ON A CLASS OF THREE-DIMENSIONAL TRANS-SASAKIAN MANIFOLDS

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ABSTRACT. The object of the present paper is to study 3-dimensional trans-Sasakian manifolds with conservative curvature tensor and also 3-dimensional conformally flat trans-Sasakian manifolds. Next we consider compact connected $\eta$-Einstein 3-dimensional trans-Sasakian manifolds. Finally, an example of a 3-dimensional trans-Sasakian manifold is given, which verifies our results.

1. Introduction

Trans-Sasakian manifolds arose in a natural way from the classification of almost contact metric structures by D. Chinea and C. Gonzales [6] and they appear as a natural generalization of both Sasakian and Kenmotsu manifolds. Again in the Gray-Hervella classification of almost Hermite manifolds [10], there appears a class $W_4$ of Hermitian manifolds which are closely related to locally conformally Kähler manifolds. An almost contact metric structure on a manifold $M$ is called a trans-Sasakian structure [17] if the product manifold $M \times \mathbb{R}$ belongs to the class $W_4$. The class $C_6 \oplus C_5$ ([15], [16]) coincides with the class of trans-Sasakian structures of type $(\alpha,\beta)$. In [16], the local nature of the two subclasses $C_5$ and $C_6$ of trans-Sasakian structures is characterized completely. In [7], some curvature identities and sectional curvatures for $C_5$, $C_6$ and trans-Sasakian manifolds are obtained. It is known that [12] trans-Sasakian structures of type $(0,0)$, $(0,\beta)$, and $(\alpha,0)$ are cosymplectic, $\beta$-Kenmotsu and $\alpha$-Sasakian, respectively, where $\alpha, \beta \in \mathbb{R}$.

The local structure of trans-Sasakian manifolds of dimension $n \geq 5$ has been completely characterized by J. C. Marrero [15]. He proved that a trans-Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or $\alpha$-Sasakian or $\beta$-Kenmotsu manifold. Three-dimensional trans-Sasakian manifolds have been studied by De and Tripathi [9], De and Sarkar [8], Kim, Prasad and Tripathi [14], Bagewadi and Venkatesha [1], Shukla and Singh [18] and many others. In

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[13] Jun and Kim studied 3-dimensional almost contact metric manifolds. The curvature tensor $R$ in a Riemannian manifold is said to be conservative [11], that is, $\text{div} \, R = 0$ if and only if $(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z)$ where $S$ is the Ricci tensor of the manifold. Moreover, Boyer and Galicki [5] proved that if $M$ is a compact $\eta$-Einstein K-contact manifold with Ricci tensor $S = ag + b\eta \otimes \eta$, and if $a \geq -2$, then $M$ is Sasakian. Motivated by these works in this paper we study some curvature conditions in a 3-dimensional trans-Sasakian manifold.

The paper is organized as follows. In Section 2, some preliminary results are recalled. After preliminaries in Section 3, we give an example of a 3-dimensional trans-Sasakian manifold of type $(\alpha, \beta)$. Then we study 3-dimensional connected trans-Sasakian manifold with conservative curvature tensor. In the next section, we study 3-dimensional conformally flat connected trans-Sasakian manifold. In Section 6, we prove that if a compact connected 3-dimensional trans-Sasakian manifold is $\eta$-Einstein with constant coefficients, then it is either $\alpha$-Sasakian or $\beta$-Kenmotsu. Finally, we construct an example of a 3-dimensional trans-Sasakian manifold with constant function $\alpha, \beta$ on $M$.

2. Preliminaries

Let $M$ be a connected almost contact metric manifold with an almost contact metric structure $(\phi, \xi, \eta, g)$, that is, $\phi$ is an $(1,1)$ tensor field, $\xi$ is a vector field, $\eta$ is a 1-form and $g$ is a compatible Riemannian metric such that

(2.1) \[ \phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta\phi = 0, \]

(2.2) \[ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \]

(2.3) \[ g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X) \]

for all $X$ and $Y$ tangent to $M$ ([2], [3]).

The fundamental 2-form $\Phi$ of the manifold is defined by

(2.4) \[ \Phi(X, Y) = g(X, \phi Y) \]

for all $X$ and $Y$ tangent to $M$.

