SEMI-RIEMANNIAN SUBMANIFOLDS OF A SEMI-RIEMANNIAN MANIFOLD WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract. We study some properties of a semi-Riemannian submanifold of a semi-Riemannian manifold with a semi-symmetric non-metric connection. Then, we prove that the Ricci tensor of a semi-Riemannian submanifold of a semi-Riemannian space form admitting a semi-symmetric non-metric connection is symmetric but is not parallel. Last, we give the conditions under which a totally umbilical semi-Riemannian submanifold with a semi-symmetric non-metric connection is projectively flat.

1. Introduction

The notion of a semi-symmetric linear connection on a differentiable manifold was initiated by Friedmann and Schouten [5] in 1924. In 1992, Agashe and Chafle [1] defined a semi-symmetric non-metric connection on a Riemannian manifold and studied the Weyl projective curvature tensor with respect this connection. Moreover, in 1994 they considered in [2] a submanifold admitting a semi-symmetric non-metric connection and studied some of its properties when the ambient manifold is a space form admitting a semi-symmetric non-metric connection. In 1995, the properties of hypersurfaces of a Riemannian manifold with a semi-symmetric non-metric connection were studied by De and Kamilya [4]. In 2000, Sengupta, De and Binh [9] defined a semi-symmetric non-metric connection which generalized the notion of the semi-symmetric non-metric connection introduced by Agashe and Chafle. Later, they derived the curvature tensor and the Weyl projective curvature tensor with respect to the semi-symmetric non-metric connection. Prasad and Verma [8], in 2004, got the necessary and sufficient condition in order that the Weyl projective curvature tensor of a semi-symmetric non-metric connection is equal to the Weyl projective curvature of the Riemannian connection. Moreover, they showed that if the curvature tensor with respect to the semi-symmetric non-metric connection
vanishes, then the Riemannian manifold is projectively flat. Yücesan and Yaşar [11] studied non-degenerate hypersurfaces of a semi-Riemannian manifold with a semi-symmetric non-metric connection and got the conditions under which a non-degenerate hypersurface with a semi-symmetric non-metric connection is projectively flat.

This paper is organized as follows: In Section 2, we consider a semi-Riemannian submanifold immersed in an ambient semi-Riemannian manifold. Then we determine the semi-symmetric non-metric connection, and give the equations of Gauss and Weingarten for a semi-Riemannian submanifold of a semi-Riemannian manifold with a semi-symmetric non-metric connection. Furthermore, we show that on a semi-Riemannian submanifold the connection induced from the semi-symmetric non-metric connection is also a semi-symmetric non-metric connection. In Section 3, by using the equations stated above, we derive Gauss curvature and Codazzi-Mainardi equations with respect to the semi-symmetric non-metric connection. In Section 4, we show that the Ricci tensor of a semi-Riemannian submanifold of a semi-Riemannian space form admitting a semi-symmetric non-metric connection is symmetric, but is not parallel. In the last section, we prove that a totally umbilical semi-Riemannian submanifold in a projectively flat semi-Riemannian manifold with a semi-symmetric non-metric connection is projectively flat.

2. Semi-symmetric non-metric connection

We suppose that $M$ is an $n$-dimensional semi-Riemannian manifold of an $(n+p)$-dimensional semi-Riemannian manifold $\tilde{M}$ with semi-Riemannian metric $\tilde{g}$ of index $0 \leq \nu \leq n + p$. Let us denote by $g$ the induced semi-Riemannian metric tensor on $M$ from $\tilde{g}$ on $\tilde{M}$. As $M$ has codimension $p$, we can locally choose $p$ cross sections $\xi_{\alpha}$, $1 \leq \alpha \leq p$, of the normal bundle $TM_{\perp}$ of $M$ in $\tilde{M}$ which are orthonormal at each point of $M$. The index of $\tilde{g}$ restricted to $TM_{\perp}$ is called the co-index of $M$ in $\tilde{M}$ and $\text{ind} \tilde{M} = \nu = \text{ind} M + \text{coind} M$ (see [7]).

A linear connection $\tilde{\nabla}$ on $\tilde{M}$ is called a semi-symmetric non-metric connection if its torsion tensor $\tilde{T}$ satisfies

$$\tilde{T}(\tilde{X}, \tilde{Y}) = \tilde{\pi}(\tilde{Y})\tilde{X} - \tilde{\pi}(\tilde{X})\tilde{Y}$$

and

$$\left(\tilde{\nabla}_{\tilde{X}} \tilde{g}\right)\tilde{Y} = -\tilde{\pi}(\tilde{Y})\tilde{g}(\tilde{X}, \tilde{Z}) - \tilde{\pi}(\tilde{Z})\tilde{g}(\tilde{X}, \tilde{Y})$$

for $\tilde{X}, \tilde{Y} \in \chi(\tilde{M})$, where $\tilde{\pi}$ is a 1-form on $\tilde{M}$ (see [1]).

