ON MINIMAL SEMICONTINUOUS FUNCTIONS

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Abstract. In this paper, we introduce the notions of minimal semicontinuity, strongly \( m \)-semiclosed graph, \( m \)-semiclosed graph, \( m \)-semi-\( T_2 \), \( m \)-semicompact and investigate some properties for such notions.

1. Introduction

In [4], Popa and Noiri introduced the notion of minimal structure which is a generalization of a topology on a given nonempty set. And they introduced the notion of \( m \)-continuous function [3] as a function defined between a minimal structure and a topological space. They showed that the \( m \)-continuous functions have properties similar to those of continuous functions between topological spaces. We introduced and studied the notions of \( m \)-semiopen sets, \( m \)-semi-interior and \( m \)-semi-closure operators [2] on a space with a minimal structure. In this paper, we introduce and study the notion of \( m \)-semicontinuous function defined between a minimal structure and a topological space. We also introduce the notions of strongly \( m \)-semiclosed graph, \( m \)-semiclosed graph, \( m \)-semi-\( T_2 \), \( m \)-semicompact and investigate some properties for such notions.

2. Preliminaries

Let \( X \) be a topological space and \( A \subseteq X \). The closure of \( A \) and the interior of \( A \) are denoted by \( \overline{A} \) and \( \text{int}(A) \), respectively. A subfamily \( m_X \) of the power set \( P(X) \) of a nonempty set \( X \) is called a minimal structure [4] on \( X \) if \( \emptyset \in m_X \) and \( X \in m_X \). By \( (X, m_X) \), we denote a nonempty set \( X \) with a minimal structure \( m_X \) on \( X \). Simply we call \( (X, m_X) \) a space with a minimal structure \( m_X \) on \( X \). Let \( (X, m_X) \) be a space with a minimal structure \( m_X \) on \( X \). For a subset \( A \) of \( X \), the closure of \( A \) and the interior of \( A \) are defined as the following [4]:

\[
\begin{align*}
\text{mInt}(A) &= \bigcup \{ U : U \subseteq A, U \in m_X \}; \\
\text{mCl}(A) &= \bigcap \{ F : A \subseteq F, X - F \in m_X \}.
\end{align*}
\]

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A subset \( A \) of \( X \) is called an \textit{m-semiopen set} [2] if \( A \subseteq msCl(mInt(A)) \). The complement of an m-semiopen set is called an \textit{m-semiclosed set}. In [2], we showed that any union of m-semiopen sets is m-semiopen.

For a subset \( A \) of \( X \), the m-semi-closure of \( A \) and the m-semi-interior of \( A \), denoted by \( msCl(A) \) and \( msInt(A) \), respectively, are defined as the following:

\[
msCl(A) = \cap \{ F : A \subseteq F, F \text{ is m-semiclosed in } X \};
\]

\[
msInt(A) = \cup \{ U : U \subseteq A, U \text{ is m-semiopen in } X \}.
\]

**Theorem 2.1** ([2]). Let \((X, m_X)\) be a space with a minimal structure \( m_X \) on \( X \) and \( A \subseteq X \). Then

1. \( msInt(A) \subseteq A \subseteq msCl(A) \).
2. If \( A \subseteq B \), then \( msInt(A) \subseteq msInt(B) \) and \( msCl(A) \subseteq msCl(B) \).
3. \( A \) is m-semiopen if and only if \( msInt(A) = A \).
4. \( F \) is m-semiclosed if and only if \( msCl(F) = F \).
5. \( msInt(msInt(A)) = msInt(A) \) and \( msCl(msCl(A)) = msCl(A) \).
6. \( msCl(X - A) = X - msInt(A) \) and \( msInt(X - A) = X - msCl(A) \).

Let \( f : (X, m_X) \rightarrow (Y, \tau) \) be a function between a space \((X, m_X)\) with minimal structure \( m_X \) and a topological space \((Y, \tau)\). Then \( f \) is said to be \textit{m-continuous} [3] if for each \( x \) and each open set \( V \) containing \( f(x) \), there exists an m-open set \( U \) containing \( x \) such that \( f(U) \subseteq V \).