An almost contact metric structure $(\phi, \xi, \eta, g)$ on a connected manifold $M$ is called trans-Sasakian structure [17] if $(M \times \mathbb{R}, J, G)$ belongs to the class $W_4$ [10], where $J$ is the almost complex structure on $M \times \mathbb{R}$ defined by

\[ J \left( X, f \frac{d}{df} \right) = \left( \phi X - f\xi, \eta(X) \frac{d}{df} \right) \]

for any vector fields $X$ on $M$, $f$ is a smooth function on $M \times \mathbb{R}$ and $G$ is the product metric on $M \times \mathbb{R}$. This may be expressed by the condition [4]

(2.5) \[ (\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \]

for smooth functions $\alpha$ and $\beta$ on $M$. Hence we say that the trans-Sasakian structure is of type $(\alpha, \beta)$. From (2.5) it follows that

(2.6) \[ \nabla_X \xi = -\alpha(\phi X) + \beta(X - \eta(X)\xi), \]
(2.7) \((\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y)\).

An explicit example of a 3-dimensional proper trans-Sasakian manifold is constructed in [15]. In [9], Ricci tensor and curvature tensor for 3-dimensional trans-Sasakian manifolds are studied and their explicit formulae are given. From [9] we know that for a 3-dimensional trans-Sasakian manifold

\begin{equation}
2\alpha\beta + \xi \alpha = 0,
\end{equation}

\begin{equation}
S(X, \xi) = (2(\alpha^2 - \beta^2) - \xi \beta)\eta(X) - X\beta - (\phi X)\alpha,
\end{equation}

\begin{equation}
S(X, Y) = \left(\frac{r}{2} + \xi \beta - (\alpha^2 - \beta^2)\right) g(X, Y)
- \left(\frac{r}{2} + \xi \beta - 3(\alpha^2 - \beta^2)\right) \eta(X)\eta(Y)
- (Y \beta + (\phi Y)\alpha)\eta(X) - (X \beta + (\phi X)\alpha)\eta(Y),
\end{equation}

\begin{equation}
R(X, Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y)
- \eta(Y)(X\beta)\xi + \phi(X)\alpha\xi
+ \eta(X)(Y\beta)\xi + \phi(Y)\alpha\xi
- (Y \beta)X + (X \beta)Y - (\phi(Y)\alpha)X + (\phi(X)\alpha)Y,
\end{equation}

and

\begin{equation}
R(X, Y)Z = \left(\frac{r}{2} + 2\xi \beta - 2(\alpha^2 - \beta^2)\right) (g(Y, Z)X - g(X, Z)Y)
- g(Y, Z) \left[\left(\frac{r}{2} + \xi \beta - 3(\alpha^2 - \beta^2)\right) \eta(X)\xi
- \eta(X)(\phi \text{grad} \alpha - \text{grad} \beta) + (X \beta + (\phi X)\alpha)\xi\right]
+ g(X, Z) \left[\left(\frac{r}{2} + \xi \beta - 3(\alpha^2 - \beta^2)\right) \eta(Y)\xi
- \eta(Y)(\phi \text{grad} \beta + Y \beta + (\phi Y)\alpha)\xi\right]
- \left[(Z \beta + (\phi Z)\alpha)\eta(X) + (X \beta + (\phi X)\alpha)\eta(Z)
+ \left(\frac{r}{2} + \xi \beta - 3(\alpha^2 - \beta^2)\right) \eta(Y)\eta(Z)\right]X
+ \left[(Z \beta + (\phi Z)\alpha)\eta(X) + (X \beta + (\phi X)\alpha)\eta(Z)
+ \left(\frac{r}{2} + \xi \beta - 3(\alpha^2 - \beta^2)\right) \eta(X)\eta(Z)\right]Y,
\end{equation}

where \(S\) is the Ricci tensor of type \((0, 2)\) and \(R\) is the curvature tensor of type \((1, 3)\) and \(r\) is the scalar curvature of the manifold \(M\).
3. Example of a 3-dimensional trans-Sasakian manifold of type $(\alpha, \beta)$

We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where $(x, y, z)$ are standard co-ordinate of $\mathbb{R}^3$.

The vector fields

$$e_1 = z \left( \frac{\partial}{\partial x} + y \frac{\partial}{\partial z} \right), \quad e_2 = z \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of $M$.

