R(g, g')-CONTINUITY ON GENERALIZED TOPOLOGICAL SPACES

YOUNG KEY KIM AND WON KEUN MIN

Abstract. We introduce the notion of R(g, g')-continuity on generalized topological spaces, which is a strong form of (g, g')-continuity. We investigate some properties and relationships among R(g, g')-continuity, (g, g')-continuity and some strong forms of (g, g')-continuity.

1. Introduction

Császár [1] introduced the notion of generalized topological spaces. He also introduced the notions of continuous functions and associated interior and closure operators on generalized topological spaces. Characterizations for the generalized continuous (= (g, g')-continuous) function were investigated in [1, 3]. In [5], we introduced and investigated the notions of super (g, g')-continuous functions and strongly θ(g, g')-continuous functions on generalized topological spaces. The purpose of this paper is to introduce the notion of R(g, g')-continuity on generalized topological spaces, which is a strong form of (g, g')-continuity. We investigate some properties and relationships among R(g, g')-continuity (g, g')-continuity and some strong forms of (g, g')-continuity.

2. Preliminaries

We recall some notions and notations defined in [1]. Let X be a nonempty set and g be a collection of subsets of X. Then g is called a generalized topology (simply GT) on X if and only if ∅ ∈ g and \( G_i \in g \) for \( i \in I \neq \emptyset \) implies \( G = \bigcup_{i \in I} G_i \in g \). We call the pair \((X, g)\) a generalized topological space on X. We denote \( M_g = \bigcup\{ A \subseteq X : A \in g \} \). A generalized topology g on X is called strong [2] if \( X \in g \). The elements of g are called g-open sets and the complements are called g-closed sets. The generalized-closure of a subset \( S \) of \( X \), denoted by \( c_g(S) \), is the intersection of generalized closed sets including \( S \).

Received March 16, 2011.
2010 Mathematics Subject Classification. 54A05.
Key words and phrases. (g, g')-continuous, super (g, g')-continuous, strongly θ(g, g')-continuous, R(g, g')-continuous, G-regular.

©2012 The Korean Mathematical Society

809
And the interior of $S$, denoted by $i_g(S)$, the union of generalized open sets included in $S$.

Let $g$ and $g'$ be generalized topologies on $X$ and $Y$, respectively. Then a function $f : (X, g) \to (Y, g')$ is said to be

(1) *(g, g')-continuous* [1] if $G \in g'$ implies that $f^{-1}(G) \in g$;

(2) *super (g, g')-continuous* [5] if for each $x \in X$ and each $g'$-open set $V$ containing $f(x)$, there exists a $g$-open set $U$ containing $x$ such that $f(i_g(c_g(U))) \subseteq V$;

(3) *strongly $\theta(g, g')$-continuous* [5] if for each $x \in X$ and each $g'$-open set $V$ of $f(x)$, there exists a $g$-open set $U$ of $x$ such that $f(c_g(U)) \subseteq V$.

3. $R(g, g')$-continuous functions

**Definition 3.1.** Let $(X, g)$ and $(Y, g')$ be generalized topological spaces. Then a function $f : X \to Y$ is said to be $R(g, g')$-continuous if for each $x \in X$ and each $g'$-open set $V$ containing $f(x)$, there is a $g$-open set $U$ containing $x$ such that $c_g(f(U)) \subseteq V$.

**Theorem 3.2.** Let $f : X \to Y$ be a $R(g, g')$-continuous function on GTS's $(X, g)$ and $(Y, g')$. Then if $f(M_g) \subseteq M_{g'}$, then $f(c_g(U)) \subseteq c_{g'}(f(U))$ for every $g$-open set $U \subseteq X$.