We define a linear connection $\tilde{\nabla}$ on $\tilde{M}$ given by

$$\tilde{\nabla}_{\tilde{X}} \tilde{Y} = \tilde{\nabla}_{\tilde{X}} \tilde{Y} + \tilde{\pi}(\tilde{Y})\tilde{X}$$

(2.1)

for $\tilde{X}, \tilde{Y} \in \chi(\tilde{M})$, where $\tilde{\nabla}$ denotes the Levi-Civita connection with respect to $\tilde{g}$ and $\tilde{\pi}$ is a 1-form associated to a vector field $\tilde{Q}$ by $\tilde{g}(\tilde{Q}, \tilde{X}) = \tilde{\pi}(\tilde{X})$ for
\( \tilde{X} \in \chi(\tilde{M}) \). Then \( \tilde{\nabla} \) is a semi-symmetric non-metric connection on \( \tilde{M} \). On \( M \) we define a vector field \( Q \) and real valued functions \( \mu_\alpha, 1 \leq \alpha \leq p \), by decomposing \( \tilde{Q} \) into its unique tangential and normal components, thus

\[
(2.2) \quad \tilde{Q} = Q + \sum_{\alpha=1}^{p} \mu_\alpha \xi_\alpha.
\]

If we denote by \( \overset{\circ}{\nabla} \) the induced Levi-Civita connection on \( M \) from \( \overset{\circ}{\nabla} \) on \( \tilde{M} \), then we have the Gauss equation with respect to \( \overset{\circ}{\nabla} \) given by

\[
(2.3) \quad \overset{\circ}{\nabla}_X Y = \overset{\circ}{\nabla}_X Y + \sum_{\alpha=1}^{p} \overset{\circ}{h}_\alpha(X, Y) \xi_\alpha
\]

for \( X, Y \in \chi(M) \), where \( \overset{\circ}{h}_\alpha \), \( 1 \leq \alpha \leq p \), are the second fundamental forms on \( M \) [7]. Let \( \tilde{\nabla} \) on \( M \) be induced connection from the semi-symmetric non-metric connection \( \tilde{\nabla} \) on \( \tilde{M} \). Thus, the equation given by

\[
(2.4) \quad \tilde{\nabla}_X Y = \nabla_X Y + \sum_{\alpha=1}^{p} h_\alpha(X, Y) \xi_\alpha,
\]

will be called the Gauss equation with respect to \( \tilde{\nabla} \) for \( X, Y \in \chi(M) \), where \( h_\alpha \), \( 1 \leq \alpha \leq p \), are tensors of type \((0, 2)\) on \( M \).

Substituting (2.3) and (2.4) into (2.1), we see that

\[
\nabla_X Y + \sum_{\alpha=1}^{p} h_\alpha(X, Y) \xi_\alpha = \overset{\circ}{\nabla}_X Y + \sum_{\alpha=1}^{p} \overset{\circ}{h}_\alpha(X, Y) \xi_\alpha + \pi(Y)X
\]

from which we get

\[
(2.5) \quad \nabla_X Y = \overset{\circ}{\nabla}_X Y + \pi(Y)X,
\]

where

\[
\pi(Y) = g(Y, Q),
\]

and we obtain

\[
(2.6) \quad h_\alpha = \overset{\circ}{h}_\alpha, \quad 1 \leq \alpha \leq p,
\]

for \( X, Y \in \chi(M) \). By using (2.5), we deduce that

\[
(2.7) \quad (\nabla_X g)(Y, Z) = -\pi(Y)g(X, Z) - \pi(Z)g(X, Y)
\]

for \( X, Y, Z \in \chi(M) \).

Also, from (2.5) the torsion tensor of the connection \( \nabla \), denoted by \( T \), can be obtained as

\[
(2.8) \quad T(X, Y) = \pi(Y)X - \pi(X)Y.
\]

Then from (2.7) and (2.8) we have the following theorem:
Theorem 1. The induced connection on a semi-Riemannian submanifold of a semi-Riemannian manifold with a semi-symmetric non-metric connection is also a semi-symmetric non-metric connection.

The Weingarten equation with respect to $\nabla$ is given by

\[ \nabla_X \xi_\alpha = A_\xi \xi_\alpha (X) + D_X \xi_\alpha, \quad 1 \leq \alpha \leq p, \]

for $X \in \chi(M)$, where $D$ is a metric connection on the normal bundle $TM^\perp$ with respect to the fibre metric induced from $\tilde{g}$, and the $(1,1)$ tensor fields $A_\xi \xi_\alpha, 1 \leq \alpha \leq p$, on $M$ such that

\[ h_\alpha (X, Y) = \varepsilon_\alpha g (A_\xi \xi_\alpha (X), Y) \]

are called the shape operators of $M \subset \tilde{M}$ (see [7]).