Let $g$ be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0, \quad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$ 

Let $\eta$ be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$. Let $\phi$ be the $(1,1)$ tensor field defined by

$$\phi(e_1) = e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = 0.$$ 

Then using the linearity of $\phi$ and $g$, we have

$$\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3, \quad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$, the set of all smooth vector fields on $M$.

Then for $e_3 = \xi$, the structure $(\phi, \xi, \eta, g)$ defines an almost contact metric structure on $M$.

Let $\nabla$ be the Levi-Civita connection with respect to metric $g$ and $R$ be the curvature tensor of $g$. Then we have

$$[e_1, e_2] = ye_2 - z^2 e_3, \quad [e_1, e_3] = -\frac{1}{z} e_1 \text{ and } [e_2, e_3] = -\frac{1}{z} e_2.$$ 

Taking $e_3 = \xi$ and using Koszul formula for the Riemannian metric $g$, we can easily calculate

$$\nabla_{e_3} e_3 = -\frac{1}{z} e_1 + \frac{1}{z^2} e_2, \quad \nabla_{e_1} e_2 = -\frac{1}{2} z^2 e_3, \quad \nabla_{e_2} e_1 = \frac{1}{z} e_3, \quad \nabla_{e_2} e_2 = ye_1 + \frac{1}{z} e_3, \quad \nabla_{e_3} e_1 = \frac{1}{2} z^2 e_3 - ye_2.$$ 

$$\nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_2 = -\frac{1}{2} z^2 e_1, \quad \nabla_{e_3} e_1 = \frac{1}{2} z^2 e_2.$$ 

From the above it can be easily seen that $(\phi, \xi, \eta, g)$ is a trans-Sasakian structure on $M$. Consequently $M^3(\phi, \xi, \eta, g)$ is a trans-Sasakian manifold with $\alpha = -\frac{1}{2} z^2 \neq 0$ and $\beta = -\frac{1}{2} \neq 0$. 

4. 3-Dimensional connected trans-Sasakian manifolds with conservative curvature tensor

Let $M$ be a 3-dimensional connected trans-Sasakian manifold with conservative curvature tensor [11], that is, $\text{div} R = 0$. Then its Ricci tensor is given by $(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z)$. From this we obtain $r = \text{constant}$. We know that

$$
(\nabla_X S)(Y, Z) = \nabla_X S(Y, Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z).
$$

Using (2.10) we have

$$
(\nabla_X S)(Y, Z) = \left[ \frac{dr(X)}{2} + \nabla_X (\xi \beta) - 2\alpha\alpha(X) + 2\beta d\beta(X) \right] g(Y, Z)
$$

$$
+ \left( \frac{r}{2} + \xi \beta - (\alpha^2 - \beta^2) \right) \nabla_X g(Y, Z)
$$

$$
- \left( \frac{dr(X)}{2} + \nabla_X (\xi \beta) - 6\alpha\alpha(X) + 6\beta d\beta(X) \right) \eta(Y)\eta(Z)
$$

$$
- \left( \frac{r}{2} + \xi \beta - 3(\alpha^2 - \beta^2) \right) [\eta(\nabla_X \eta(Y)\eta(Z) + \eta(Y)\nabla_X \eta(Z)]
$$

$$
- (\nabla_X (Z\beta + (\phi Z)\alpha))\eta(Y) - (Z\beta + (\phi Z)\alpha)\nabla_X \eta(Y)
$$

$$
- (\nabla_X (Y\beta + (\phi Y)\alpha))\eta(Z) - (Y\beta + (\phi Y)\alpha)\nabla_X \eta(Z)
$$

$$
- \left( \frac{r}{2} + \xi \beta - (\alpha^2 - \beta^2) \right) g(\nabla_X Y, Z)
$$

$$
+ \left( \frac{r}{2} + \xi \beta - 3(\alpha^2 - \beta^2) \right) \eta(\nabla_X Y)\eta(Z)
$$

$$
+ (Z\beta + (\phi Z)\alpha)\eta(\nabla_X Y) + ((\nabla_X Y)\beta + (\phi(\nabla_X Y))\alpha)\eta(Z)
$$

$$
- \left( \frac{r}{2} + \xi \beta - (\alpha^2 - \beta^2) \right) g(Y, \nabla_X Z)
$$

$$
+ \left( \frac{r}{2} + \xi \beta - 3(\alpha^2 - \beta^2) \right) \eta(Y)\eta(\nabla_X Z)
$$

$$
+ ((\nabla_X Z)\beta + (\phi(\nabla_X Z))\alpha)\eta(Y) + (Y\beta + (\phi Y)\alpha)\eta(\nabla_X Z) r.
$$