**Proof.** Let $U$ be a $g$-open set in $X$. For each $x \in c_g(U)$, let $V$ be any $g'$-open set containing $f(x)$. Since $f$ is $R(g, g')$-continuous, there exists a $g$-open set $G$ containing $x$ such that $c_g(f(G)) \cap M_{g'} \subseteq V$. Furthermore, since $x \in c_g(U)$ and a $g$-open set $G$ contains $x$, $U \cap G \neq \emptyset$. From $f(M_g) \subseteq M_{g'}$, it follows

\[
\emptyset \neq f(U \cap G) \subseteq f(U) \cap f(G) \subseteq f(U) \cap c_{g'}(f(G)) = (f(U) \cap M_{g'}) \cap c_{g'}(f(G)) \subseteq f(U) \cap V.
\]

So $f(U) \cap V \neq \emptyset$ and $f(x) \in c_{g'}(f(U))$. This implies $f(c_g(U)) \subseteq c_{g'}(f(U))$. □

**Theorem 3.3.** Let $f : (X, g) \to (Y, g')$ be a function on GTS’s $(X, g)$ and $(Y, g')$. If $f(M_g) \subseteq M_{g'}$, then the following are equivalent:

1. $f$ is $R(g, g')$-continuous.

2. For each point $x \in X$ and a $g'$-open set $V$ containing $f(x)$, there is a $g$-open set $U$ containing $x$ such that $c_g(f(c_g(U))) \cap M_{g'} \subseteq V$.

3. For each point $x \in X$ and a $g'$-closed set $F$ with $f(x) \notin F$, there is a $g$-open set $U$ containing $x$ and a $g'$-open set $V$ such that $F \cap M_{g'} \subseteq V$ and $f(c_g(U)) \cap V = \emptyset$.

4. For each point $x \in X$ and a $g'$-closed set $F$ with $f(x) \notin F$, there is a $g$-open set $U$ containing $x$ and a $g'$-open set $V$ such that $F \cap M_{g'} \subseteq V$ and $f(U) \cap V = \emptyset$.

**Proof.** (1) $\Rightarrow$ (2) For $x \in X$, let $V$ be a $g'$-open set containing $f(x)$. Then there is a $g$-open set $U$ containing $x$ such that $c_g(f(U)) \cap M_{g'} \subseteq V$. By
Theorem 3.2, we have \( f(c_g(U)) \subseteq c_{g'}(f(U)) \). It implies \( c_{g'}(f(c_g(U))) \cap M_{g'} \subseteq c_{g'}(f(U)) \cap M_{g'} \subseteq V \).

(2) \( \Rightarrow \) (3) For \( x \in X \), let \( F \) be a \( g' \)-closed set with \( f(x) \notin F \). Since \( f(x) \in Y - F \) and \( Y - F \) is \( g' \)-open, by (2), there is a \( g \)-open set \( U \) containing \( x \) such that \( c_{g'}(f(c_g(U))) \cap M_{g'} \subseteq Y - F \). Set \( V = Y - (c_{g'}(f(c_g(U)))) \). Then \( V \) is a \( g' \)-open set such that \( F \cap M_{g'} \subseteq V \) and \( f(c_g(U)) \cap V = \emptyset \).

(3) \( \Rightarrow \) (4) It is obvious.

(4) \( \Rightarrow \) (1) Let \( x \in X \) and \( V \) a \( g' \)-open set containing \( f(x) \). Then \( Y - V \) is a \( g' \)-closed set and \( f(x) \notin Y - V \). By (4), there is a \( g \)-open set \( U \) containing \( x \) and a \( g' \)-open set \( W \) such that \( (Y - V) \cap M_{g'} \subseteq W \) and \( f(U) \cap W = \emptyset \). So \( c_{g'}(f(U)) \cap M_{g'} \subseteq c_{g'}(Y - W) \cap M_{g'} = (Y - W) \cap M_{g'} \subseteq V \), and hence \( f \) is \( R(g, g') \)-continuous.

\[ \square \]

Corollary 3.4. Let \( f : (X, g) \to (Y, g') \) be a function on GTS’s \((X, g)\) and \((Y, g')\). If \( Y \) is strong, then the following are equivalent:

1. \( f \) is \( R(g, g') \)-continuous.
2. For each point \( x \in X \) and a \( g' \)-open set \( V \) containing \( f(x) \), there is a \( g \)-open set \( U \) containing \( x \) such that \( c_{g'}(f(c_g(U))) \subseteq V \).
3. For each point \( x \in X \) and a \( g' \)-closed set \( F \) with \( f(x) \notin F \), there is a \( g \)-open set \( U \) containing \( x \) and a \( g' \)-open set \( V \) such that \( F \subseteq V \) and \( f(c_g(U)) \cap V = \emptyset \).
4. For each point \( x \in X \) and a \( g' \)-closed set \( F \) with \( f(x) \notin F \), there is a \( g \)-open set \( U \) containing \( x \) and a \( g' \)-open set \( V \) such that \( F \subseteq V \) and \( f(U) \cap V = \emptyset \).

Theorem 3.5. Let \( f : X \to Y \) be a function on GTS’s \((X, g)\) and \((Y, g')\). Then if \( f \) is \( R(g, g') \)-continuous and \( Y \) is strong, then it is strongly \( \theta(g, g') \)-continuous.

Proof. It follows from Corollary 3.4(2).

Remark 3.6. The converse of Theorem 3.5 is not true in general as shown by the next example.

Example 3.7. Let \( X = \{a, b, c\} \) and \( Y = \{1, 2, 3\} \). Consider generalized topologies \( g = (\emptyset, \{a\}) \) on \( X \) and \( g' = (\emptyset, \{1\}, Y) \) on \( Y \). Let us define a function \( f : X \to Y \) as \( f(a) = f(b) = f(c) = 1 \). Then \( f \) is strongly \( \theta(g, g') \)-continuous. But since \( c_{g'}(f(\{a\})) = c_{g'}(\{1\}) = Y \), \( f \) can not be \( R(g, g') \)-continuous.

From Remark 3.8 of [5] and Theorem 3.5, we have the implications:

\( R(g, g') \)-continuous \( \Rightarrow \) strongly \( \theta(g, g') \)-continuous \( \Rightarrow \) super \( (g, g') \)-continuous \( \Rightarrow \) \( (g, g') \)-continuous.

Definition 3.8. Let \((X, g)\) and \((Y, g')\) be generalized topological spaces. Then a function \( f : X \to Y \) is said to be weakly \((g, g')\)-closed if for each \( g \)-closed set \( F \) in \( X \), \( c_{g'}(f(\{i_g(F)\})) \subseteq f(F) \).

Lemma 3.9. Let \((X, g)\) and \((Y, g')\) be GTS’s. Then if a function \( f : X \to Y \) is weakly \((g, g')\)-closed, then \( c_{g'}(f(U)) \subseteq f(c_g(U)) \) for every \( g \)-open set \( U \) in \( X \).
Proof. For any \( g \)-open set \( U \subseteq X \), since \( c_g(U) \) is \( g \)-closed and \( U \subseteq i_g(c_g(U)) \), it is obtained. \( \square \)

Theorem 3.10. Let \((X, g)\) and \((Y, g')\) be GTS’s. Then if a function \( f : X \to Y \) is weakly \((g, g')\)-closed and strongly \( g(g')\)-continuous, then it is \( R(g, g')\)-continuous.

Proof. For \( x \in X \), let \( V \) be a \( g' \)-open set containing \( f(x) \). Then from the strong \( \theta(g, g')\)-continuity of \( f \), there exists a \( g \)-open set \( U \) of \( x \) such that \( f(c_g(U)) \subseteq V \). From Lemma 3.9, it follows \( c'_g(f(U)) \cap M_{g'} \subseteq f(c_g(U)) \cap M_{g'} \subseteq V \). Hence by Theorem 3.3(2), \( f \) is \( R(g, g')\)-continuous. \( \square \)

Definition 3.11. Let \((X, g)\) be a generalized topological space. Then \( X \) is said to be relative \( G\)-regular (simply, \( G\)-regular) [4] on \( M_g \) if for \( x \in M_g \) and a \( g \)-closed set \( F \) with \( x \notin F \), there exist \( U, V \in g \) such that \( x \in U \), \( F \cap M_g \subseteq V \) and \( U \cap V = \emptyset \).