By virtue of (2.1) and (2.2), we get

\[ \nabla_X \xi_\alpha = \nabla_X \xi_\alpha + \varepsilon_\alpha \mu_\alpha X, \quad 1 \leq \alpha \leq p. \]

From the above and (2.9) it follows that

\[ \nabla_X \xi_\alpha = -(A_\xi \xi_\alpha - \varepsilon_\alpha \mu_\alpha I) (X) + D_X \xi_\alpha, \quad 1 \leq \alpha \leq p, \]

where $I$ is the identity tensor and

\[ \varepsilon_\alpha = \begin{cases} -1, & \xi_\alpha \text{ is timelike,} \\ +1, & \xi_\alpha \text{ is spacelike.} \end{cases} \]

Let the shape operators $A_\xi \xi_\alpha, 1 \leq \alpha \leq p$, of type $(1,1)$ on $M$ be denoted by

\[ A_\xi \xi_\alpha = A_\xi \xi_\alpha - \varepsilon_\alpha \mu_\alpha I, \quad 1 \leq \alpha \leq p. \]

So, equation (2.11), called the Weingarten equation with respect to $\nabla$, can be rewritten as

\[ \nabla_X \xi_\alpha = -A_\xi \xi_\alpha (X) + D_X \xi_\alpha, \quad 1 \leq \alpha \leq p, \]

for $X \in \chi(M)$.

By using (2.6), (2.10) and (2.12), we have

\[ \varepsilon_\alpha h_\alpha (X, Y) = g (A_\xi \xi_\alpha X, Y) + \varepsilon_\alpha \mu_\alpha g (X, Y). \]

Let $\xi = \sum_{\alpha=1}^{p} a_\alpha \xi_\alpha, \eta = \sum_{\alpha=1}^{p} b_\alpha \xi_\alpha$ be two normal vector fields on $M$. Then from (2.12), we see that

\[ A_\xi A_\eta = A_\xi A_\eta - \varepsilon_\alpha a_\alpha \mu_\alpha A_\eta - \varepsilon_\alpha b_\alpha \mu_\alpha A_\xi + a_\alpha b_\alpha \mu_\delta I. \]

Thus,

\[ [A_\xi, A_\eta] = [A_\xi, A_\eta], \]
and
\[ g([A_\xi, A_\eta]X, Y) = g([A_\xi, A_\eta]X, Y) \]
for all \( X, Y \in \chi(M) \). Hence we have:

**Theorem 2.** Let \( M \) be a semi-Riemannian submanifold of a semi-Riemannian manifold \( \tilde{M} \) admitting a semi-symmetric non-metric connection. Then the second fundamental tensors with respect to the semi-symmetric non-metric connection are simultaneously diagonalizable if and only if second fundamental tensors with respect to the Levi-Civita connection are simultaneously diagonalizable.

Let \( E_1, \ldots, E_\nu, \ldots, E_n \) be the principal vector fields on \( M \) corresponding to the unit normal section \( \xi = \sum_{\alpha=1}^p a_\alpha \xi_\alpha \) with respect to \( \tilde{\nabla} \). Then by using (2.12), we have

\[ A_\xi(E_i) = (\tilde{k}_i - \varepsilon_\alpha a_\alpha \mu_\alpha)E_i, \quad 1 \leq i \leq n, \]
where \( \tilde{k}_i, 1 \leq i \leq n, \) are the principal curvatures corresponding to the unit normal section \( \xi \) with respect to the Levi-Civita connection \( \nabla \). Taking

\[ k_i = \tilde{k}_i - \varepsilon_\alpha a_\alpha \mu_\alpha, \quad 1 \leq i \leq n, \quad 1 \leq \alpha \leq p. \]
So, by (2.16), equation (2.15) is reformed as

\[ A_\xi(E_i) = k_i E_i, \quad 1 \leq i \leq n, \]
where \( k_i, 1 \leq i \leq n, \) are the principal curvatures of the unit normal section \( \xi \) with respect to the semi-symmetric non-metric connection \( \tilde{\nabla} \).

From (2.15), (2.16) and (2.17), we assert the following:

**Theorem 3.** The principal directions of the unit normal direction \( \xi \) with respect to the Levi-Civita connection \( \nabla \) and the semi-symmetric non-metric connection \( \tilde{\nabla} \) coincides and corresponding principal curvatures are equal if and only if \( \xi \) is orthogonal to \( \tilde{Q} \).

The mean curvature vector field of \( M \) with respect to \( \nabla \) is given by

\[ \nabla H = \frac{1}{n} \sum_{i=1}^n \varepsilon_i \sum_{\alpha=1}^p h_\alpha(E_i, E_i)\xi_\alpha, \]
where
\[ \varepsilon_i = \begin{cases} -1, & E_i \text{ is timelike}, \\ +1, & E_i \text{ is spacelike} \end{cases} \]
(see [7]). We define similarly the mean curvature vector field of \( M \) with respect to \( \tilde{\nabla} \) by

\[ \tilde{H} = \frac{1}{n} \sum_{i=1}^n \varepsilon_i \sum_{\alpha=1}^p h_\alpha(E_i, E_i)\xi_\alpha. \]
From (2.6), (2.18) and (2.19), \( H = \hat{H} \). Hence we have:

**Lemma 4.** A semi-Riemannian submanifold \( M \) of a semi-Riemannian manifold \( \tilde{M} \) admitting a semi-symmetric non-metric connection is totally geodesic with respect to the semi-symmetric non-metric connection if and only if it is totally geodesic with respect to the Levi-Civita connection.