The above relation can be written as

$$
(\nabla_X S)(Y, Z) = \left[ \frac{dr(X)}{2} + \nabla_X (\xi \beta) - 2\alpha\alpha(X) + 2\beta d\beta(X) \right] g(Y, Z)
$$

$$
- \left( \frac{dr(X)}{2} + \nabla_X (\xi \beta) - 6\alpha\alpha(X) + 6\beta d\beta(X) \right) \eta(Y)\eta(Z)
$$

$$
- \left( \frac{r}{2} + \xi \beta - 3(\alpha^2 - \beta^2) \right) [\eta(\nabla_X \eta(Y)\eta(Z) + \eta(Y)\nabla_X \eta(Z)]
$$
\[-(\nabla_X (Z\beta + (\phi Z)\alpha)\eta(Y) - (Z\beta + (\phi Z)\alpha)(\nabla_X \eta)(Y)\]
\[-(\nabla_X (Y\beta + (\phi Y)\alpha)\eta(Z) - (Y\beta + (\phi Y)\alpha)(\nabla_X \eta)(Z)\]
\[+ ((\phi(\nabla_X Y))\alpha)\eta(Z) + ((\phi(\nabla_X Z))\alpha)\eta(Y).\]

Now from (4.3) we have
\[
(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = \frac{dr(X)}{2} + \nabla_X (\xi\beta) - 2\alpha d\alpha(X) + 2\beta d\beta(X) \]
\[\quad g(Y, Z) - \frac{dr(Y)}{2} + \nabla_Y (\xi\beta) - 2\alpha d\alpha(Y) + 2\beta d\beta(Y) \]
\[\quad g(X, Z) - \frac{dr(X)}{2} + \nabla_X (\xi\beta) - 6\alpha d\alpha(X) + 6\beta d\beta(X) \]
\[\quad \eta(Y)\eta(Z) + \frac{dr(Y)}{2} + \nabla_Y (\xi\beta) - 6\alpha d\alpha(Y) + 6\beta d\beta(Y) \]
\[\quad \eta(X)\eta(Z).\]

(4.4)
\[-\left(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\right) \right\}
\[-(\nabla_X \eta)(Y)\eta(Z) + \eta(Y)(\nabla_X \eta)(Z)\]
\[-(\nabla_Y \eta)(X)\eta(Z) - \eta(X)(\nabla_Y \eta)(Z)\]
\[-(\nabla_X (Z\beta + (\phi Z)\alpha))\eta(Y) + (\nabla_Y (Z\beta + (\phi Z)\alpha))\eta(X)\]
\[-(\nabla_X (Y\beta + (\phi Y)\alpha))\eta(Z) + (\nabla_Y (X\beta + (\phi X)\alpha))\eta(Z)\]
\[-(Z\beta + (\phi Z)\alpha)(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X)\]
\[-(Y\beta + (\phi Y)\alpha)(\nabla_X \eta)(Z)\]
\[+ (X\beta + (\phi X)\alpha)(\nabla_Y \eta)(Z)\]
\[+ ((\phi(\nabla_X Y))\alpha)\eta(Z) - ((\phi(\nabla_Y X))\alpha)\eta(Z)\]
\[+ ((\phi(\nabla_X Z))\alpha)\eta(Y) - ((\phi(\nabla_Y Z))\alpha)\eta(X).\]

Suppose div\( R = 0 \) and \( \alpha, \beta \) are constants. Then using (2.7) in (4.4) and using \( r = \) constant, we obtain
\[
\left(\frac{r}{2} - 3(\alpha^2 - \beta^2)\right) [-\alpha g(\phi X, Y)\eta(Z)\]
\[-\alpha g(\phi X, Z)\eta(Y) + \alpha g(\phi Y, X)\eta(Z)\]
\[+ \alpha g(\phi Y, Z)\eta(X) + \beta g(\phi X, \phi Z)\eta(Y) - \beta g(\phi Y, \phi Z)\eta(X)] = 0.\]

(4.5)