Theorem 3.12 ([4]). Let \((X, g)\) be a GTS. Then \( X \) is \( G\)-regular if and only if for \( x \in M_g \) and a \( g \)-open set \( U \) containing \( x \), there is a \( g \)-open set \( V \) containing \( x \) such that \( x \in V \subseteq c_g V \cap M_g \subseteq U \).

Theorem 3.13. Let \((X, g)\) and \((Y, g')\) be GTS’s. Then a function \( f : X \to Y \) is strongly \( \theta(g, g')\)-continuous and \( Y \) is \( G\)-regular, then it is \( R(g, g')\)-continuous.

Proof. For \( x \in X \), let \( V \) be a \( g' \)-open set containing \( f(x) \). Since \( Y \) is \( G\)-regular, for the \( g' \)-open set \( V \) containing \( f(x) \), there is a \( g' \)-open set \( W \) containing \( f(x) \) such that \( f(x) \in W \subseteq c'_g W \cap M_{g'} \subseteq V \). For the \( g' \)-open set \( W \) containing \( f(x) \), from the strong \( \theta(g, g')\)-continuity of \( f \), there exists a \( g \)-open set \( U \) of \( x \) such that \( f(c_g(U)) \subseteq W \). This implies \( c'_g(f(c_g(U))) \cap M_{g'} \subseteq c'_g(W) \cap M_{g'} \subseteq V \). By Theorem 3.3(2), \( R(g, g')\)-continuous. \( \square \)

From Corollary 3.13 of [5], Lemma 3.5 and Theorem 3.13, the following corollary is easily obtained:

Corollary 3.14. Let \( f : X \to Y \) be a function between two GTS’s \((X, g)\) and \((Y, g')\). Then if \( Y \) is \( G\)-regular and strong, then the following things are equivalent:

1. \( R(g, g')\)-continuity.
2. strongly \( \theta(g, g')\)-continuity.
3. \((g, g')\)-continuity.

Let \((X, g)\) and \((Y, g')\) be GTS’s. Then a function \( f : X \to Y \) is said to be \((g, g')\)-open [3] if for every \( g \)-open set \( G \) in \( X \), \( f(G) \) is \( g' \)-open in \( Y \).

Theorem 3.15. Let \((X, g)\) and \((Y, g')\) be GTS’s and \( f(M_g) = M_{g'} \). Then if a function \( f : X \to Y \) is \((g, g')\)-open and \( R(g, g')\)-continuous, then \( Y \) is \( G\)-regular.
Proof. Let \( y \in M'_g \) and \( V \) any \( g' \)-open set containing \( y \). Let \( f(x) = y \) for \( x \in X \). Then since \( f \) is \( R(g,g') \)-continuous, there exists a \( g \)-open set \( U \) containing \( x \) such that \( c'_g(f(U)) \cap M'_{g'} \subseteq V \). Since \( f \) is \( (g,g') \)-open, \( f(U) \) is a \( g' \)-open set containing \( y \), and so \( f(U) = f(U) \cap M'_{g'} \subseteq c'_g(f(U)) \cap M'_{g'} \subseteq V \). Therefore, since \( f(U) \) is a \( g' \)-open set containing \( y \), by Theorem 3.12, \( Y \) is \( G \)-regular. \( \square \)

References


Young Key Kim
Department of Mathematics
MyongJi University
Yongin 449-728, Korea
E-mail address: ykkim@mju.ac.kr

Won Keun Min
Department of Mathematics
Kangwon National University
Chunchon 200-701, Korea
E-mail address: wkmin@kangwon.ac.kr