**Lemma 5.** A semi-Riemannian submanifold \( M \) of a semi-Riemannian manifold \( \tilde{M} \) admitting a semi-symmetric non-metric connection is totally umbilical with respect to the semi-symmetric non-metric connection if and only if it is totally umbilical with respect to the Levi-Civita connection.

3. The Gauss curvature and Codazzi-Mainardi equations

We denote by

\[
\hat{R}(\hat{X}, \hat{Y})\hat{Z} = \nabla_{\hat{X}} \nabla_{\hat{Y}} \hat{Z} - \nabla_{\nabla_{\hat{Y}} \hat{X}} \hat{Z} - \nabla_{[\hat{X}, \hat{Y}]} \hat{Z}
\]

and

\[
\hat{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_{\nabla_Y X} Z - \nabla_{[X, Y]} Z,
\]

the curvature tensors of \( \tilde{M} \) and \( M \) with respect to \( \hat{\nabla} \) and \( \nabla \), respectively, where \( \hat{X}, \hat{Y}, \hat{Z} \in \chi(M) \) and \( X, Y, Z \in \chi(M) \). Then the Gauss curvature and Codazzi-Mainardi equations with respect to \( \hat{\nabla} \) and \( \nabla \), respectively, are given by

\[
\hat{R}(X, Y, Z, W) = \hat{R}(X, Y, Z, W) + \sum_{\alpha=1}^{p} \varepsilon_\alpha \{ h_\alpha(X, Z) h_\alpha(Y, W) - h_\alpha(Y, Z) h_\alpha(X, W) \},
\]

and

\[
\hat{R}(X, Y, Z, \xi_\alpha) = \varepsilon_\alpha \{ (\nabla_X h_\alpha)(Y, Z) - (\nabla_Y h_\alpha)(X, Z) \}
+ \sum_{\beta=1}^{p} \bar{g}(\hat{h}_\beta(Y, Z) D_X \xi_\beta - \hat{h}_\beta(X, Z) D_Y \xi_\beta, \xi_\alpha)
\]

for \( X, Y, Z \in \chi(M) \) (see [7]), where

\[
\hat{R}(X, Y, Z, W) = \hat{g}(\hat{R}(X, Y)Z, W), \quad \hat{R}(X, Y, Z, W) = \bar{g}(\hat{R}(X, Y)Z, W).
\]

Now we shall find the Gauss curvature and the Codazzi-Mainardi equations with respect to the semi-symmetric non-metric connections \( \hat{\nabla} \) and \( \nabla \). The curvature tensors of \( \tilde{M} \) and \( M \) with respect to \( \hat{\nabla} \) and \( \nabla \), respectively, are defined by

\[
\hat{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_{\nabla_Y X} Z - \nabla_{[X, Y]} Z
\]
and
\[ R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z \]
for \( X, Y, Z \in \chi(M) \).

By using (2.4) and (2.13), we have the curvature tensor of the semi-symmetric non-metric connection \( \tilde{\nabla} \) given by
\[ (3.1) \]
\[ \tilde{R}(X, Y)Z = R(X, Y)Z + \sum_{\alpha=1}^{p} h_{\alpha}(X, Z)A_{\xi_{\alpha}} Y - h_{\alpha}(Y, Z)A_{\xi_{\alpha}} X \]
\[ +(\nabla_X h_{\alpha})(Y, Z)\xi_{\alpha} - (\nabla_Y h_{\alpha})(X, Z)\xi_{\alpha} + h_{\alpha}(\pi(Y)X - \pi(X)Y, Z)\xi_{\alpha} \]
\[ + h_{\alpha}(Y, Z)D_X \xi_{\alpha} - h_{\alpha}(X, Z)D_Y \xi_{\alpha} \] .

From (3.1), the Gauss curvature equation and the Codazzi-Mainardi equation with respect to \( \tilde{\nabla} \) and \( \nabla \), respectively, are obtained as:
\[ (3.2) \]
\[ \tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + \sum_{\alpha=1}^{p} \varepsilon_{\alpha} \{ h_{\alpha}(X, Z)h_{\alpha}(Y, W) \]
\[ - h_{\alpha}(Y, Z)h_{\alpha}(X, W) + \mu_{\alpha} h_{\alpha}(Y, Z)g(X, W) \]
\[ - \mu_{\alpha} h_{\alpha}(X, Z)g(Y, W) \} , \]
and
\[ \tilde{R}(X, Y, Z, \xi_{\alpha}) = \varepsilon_{\alpha} \{ (\nabla_X h_{\alpha})(Y, Z) - (\nabla_Y h_{\alpha})(X, Z) \]
\[ + h_{\alpha}(\pi(Y)X - \pi(X)Y, Z) \} + \sum_{\beta=1}^{p} \tilde{g}(h_{\beta}(Y, Z)D_X \xi_{\beta} \]
\[ - h_{\beta}(X, Z)D_Y \xi_{\beta} , \xi_{\alpha} \] for \( X, Y, Z \in \chi(M) \).