Let \( \{e_0, e_1, e_2\} \) be a local \( \phi \)-basis, that is, an orthonormal frame such that \( e_0 = \xi \) and \( e_2 = \phi e_1 \). In (4.5) putting \( X = e_1, Y = e_2 \), we get
\[
2\alpha \left[\left(\frac{r}{2} - 3(\alpha^2 - \beta^2)\right) \right] \eta(Z) = 0.\]

(4.6)

This implies either \( \alpha = 0 \) or \( r = 6(\alpha^2 - \beta^2) \), or both holds. If \( r = 6(\alpha^2 - \beta^2) \), then from (2.10) it follows that
\[
S(X, Y) = 2(\alpha^2 - \beta^2)g(X, Y).\]

(4.7)
This implies that the manifold is an Einstein manifold. This leads to the following theorem:

**Theorem 4.1.** If a 3-dimensional connected trans-Sasakian manifold is of conservative curvature tensor, then the manifold is either a $\beta$-Kenmotsu manifold or an Einstein manifold or both holds provided $\alpha, \beta = \text{constant}$.

If the manifold is an Einstein manifold, then the manifold is of conservative curvature tensor. Hence we obtain the following:

**Corollary 1.** A 3-dimensional connected trans-Sasakian manifold which is not a $\beta$-Kenmotsu manifold is of conservative curvature tensor if and only if the manifold is an Einstein manifold provided $\alpha, \beta = \text{constant}$.

5. 3-Dimensional conformally flat connected trans-Sasakian manifolds

Let $M$ be a 3-dimensional conformally flat connected trans-Sasakian manifold. At first we prove the following:

**Lemma 5.1.** Let $M$ be a 3-dimensional connected trans-Sasakian manifold with $\alpha, \beta = \text{constant}$. If there exist functions $L$ and $N$ on $M$ such that

$$\nabla_X Q(Y) - \nabla_Y Q(X) = LX + NY, \; X, Y \in \chi(M),$$

then either $\alpha = 0$ or

$$QX = 2(\alpha^2 - \beta^2)X.$$  

**Proof.** We have from (2.10),

$$QX = aX + b\eta(X)\xi,$$

where $a = (\frac{\tau}{3} - (\alpha^2 - \beta^2))$ and $b = -((\frac{\tau}{3} - 3(\alpha^2 - \beta^2))$ and thus using (5.3) we have

$$QX = 2(\alpha^2 - \beta^2)X.$$

Replacing $X$ by $\phi X$ and $Y$ by $\phi Y$ in (5.3) we get

$$Q\phi X = (\phi X\alpha)\phi Y - (\phi Y\alpha)\phi X - 2abg(\phi^2 X, \phi Y)\xi.$$  

From (5.1) and (5.5), we obtain

$$L + (\phi Y)a)\phi X + (N - (\phi X)a)\phi Y = 2abg(\phi^2 X, \phi Y)\xi.$$  

Using (2.1) in (5.6) yields

$$2abg(X, \phi Y) = 0,$$

which implies either $\alpha = 0$ or $b = 0$. Thus from the definition of $\eta$-Einstein manifold, we get $QX = aX$ and hence $QX = 2(\alpha^2 - \beta^2)X$. □
It is classical that on a 3-dimensional conformally flat Riemannian manifold [19], we have

\[
(\nabla X Q)Y - (\nabla Y Q)X = \frac{1}{4}(dr(X)Y - dr(Y)X).
\]

Then by Lemma 5.1 we get either \( \alpha = 0 \) or \( QX = 2(\alpha^2 - \beta^2)X \). This leads to the following theorem:

**Theorem 5.1.** A 3-dimensional conformally flat connected trans-Sasakian manifold is either a \( \beta \)-Kenmotsu manifold or an Einstein manifold.

Since an Einstein manifold is of conservative curvature tensor, hence we obtain the following:

**Corollary 2.** In a 3-dimensional conformally flat connected trans-Sasakian manifold which is not a \( \beta \)-Kenmotsu manifold, the curvature tensor is conservative.