### 3. Minimal semicontinuous functions

**Definition 3.1.** Let \( f : (X, m_X) \rightarrow (Y, \tau) \) be a function between a space \( X \) with a minimal structure \( m_X \) and a topological space \( Y \). Then \( f \) is said to be \textit{minimal semicontinuous} (briefly \textit{m-semicontinuous}) if for each \( x \) and each open set \( V \) containing \( f(x) \), there exists an m-semiopen set \( U \) containing \( x \) such that \( f(U) \subseteq V \).

\[ m - \text{continuity} \Rightarrow m - \text{semicontinuity} \]

In the above diagram, the converse may not be true.

**Example 3.2.** Let \( X = \{a, b, c\} \). Consider a minimal structure \( m_X = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \) and a topology \( \tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\} \). Let \( f : (X, m_X) \rightarrow (X, \tau) \) be the identity function. Then \( f \) is m-semicontinuous but not m-continuous.

**Theorem 3.3.** Let \( f : (X, m_X) \rightarrow (Y, \tau) \) be a function between a space \( X \) with a minimal structure \( m_X \) and a topological space \((Y, \tau)\). Then the following statements are equivalent:

1. \( f \) is m-semicontinuous.
2. For each open set \( V \) in \( Y \), \( f^{-1}(V) \) is m-semiopen.
3. For each closed set \( B \) in \( Y \), \( f^{-1}(B) \) is m-semiclosed.
4. \( f(msCl(A)) \subseteq cl(f(A)) \) for \( A \subseteq X \).
5. \( msCl(f^{-1}(B)) \subseteq f^{-1}(cl(B)) \) for \( B \subseteq Y \).
6. \( f^{-1}(int(B)) \subseteq msInt(f^{-1}(B)) \) for \( B \subseteq Y \).
Proof. (1) ⇒ (2) Let $V$ be an open set in $Y$ and $x \in f^{-1}(V)$. By hypothesis, there exists an $m$-semiopen set $U$ containing $x$ such that $f(U) \subseteq V$. So we have $x \in U \subseteq f^{-1}(V)$ for all $x \in f^{-1}(V)$. Hence $f^{-1}(V)$ is $m$-semiopen.

(2) ⇒ (3) Obvious.

(3) ⇒ (4) For $A \subseteq X$,

\[
f^{-1}(\text{cl}(f(A))) = f^{-1}(\cap\{F \subseteq Y : f(A) \subseteq F \text{ and } F \text{ is closed}\})
\]

\[
= \cap \{f^{-1}(F) \subseteq X : A \subseteq f^{-1}(F) \text{ and } f^{-1}(F) \text{ is } m\text{-semiclosed}\}
\]

\[
\supseteq \cap \{K \subseteq X : A \subseteq K \text{ and } K \text{ is } m\text{-semiclosed}\}
\]

\[
= m\text{Cl}(A).
\]

Hence $f(m\text{Cl}(A)) \subseteq \text{cl}(f(A))$.

(4) ⇒ (5) Obvious.

(5) ⇒ (6) It follows from Theorem 2.1(6).

(6) ⇒ (1) Let $x \in X$ and $V$ an open set containing $f(x)$. Then from (6), it follows $x \in f^{-1}(V) = f^{-1}(\text{int}(V)) \subseteq m\text{Int}(f^{-1}(V))$. So there exists an $m$-semiopen set $U$ containing $x$ such that $x \in U \subseteq f^{-1}(V)$. Hence this implies $f$ is $m$-semicontinuous. □

Lemma 3.4 ([2]). Let $(X, m_X)$ be a space with a minimal structure $m_X$ on $X$ and $A \subseteq X$. Then

1. $m\text{Int}(m\text{Cl}(A)) \subseteq m\text{Int}(m\text{Cl}(m\text{Cl}(A))) \subseteq m\text{Cl}(A)$.
2. $m\text{Int}(A) \subseteq m\text{Cl}(m\text{Int}(m\text{Int}(A))) \subseteq m\text{Int}(m\text{Cl}(A))$.
3. $A$ is $m$-semiclosed if and only if $m\text{Int}(m\text{Cl}(A)) \subseteq A$.