From (2.14) and (3.2), we get
\[ \tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + \sum_{\alpha=1}^{p} \varepsilon_{\alpha} g(A_{\xi_{\alpha}}(X), X)g(A_{\xi_{\alpha}}(Y), Y) \]
\[ - g(A_{\xi_{\alpha}}(X), Y)^2 \} + \sum_{\alpha=1}^{p} \mu_{\alpha} g(A_{\xi_{\alpha}}(Y), Y)g(X, X) \]
\[ - \mu_{\alpha} g(A_{\xi_{\alpha}}(X), Y)g(X, Y) \} \]
for \( X, Y \in \chi(M) \). Therefore we have the following theorem:

**Theorem 6.** Let \( \mathcal{P} \) be a 2-dimensional non-degenerate subspace of \( T_xM \), and let \( \tilde{K}(\mathcal{P}) \) and \( K(\mathcal{P}) \) be the sectional curvatures of \( \mathcal{P} \) in \( \tilde{M} \) and \( M \) with respect to the semi-symmetric non-metric connections \( \tilde{\nabla} \) and \( \nabla \), respectively. If \( X \)
and $Y$ form an orthonormal base of $\mathcal{P}$, then
\[
\tilde{K}(\mathcal{P}) = K(\mathcal{P}) + \frac{1}{g(X, X)g(Y, Y)} \sum_{\alpha=1}^{p} \{ g(A_{\xi\alpha}(X), X)g(A_{\xi\alpha}(Y), Y) \\
- g(A_{\xi\alpha}(X), Y)^2 + \mu_{\alpha} g(A_{\xi\alpha}(Y), Y)g(X, X) \}.
\]

As an immediate consequences of Theorem 6 we obtain:

**Corollary 7.** If $\tilde{M}$ is a 3-dimensional flat Lorentz manifold and $M$ is a spacelike or timelike surface in $\tilde{M}$, then there exists a semi-symmetric non-metric connection $\tilde{\nabla}$ on $M$ for which $\det \tilde{A}_{\xi}$ is an intrinsic invariant of $M$, and when $\tilde{Q}$ is tangent to $M$, $\det \tilde{A}_{\xi}$ is equal to $\det \circ \tilde{A}_{\xi}$ which is the Gauss curvature of $M$.

### 4. The equation of Ricci with respect to a semi-symmetric non-metric connection

Let $\xi$ be a normal vector field on $M$. We get
\[
\tilde{R}(X, Y)\xi = \tilde{\nabla}_X \tilde{\nabla}_Y \xi - \tilde{\nabla}_Y \tilde{\nabla}_X \xi - \tilde{\nabla}_{[X, Y]} \xi,
\]
where $X, Y \in \chi(M)$. Using (2.12) and (4.1), we have
\[
\tilde{R}(X, Y)\xi = R^N(X, Y)\xi + \sum_{\alpha=1}^{p} \{ h_{\alpha}(A_{\xi\alpha}X, Y) - h_{\alpha}(A_{\xi\alpha}Y, X) \} \xi_{\alpha} \\
+ A_{D_{\xi}X}Y - A_{D_{\xi}Y}X - Tor_{A_{\xi}}(X, Y),
\]
where $R^N$ is the curvature tensor of the normal connection. Using (2.13) and (4.2), we obtain
\[
\tilde{R}(X, Y, \xi, \eta) = R^N(X, Y, \xi, \eta) - g([A_{\xi}, A_{\eta}]X, Y),
\]
where $\eta$ is a normal vector field on $M$. Equation (4.3) is called the equation of Ricci with respect to the semi-symmetric non-metric connection $\tilde{\nabla}$.

A relation between the curvature tensor of the semi-symmetric non-metric connection $\tilde{\nabla}$ and the Levi-Civita connection $\circ \tilde{\nabla}$ is given by
\[
\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = \circ \tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} + \tilde{\alpha}(\tilde{X}, \tilde{Z})\tilde{Y} - \tilde{\alpha}(\tilde{Y}, \tilde{Z})\tilde{X},
\]
where $\tilde{\alpha}$ is a tensor of type $(0, 2)$ defined by
\[
\tilde{\alpha}(\tilde{X}, \tilde{Y}) = \circ \tilde{(\nabla}_{\tilde{X})\tilde{\pi}}\tilde{Y} - \tilde{\pi}(\tilde{X})\circ \tilde{\pi}(\tilde{Y}) \\
= (\nabla_{\tilde{X}}\tilde{\pi})\tilde{Y}
\]
for any vector fields $\tilde{X}, \tilde{Y}, \tilde{Z}$ on $\tilde{M}$.
Presently, we consider the semi-Riemannian manifold $\tilde{M}$ with constant curvature $k$. Then we have (see [7])

\begin{equation}
\tilde{R}(\tilde{X}, \tilde{Y}) \tilde{Z} = k \{ \tilde{g}(\tilde{Y}, \tilde{Z}) \tilde{X} - \tilde{g}(\tilde{X}, \tilde{Z}) \tilde{Y} \}. \tag{4.6}
\end{equation}