### 6. Compact connected \( \eta \)-Einstein manifolds

Let \( M \) be a 3-dimensional compact connected trans-Sasakian manifold. If the manifold is \( \eta \)-Einstein, then the Ricci tensor \( S \) of type \( (0, 2) \) of the manifold is given by

\[
S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),
\]

where \( a, b \) are smooth functions on \( M \). Here we suppose that \( a \) and \( b \) are constants. Putting \( Y = \xi \) in (6.1) and using (2.9), we get

\[
X\beta + (\phi X)\alpha + [(a + b) - 2(\alpha^2 - \beta^2) + \xi\beta]\eta(X) = 0.
\]

For \( X = \xi \), (6.2) yields

\[
\xi\beta = (\alpha^2 - \beta^2) - \frac{(a + b)}{2}.
\]

By virtue of (6.2) and (6.3), it follows that

\[
X\beta + (\phi X)\alpha + \left[\frac{(a + b)}{2} - \alpha^2 + \beta^2\right]\eta(X) = 0.
\]

The gradient of the function \( \beta \) is related to the exterior derivative \( d\beta \) by the formula

\[
d\beta(X) = g(\text{grad}\beta, X).
\]

Using (6.5) in (6.4) we obtain

\[
d\beta(X) + g(\text{grad}\alpha, \phi X) + \left[\frac{(a + b)}{2} - \alpha^2 + \beta^2\right]\eta(X) = 0.
\]
Differentiating (6.6) covariantly with respect to \( Y \) we get

\[
(\nabla_Y d\beta)(X) + g(\nabla_Y \text{grad} \alpha, \phi X) + g(\text{grad} \alpha, (\nabla_Y \phi)X)
\]

(6.7)

\[
+ Y(\beta^2 - \alpha^2)\eta(X) + \left[ \frac{(a+b)}{2} - \alpha^2 + \beta^2 \right] (\nabla_Y \eta)(X) = 0.
\]

Interchanging \( X \) and \( Y \) in (6.7), we get

\[
(\nabla_X d\beta)(Y) + g(\nabla_X \text{grad} \alpha, \phi Y) + g(\text{grad} \alpha, (\nabla_X \phi)Y)
\]

(6.8)

\[
+ X(\beta^2 - \alpha^2)\eta(Y) + \left[ \frac{(a+b)}{2} - \alpha^2 + \beta^2 \right] (\nabla_X \eta)(Y) = 0.
\]

Subtracting (6.7) from (6.8) we get

\[
g(\nabla_X \text{grad} \alpha, \phi Y) - g(\nabla_Y \text{grad} \alpha, \phi X) + [(\nabla_X \phi)Y - (\nabla_Y \phi)X] \alpha
\]

\[
+ [X(\beta^2 - \alpha^2)\eta(Y) - Y(\beta^2 - \alpha^2)\eta(X)]
\]

\[
+ \left[ \frac{(a+b)}{2} - \alpha^2 + \beta^2 \right] [(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X)] = 0.
\]

From (2.7) and (2.4) we get

\[
(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = 2\alpha \Phi(X,Y).
\]

Using (6.10) in (6.9) we have

\[
g(\nabla_X \text{grad} \alpha, \phi Y) - g(\nabla_Y \text{grad} \alpha, \phi X) + [(\nabla_X \phi)Y - (\nabla_Y \phi)X] \alpha
\]

\[
+ [X(\beta^2 - \alpha^2)\eta(Y) - Y(\beta^2 - \alpha^2)\eta(X)]
\]

\[
+ 2 \left[ \frac{(a+b)}{2} - \alpha^2 + \beta^2 \right] \Phi(X,Y) = 0.
\]

Let \( \{ e_0, e_1, e_2 \} \) be a local \( \phi \)-basis, that is, an orthonormal frame such that \( e_0 = \xi \) and \( e_2 = \phi e_1 \). In (2.5) putting \( X = e_1, Y = e_2 \), we get

\[
(\nabla_{e_1} \phi)e_2 = \alpha g(e_1, e_2)\xi - \eta(e_2)e_1 + \beta (g(\phi e_1, e_2)\xi - \eta(e_2)\phi e_1)
\]

\[
= \beta g(\phi e_1, e_2)\xi = \beta \xi.
\]

Similarly,

\[
(\nabla_{e_2} \phi)e_1 = -\beta \xi.
\]

Now,

\[
\Phi(e_1, e_2) = g(e_1, \phi e_2) = g(e_1, \phi^2 e_1) = -1.
\]