From Theorem 3.3 and Lemma 3.4, we have the next theorem.

Theorem 3.5. Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space $X$ with a minimal structure $m_X$ and a topological space $(Y, \tau)$. Then the following statements are equivalent:

1. $f$ is $m$-semicontinuous.
2. $f^{-1}(V) \subseteq m\text{Cl}(m\text{Int}(f^{-1}(V)))$ for each open set $V$ in $Y$.
3. $m\text{Int}(m\text{Cl}(m\text{Int}(f^{-1}(F)))) \subseteq f^{-1}(F)$ for each closed set $F$ in $Y$.
4. $f(m\text{Int}(m\text{Cl}(A))) \subseteq \text{cl}(f(A))$ for $A \subseteq X$.
5. $m\text{Int}(m\text{Cl}(f^{-1}(B))) \subseteq f^{-1}(\text{cl}(B))$ for $B \subseteq Y$.
6. $f^{-1}(\text{int}(B)) \subseteq m\text{Cl}(m\text{Int}(f^{-1}(B)))$ for $B \subseteq Y$.

Definition 3.6. Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space $(X, m_X)$ with a minimal structure $m_X$ and a topological space $(Y, \tau)$. Then $f$ has a strongly $m$-semiclosed graph (resp., an $m$-semiclosed graph) if for each $(x, y) \in (X \times Y) - G(f)$, there exist an $m$-semiopen set $U$ containing $x$ and an open set $V$ containing $y$ such that $(U \times \text{cl}(V)) \cap G(f) = \emptyset$ (resp., $(U \times V) \cap G(f) = \emptyset$).
Theorem 3.12. Let \( f : (X, m_X) \to (Y, \tau) \) be a function between a space \((X, m_X)\) with a minimal structure \(m_X\) and a topological space \((Y, \tau)\). Then \( f \) has a strongly \(m\)-semi-closed graph (resp., an \(m\)-semi-closed graph) if and only if for each \((x, y) \in (X \times Y) - G(f)\), there exist an \(m\)-semiopen set \(U\) containing \(x\) and an open set \(V\) containing \(y\) such that \(f(U) \cap cl(V) = \emptyset\) (resp., \(f(U) \cap V = \emptyset\)).

Theorem 3.8. Let \( f : (X, m_X) \to (Y, \tau) \) be a function between a space \((X, m_X)\) with a minimal structure \(m_X\) and a topological space \((Y, \tau)\). If \( f \) is \(m\)-semi-continuous and \((Y, \tau)\) is T2, then \( f \) has a strongly \(m\)-semi-closed graph.

**Proof.** Let \((x, y) \in (X \times Y) - G(f)\); then \(f(x) \neq y\). Since \(Y\) is T2, there are disjoint open sets \(U, V\) such that \(f(x) \in U\), \(y \in V\). This implies \(cl(V) \cap U = \emptyset\).

And for \(f(x) \in U\), from \(m\)-semi-continuity of \(f\), there exists an \(m\)-semiopen set \(G\) containing \(x\) such that \(f(G) \subseteq U\). Consequently, we can say that there exist an open set \(V\) and \(m\)-semiopen set \(G\) containing \(y, x\), respectively, such that \(f(G) \cap cl(V) = \emptyset\) and so by Lemma 3.7, \( f \) has a strongly \(m\)-semi-closed graph. \(\square\)

Corollary 3.9. Let \( f : (X, m_X) \to (Y, \tau) \) be a function between a space \((X, m_X)\) with a minimal structure \(m_X\) and a topological space \((Y, \tau)\). If \( f \) is \(m\)-semi-continuous and \((Y, \tau)\) is T2, then \( f \) has an \(m\)-semi-closed graph.

Theorem 3.10. Let \( f : (X, m_X) \to (Y, \tau) \) be a function between a space \((X, m_X)\) with a minimal structure \(m_X\) and a topological space \((Y, \tau)\). If \( f \) is a surjective function with a strongly \(m\)-semi-closed graph, then \( Y \) is T2.