From (4.4), (4.5) and (4.6), we have

\[ \tilde{R}(X, Y) \xi = (\tilde{\nabla}_X \tilde{\pi}) \xi Y - (\tilde{\nabla}_Y \tilde{\pi}) \xi X \]

for any vector fields $X$, $Y$ and a normal vector field $\xi$ on $M$. Thus, $\tilde{R}(X, Y) \xi$ is tangent to $M$ and hence equation (4.3) reduces to

\[ R^N(X, Y, \xi, \eta) = g([A\xi, A\eta])X, Y). \]

The normal connection $D$ in the normal bundle $TM^\perp$ is said to be flat if

\[ R^N(X, Y) = D_X D_Y - D_Y D_X - D_{[X, Y]} \]

vanishes identically on $M$.

Hence we have:

**Corollary 8.** Let $M$ be a semi-Riemannian submanifold of a semi-Riemannian manifold $\tilde{M}$ with constant curvature admitting a semi-symmetric non-metric connection. Then the normal connection $D$ in the normal bundle $TM^\perp$ is flat if and only if all the second fundamental tensors with respect to the semi-symmetric non-metric connection are simultaneously diagonalizable.

**Theorem 9.** The Ricci tensor of a semi-Riemannian submanifold $M$ with respect to the semi-symmetric non-metric connection is symmetric if and only if $\pi$ is closed.

**Proof.** The Ricci tensor of a semi-Riemannian submanifold $M$ with respect to the semi-symmetric non-metric connection is given by

\begin{equation}
Ric(X, Y) = \sum_{i=1}^{n} \varepsilon_i g(R(E_i, X)Y, E_i) \tag{4.7}
\end{equation}

for $\forall X, Y \in \chi(M)$. Then using (4.4) in (4.7), we obtain

\[ Ric(X, Y) = \tilde{Ric}(X, Y) - (n - 1)\alpha(X, Y), \]

where $\tilde{Ric}$ denotes the Ricci tensor of $M$ with respect to the Levi-Civita connection and

\[ \alpha(X, Y) = (\nabla_X \pi) Y. \]

From above it follows that

\[ Ric(X, Y) - Ric(Y, X) = (n - 1)(\alpha(Y, X) - \alpha(X, Y)) = 2(n - 1)d\pi(Y, X) \]

which completes the proof. \qed
Theorem 10. Let $M$ be a semi-Riemannian submanifold of a semi-Riemannian manifold $\tilde{M}$. If $\tilde{\text{Ric}}$ and $\text{Ric}$ are the Ricci tensor of $\tilde{M}$ and $M$ with respect to the semi-symmetric non-metric connection, respectively, then for $\forall X, Y \in \chi(M)$

$$\tilde{\text{Ric}}(X, Y) = \text{Ric}(X, Y) - \sum_{\alpha=1}^{p} \varepsilon_{\alpha} f_{\alpha} h_{\alpha}(X, Y) + h_{\alpha}(A_{\xi_{\alpha}} X, Y)$$

$$+ n \varepsilon_{\alpha} \mu_{\alpha} h_{\alpha}(X, Y) + \varepsilon_{\alpha} \tilde{g}(\tilde{R}(\xi_{\alpha}, X)Y, \xi_{\alpha}),$$

where if $\xi_{\alpha}$ is spacelike, $\varepsilon = +1$ or if $\xi_{\alpha}$ is timelike, $\varepsilon = -1$ and $f_{\alpha} = \sum_{i=1}^{n} \varepsilon_{i} h_{\alpha}(E_{i}, E_{i})$.

Proof. Let $\{E_{1}, \ldots, E_{\nu+1}, \ldots, E_{n}, \xi_{1}, \ldots, \xi_{p}\}$ be an orthonormal basis of $\chi(\tilde{M})$. Then the Ricci curvature of $\tilde{M}$ with respect to the semi-symmetric non-metric connection is given by

$$\tilde{\text{Ric}}(X, Y) = \sum_{i=1}^{n} \varepsilon_{i} \tilde{g}(\tilde{R}(E_{i}, X)Y, E_{i}) + \sum_{\alpha=1}^{p} \varepsilon_{\alpha} \tilde{g}(\tilde{R}(\xi_{\alpha}, X)Y, \xi_{\alpha})$$

for $\forall X, Y \in \chi(M)$. By taking account of (4.9), (3.2), (2.14) and considering the symmetry of shape operators we get (4.8). □