In (6.11) putting \( X = e_1 \) and \( Y = e_2 \) and using (6.12), (6.13) and (6.14) we obtain

\[
g(\nabla_{e_1} \text{grad} \alpha, e_1) + g(\nabla_{e_2} \text{grad} \alpha, e_2) = 2\beta \xi \alpha - 2\alpha \left[ \frac{(a+b)}{2} - \alpha^2 + \beta^2 \right].
\]

Also (2.8) can be written as

\[
g(\text{grad} \alpha, \xi) = -2\alpha \beta.
\]
Differentiating (6.16) covariantly with respect to $\xi$ we get
\begin{equation}
(6.17)
g(\nabla_\xi \text{grad} \alpha, \xi) + g(\text{grad} \alpha, \nabla_\xi \xi) = -2\beta(\xi \alpha) - 2\alpha(\xi \beta).
\end{equation}

In view of (6.3) we can write the above relation as
\begin{equation}
(6.18)
g(\nabla_\xi \text{grad} \alpha, \xi) = -2\beta(\xi \alpha) + 2\alpha \left[ \frac{(a + b)}{2} - \alpha^2 + \beta^2 \right].
\end{equation}

From (6.15) and (6.18), we get $\Delta \alpha = 0$, where $\Delta$ is the Laplacian defined by $\Delta \alpha = \sum_{i=0}^{2} g(\nabla_{e_i} \text{grad} \alpha, e_i)$.

Since $M$ is compact, we get $\alpha$ is constant.

Now let us consider the following two cases:

Case i): In this case we suppose that $\alpha$ is a non-zero constant. Then by (2.8), $\beta = 0$ everywhere on $M$.

Case ii): In this case let $\alpha = 0$. Then from (6.4) it follows
\begin{equation}
X \beta + \left[ \frac{(a + b)}{2} + \beta^2 \right] \eta(X) = 0,
\end{equation}
that is,
\begin{equation}
g(\text{grad} \beta, X) + \left[ \frac{(a + b)}{2} + \beta^2 \right] g(X, \xi) = 0.
\end{equation}

Therefore,
\begin{equation}
(6.19)\quad \text{grad} \beta + \left[ \frac{(a + b)}{2} + \beta^2 \right] \xi = 0.
\end{equation}

Differentiating (6.19) covariantly with respect to $X$ we have
\begin{equation}
\nabla_X \text{grad} \beta + (X \beta^2) \xi + \left[ \frac{(a + b)}{2} + \beta^2 \right] \nabla_X \xi = 0.
\end{equation}

Using (2.6) we get from above
\begin{equation}
\nabla_X \text{grad} \beta + (X \beta^2) \xi + \left[ \frac{(a + b)}{2} + \beta^2 \right] (-\alpha \phi X + \beta(X - \eta(X)) \xi) = 0.
\end{equation}

Now taking inner product with $X$, we have
\begin{equation}
(6.20)\quad g(\nabla_X \text{grad} \beta, X) = -g((X \beta^2) \xi, X) - \left[ \frac{(a + b)}{2} + \beta^2 \right] (g(-\alpha \phi X, X) + \beta g(X - \eta(X)) \xi, X)).
\end{equation}

Therefore putting $X = e_i$ and taking summation over $i$, $i = 0, 1, 2$, we get from above
\begin{equation}
(6.21)\quad \Delta \beta = -2\beta \left[ \frac{(a + b)}{2} + \beta^2 \right].
\end{equation}

For $\alpha = 0$, (6.3) yields $\xi \beta = -\left( \frac{(a + b)}{2} + \beta^2 \right)$, which in view of (6.21) gives $\Delta \beta = 0$. Hence $\beta$ = constant, $M$ being compact. This leads to the following:
Theorem 6.1. If a compact 3-dimensional trans-Sasakian manifold is an \( \eta \)-Einstein manifold with constant coefficients, then it is either \( \alpha \)-Sasakian or \( \beta \)-Kenmotsu.

7. Example of a 3-dimensional trans-Sasakian manifold

We consider the 3-dimensional manifold \( M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\} \), where \( (x, y, z) \) are standard co-ordinate of \( \mathbb{R}^3 \).

The vector fields
\[
e_1 = z \frac{\partial}{\partial x}, \quad e_2 = z \frac{\partial}{\partial y}, \quad e_3 = z \frac{\partial}{\partial z}
\]
are linearly independent at each point of \( M \).

