + e$ and $x \oplus y = x + y - e$ for any $x, y \in K$.

We call such a $\beta$-algebra described in Theorem 4.1 a linear $\beta$-algebra.

If we let $\varphi : X \to X$ be a map defined by $\varphi(x) = e + bx$ for some $b \in K$. Then we have

$$\varphi(x + y) = e + b(x + y)$$

$$= (e + bx) + (e + by) - e$$

$$= \varphi(x) \ominus \varphi(y)$$

and

$$\varphi(x - y) = e + b(x - y)$$

$$= (e + bx) - (e + by) + e$$

$$= \varphi(x) \ominus \varphi(y),$$

so that $\varphi(0) = e$ implies $\varphi : (K, - , +, 0) \to (K, \ominus, \oplus, e)$ is a homomorphism of $\beta$-algebras, where “$-$” is usual subtraction in the field $K$. If $b \neq 0$, then $\psi : (K, \ominus, \oplus, e) \to (K, - , +, 0)$ defined by $\psi(x) := (x - e)/b$ is a homomorphism of
\( \beta \)-algebras and the inverse mapping of the mapping \( \varphi \), so that \((K, \ominus, \oplus, e)\) and \((K, -, +, 0)\) are isomorphic as \( \beta \)-algebras, i.e., there is only one isomorphism type in this case. We summarize:

**Proposition 4.2.** The \( \beta \)-algebra \((K, \ominus, \oplus, e)\) discussed in Theorem 4.1 is unique up to isomorphism.

**References**


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