Theorem 11. Let $M$ be a semi-Riemannian submanifold of a semi-Riemannian manifold $\tilde{M}$. If $\tilde{\rho}$ and $\rho$ are the scalar curvatures of $\tilde{M}$ and $M$ with respect to the semi-symmetric non-metric connection, respectively, then

$$\tilde{\rho} = \rho - \sum_{\alpha=1}^{p} \varepsilon_{\alpha} f_{\alpha}^{2} + n \varepsilon_{\alpha} \mu_{\alpha} f_{\alpha} + f_{\alpha}^{*} + 2 \varepsilon_{\alpha} \tilde{\text{Ric}}(\xi_{\alpha}, \xi_{\alpha}),$$

where $f_{\alpha}^{*} = \sum_{i=1}^{n} \varepsilon_{i} h_{\alpha}(A_{\xi_{\alpha}} E_{i}, E_{i})$.

Proof. Assume that $\{E_{1}, \ldots, E_{\nu+1}, \ldots, E_{n}, \xi_{1}, \ldots, \xi_{p}\}$ is an orthonormal basis of $\chi(\tilde{M})$, then the scalar curvature of $\tilde{M}$ with respect to the semi-symmetric non-metric connection is

$$\tilde{\rho} = \sum_{i=1}^{n} \varepsilon_{i} \tilde{\text{Ric}}(E_{i}, E_{i}) + \sum_{\alpha=1}^{p} \varepsilon_{\alpha} \tilde{\text{Ric}}(\xi_{\alpha}, \xi_{\alpha}).$$

By virtue of (4.8), (4.11), we obtain (4.10). □

We now assume that the 1-form $\pi$ is closed. Then we can define the sectional curvature for a section with respect to the semi-symmetric non-metric connection (see [1]).

Suppose that the semi-symmetric non-metric connection $\tilde{\nabla}$ is of constant sectional curvature, then $\tilde{R}(X, Y)Z$ should be of the form

$$\tilde{R}(X, Y)Z = c\{\tilde{g}(Y, Z)X - \tilde{g}(X, Z)Y\}$$
c being a certain scalar. Thus $\tilde{M}$ is a semi-Riemannian manifold of constant curvature $c$ with respect to semi-symmetric non-metric connection and denote it by $\tilde{M}(c)$.

**Theorem 12.** Let $M$ be a semi-Riemannian submanifold of a semi-Riemannian space form $\tilde{M}(c)$ with a semi-symmetric non-metric connection. Then we have

\[
\text{Ric}(X, Y) = c(n - 1)g(X, Y) + \sum_{\alpha=1}^{p} \{ \varepsilon_{\alpha} f_{\alpha} h_{\alpha}(X, Y) 
- h_{\alpha}(A_{\xi_{\alpha}} X, Y) - \varepsilon_{\alpha} n \mu_{\alpha} h_{\alpha}(X, Y) \}
\]

for $\forall X, Y \in \chi(M)$, where $\varepsilon_{i} = g(E_{i}, E_{i})$, $\varepsilon_{i} = 1$, if $E_{i}$ is spacelike or $\varepsilon_{i} = -1$, if $E_{i}$ is timelike, and $f_{\alpha} = \sum_{i=1}^{n} \varepsilon_{i} h_{\alpha}(E_{i}, E_{i})$.

**Proof.** Taking into account of (4.8) and (4.12), we have (4.13). $\square$

From (4.13), the following corollary can be stated as:

**Corollary 13.** Let $M$ be a semi-Riemannian submanifold of a semi-Riemannian space form $\tilde{M}(c)$ with a semi-symmetric non-metric connection. Then the Ricci tensor of $M$ is symmetric.

**Corollary 14.** Let $M$ be a semi-Riemannian submanifold of a semi-Riemannian space form $\tilde{M}(c)$ with a semi-symmetric non-metric connection. Then the Ricci tensor of $M$ is not parallel.

5. Projective curvature tensor of a semi-Riemannian submanifold with a semi-symmetric non-metric connection

We denote by

\[
\tilde{P}(\tilde{X}, \tilde{Y}) \tilde{Z} = \tilde{R}(\tilde{X}, \tilde{Y}) \tilde{Z} - \frac{1}{n + p - 1} (\tilde{\text{Ric}}(\tilde{Y}, \tilde{Z}) \tilde{X} - \tilde{\text{Ric}}(\tilde{X}, \tilde{Z}) \tilde{Y}),
\]

the Weyl projective curvature tensor of an $(n+p)$-dimensional semi-Riemannian manifold $\tilde{M}$ with respect to the Levi-Civita connection $\tilde{\nabla}$ for $\tilde{X}, \tilde{Y}, \tilde{Z} \in \chi(\tilde{M})$, where $\tilde{\text{Ric}}$ is Ricci tensor of $\tilde{M}$ with respect to the Levi-Civita connection $\tilde{\nabla}$ (see [3] and [10]).