**Proof.** Let \( y \) and \( z \) be any distinct points of \( Y \). Then there is \( x \in X \) such that \( f(x) = y \). Thus \((x, z) \in (X \times Y) - G(f)\). Since \( f \) has a strongly \(m\)-semi-closed graph, there exist an \(m\)-semiopen set \(U\) containing \(x\) and an open set \(V\) containing \(z\) such that \(f(U) \cap cl(V) = \emptyset\). So since \(f(x) = y \in f(U) \subseteq Y - cl(V)\), there exists an open set \(G\) containing \(y\) such that \(G \cap V = \emptyset\). Hence \( Y \) is T2. \(\square\)

Definition 3.11. Let \((X, m_X)\) be a space with a minimal structure \(m_X\). Then \( X \) is said to be \(m\)-semi-T2 if for any distinct points \(x\) and \(y\) of \(X\), there exist disjoint \(m\)-semiopen sets \(U, V\) such that \(x \in U\) and \(y \in V\).

Theorem 3.12. Let \( f : (X, m_X) \to (Y, \tau) \) be a function between a space \((X, m_X)\) with a minimal structure \(m_X\) and a topological space \((Y, \tau)\). If \( f \) is an injective \(m\)-semi-continuous function and \( Y \) is T2, then \( X \) is \(m\)-semi-T2.

**Proof.** Obvious. \(\square\)

Theorem 3.13. Let \( f : (X, m_X) \to (Y, \tau) \) be a function between a space \((X, m_X)\) with a minimal structure \(m_X\) and a topological space \((Y, \tau)\). If \( f \) is an injective \(m\)-semi-continuous function with an \(m\)-semi-closed graph, then \( X \) is \(m\)-semi-T2.
Proof. Let $x_1$ and $x_2$ be any distinct points of $X$. Then $f(x_1) \neq f(x_2)$, so $(x_1, f(x_2)) \in (X \times Y) - G(f)$. Since $f$ has an $m$-semiclosed graph, there exist an $m$-semiopen set $U$ containing $x_1$ and $V \in \tau$ containing $f(x_2)$ such that $f(U) \cap V = \emptyset$. Since $f$ is $m$-semicontinuous, $f^{-1}(V)$ is an $m$-semiopen set containing $x_2$ such that $U \cap f^{-1}(V) = \emptyset$. Hence $X$ is $m$-semi-$T_2$. □

Corollary 3.14. Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space $(X, m_X)$ with a minimal structure $m_X$ and a topological space $(Y, \tau)$. If $f$ is an injective $m$-semicontinuous function with a strongly $m$-semiclosed graph, then $X$ is $m$-semi-$T_2$.

Definition 3.15. A subset $A$ of a space $(X, m_X)$ with a minimal structure $m_X$ is called minimal semicompact (briefly $m$-semicompact) relative to $A$ if every collection $\{U_i : i \in J\}$ of $m$-semiopen subsets of $X$ such that $A \subseteq \bigcup \{U_i : i \in J\}$, there exists a finite subset $J_0$ of $J$ such that $A \subseteq \bigcup \{U_i : i \in J_0\}$. A subset $A$ of a minimal structure $(X, m_X)$ is said to be $m$-semicompact if $A$ is $m$-semicompact as a subspace of $X$.

Theorem 3.16. Let $f : (X, m_X) \to (Y, \tau)$ be an $m$-semicontinuous function between a space $(X, m_X)$ with a minimal structure $m_X$ and a topological space $(Y, \tau)$. If $A$ is an $m$-semicompact set, then $f(A)$ is compact.

Proof. Let $\{U_i : i \in J\}$ be an open cover of $f(A)$ in $Y$. Then since $f$ is an $m$-semicontinuous function, $\{f^{-1}(U_i) : i \in J\}$ is an $m$-semiopen cover of $A$ in $X$. By $m$-semicompactness, there exists $J_0 = \{j_1, j_2, \ldots, j_n\} \subseteq J$ such that $A \subseteq \bigcup_{j \in J_0} f^{-1}(U_j)$. Hence $f(A) \subseteq f(\bigcup_{j \in J_0} f^{-1}(U_j)) \subseteq \bigcup_{j \in J_0} U_j$. Thus $f(A)$ is compact. □

References


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