Let \( g \) be the Riemannian metric defined by
\[
g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0,
g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1
\]
that is, the form of the metric becomes
\[
g = \frac{dx^2 + dy^2 + dz^2}{z^2}.
\]

Let \( \eta \) be the 1-form defined by \( \eta(Z) = g(Z, e_3) \) for any \( Z \in \chi(M) \). Let \( \phi \) be the \((1, 1)\) tensor field defined by
\[
\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0.
\]

Then using the linearity of \( \phi \) and \( g \), we have
\[
\eta(e_3) = 1,
\]
\[
\phi^2 Z = -Z + \eta(Z)e_3,
\]
\[
g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W)
\]
for any \( Z, W \in \chi(M) \).

Then for \( e_3 = \xi \), the structure \((\phi, \xi, \eta, g)\) defines an almost contact metric structure on \( M \).

Let \( \nabla \) be the Levi-Civita connection with respect to metric \( g \). Then we have
\[
[e_1, e_3] = e_1 e_3 - e_3 e_1
\]
\[
= z \frac{\partial}{\partial x} \left(z \frac{\partial}{\partial z}\right) - z \frac{\partial}{\partial z} \left(z \frac{\partial}{\partial x}\right)
\]
\[
= z^2 \frac{\partial^2}{\partial x \partial z} - z^2 \frac{\partial^2}{\partial z \partial x} - z \frac{\partial}{\partial x}
\]
\[
= -e_1.
\]

Similarly,
\[
[e_1, e_2] = 0 \quad \text{and} \quad [e_2, e_3] = -e_2.
\]
The Riemannian connection $\nabla$ of the metric $g$ is given by

$$2g(\nabla_X Y, Z) = X g(Y, Z) + Y g(Z, X) - Z g(X, Y)$$

$$- g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]),$$

which is known as Koszul's formula.

Using (7.1) we have

$$2g(\nabla e_1 e_3, e_1) = -2g(e_1, e_1)$$

$$= 2g(-e_1, e_1).$$

Again by (7.1)

$$2g(\nabla e_1 e_3, e_2) = 0 = 2g(-e_1, e_2)$$

and

$$2g(\nabla e_1 e_3, e_3) = 0 = 2g(-e_1, e_3).$$

From (7.2), (7.3) and (7.4) we obtain

$$2g(\nabla e_1 e_3, X) = 2g(-e_1, X)$$

for all $X \in \chi(M)$.

Thus

$$\nabla e_1 e_3 = -e_1.$$

Therefore, (7.1) further yields

$$\nabla e_1 e_3 = -e_1, \quad \nabla e_1 e_1 = 0, \quad \nabla e_1 e_3 = 0,$$

$$\nabla e_2 e_3 = -e_2, \quad \nabla e_2 e_2 = 0, \quad \nabla e_2 e_1 = 0,$$

$$\nabla e_3 e_3 = 0, \quad \nabla e_3 e_2 = 0, \quad \nabla e_3 e_1 = 0.$$

(7.5) tells us that the manifold satisfies (2.6) for $\alpha = 0$ and $\beta = -1$ and $\xi = e_3$.

Hence the manifold is a trans-Sasakian manifold of type $(0, -1)$.

It is known that

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

With the help of the above results and using (7.6) it can be easily verified that

$$R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_3 = -e_2, \quad R(e_1, e_3)e_3 = -e_1,$$

$$R(e_1, e_2)e_1 = -e_1, \quad R(e_2, e_3)e_1 = e_2, \quad R(e_1, e_3)e_1 = 0,$$

$$R(e_1, e_2)e_2 = -e_2, \quad R(e_2, e_3)e_2 = e_3, \quad R(e_1, e_3)e_2 = 0.$$

From the above expressions of the curvature tensor we obtain

$$S(e_1, e_1) = g(R(e_1, e_2)e_2, e_1) + g(R(e_1, e_3)e_3, e_1)$$

$$= -2.$$

Similarly we have

$$S(e_2, e_2) = S(e_3, e_3) = -2.$$

Therefore,

$$r = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) = -6.$$
We note that here $\alpha$, $\beta$ and $r$ are all constants. $\beta \neq 0$ implies that the manifold is a $\beta$-Kenmotsu manifold. From the expressions of the Ricci tensor it follows that the manifold is an Einstein manifold. Therefore Theorem 4.1 is verified.

References


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