Analogous to this definition, the Weyl projective curvature tensor of $\tilde{M}$ with respect to the semi-symmetric non-metric connection can be defined as

\[
\tilde{P}(\tilde{X}, \tilde{Y}) \tilde{Z} = \tilde{R}(\tilde{X}, \tilde{Y}) \tilde{Z} - \frac{1}{n + p - 1} (\tilde{\text{Ric}}(\tilde{Y}, \tilde{Z}) \tilde{X} - \tilde{\text{Ric}}(\tilde{X}, \tilde{Z}) \tilde{Y})
\]

for any $\tilde{X}, \tilde{Y}, \tilde{Z} \in \chi(\tilde{M})$, where $\tilde{\text{Ric}}$ is the Ricci tensor $\tilde{M}$ with respect to the connection $\tilde{\nabla}$. Thus, from (5.1), the Weyl projective curvature tensors with
respect to the semi-symmetric non-metric connection $\tilde{\nabla}$ and induced connection $\nabla$, respectively, are given by

$$\tilde{P}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{U}) = \tilde{R}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{U}) - \frac{1}{n + p - 1} \left\{ \tilde{Ric}(\tilde{Y}, \tilde{Z})\tilde{g}(\tilde{X}, \tilde{U}) - \tilde{Ric}(\tilde{X}, \tilde{Z})\tilde{g}(\tilde{Y}, \tilde{U}) \right\}$$

(5.2)

and

$$P(X, Y, Z, U) = R(X, Y, Z, U) - \frac{1}{n - 1} \left\{ Ric(Y, Z)g(X, U) - Ric(X, Z)g(Y, U) \right\}$$

(5.3)

for $\forall X, Y, Z \in \chi(M)$, where

$$\tilde{P}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{U}) = \tilde{g}(\tilde{P}(\tilde{X}, \tilde{Y})\tilde{Z}, \tilde{U}),$$

$P(X, Y, Z, U) = g(P(X, Y)Z, U)$

and $Ric$ is the Ricci tensor of $M$ with respect to induced connection $\nabla$.

From (5.2), we obtain

$$\tilde{P}(\xi_\alpha, Y, Z, \xi_\alpha) = \tilde{R}(\xi_\alpha, Y, Z, \xi_\alpha) - \frac{\varepsilon_\alpha}{n + p - 1} \tilde{Ric}(Y, Z).$$

(5.4)

Applying (4.8) to (5.4), we have

$$Ric(Y, Z) = \frac{n + p - 2}{n + p - 1} \tilde{Ric}(Y, Z) - \sum_{\alpha=1}^{p} \varepsilon_\alpha \tilde{P}(\xi_\alpha, Y, Z, \xi_\alpha)$$

$$+ \phi_\alpha \varepsilon_\alpha h_\alpha(Y, Z) - n \mu_\alpha \varepsilon_\alpha h_\alpha(Y, Z) - h_\alpha(A_{\xi_\alpha}Y, Z).$$

(5.5)

Then, using (5.2), (5.5) and (3.2) into (5.3) we obtain

$$P(X, Y, Z, U)$$

$$= \tilde{P}(X, Y, Z, U) - \sum_{\alpha=1}^{p} \varepsilon_\alpha \{ h_\alpha(X, Z)h_\alpha(Y, U) - h_\alpha(Y, Z)h_\alpha(X, U)$$

$$+ \mu_\alpha h_\alpha(Y, Z)g(X, U) - \mu_\alpha h_\alpha(X, Z)g(Y, U) \}$$

$$+ \frac{1}{n - 1} \sum_{\alpha=1}^{p} \varepsilon_\alpha \{ \tilde{P}(\xi_\alpha, Y, Z, \xi_\alpha)g(X, U) - \tilde{P}(\xi_\alpha, X, Z, \xi_\alpha)g(Y, U) \}$$

$$+ \frac{p - 1}{(n - 1)(n + p - 1)} (\tilde{Ric}(X, Z) - \tilde{Ric}(Y, Z))$$

$$+ \frac{1}{n - 1} g(Y, U) \left\{ \sum_{\alpha=1}^{p} \varepsilon_\alpha f_\alpha h_\alpha(X, Z) - n \varepsilon_\alpha \mu_\alpha h_\alpha(X, Z) - h_\alpha(A_{\xi_\alpha}X, Z) \right\}$$

$$- \frac{1}{n - 1} g(X, U) \left\{ \sum_{\alpha=1}^{p} \varepsilon_\alpha f_\alpha h_\alpha(Y, Z) - n \varepsilon_\alpha \mu_\alpha h_\alpha(Y, Z) - h_\alpha(A_{\xi_\alpha}Y, Z) \right\}.$$

(5.6)

From (5.6), we have the following theorem:
Theorem 15. A totally umbilical semi-Riemannian submanifold in a projectively flat semi-Riemannian manifold with a semi-symmetric non-metric connection is projectively flat.

References


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