HOMOLOGY 3-SPHERES OBTAINED BY SURGERY ON EVEN NET DIAGRAMS

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Abstract. In this paper, we characterize surgery presentations for \(\mathbb{Z}\)-homology 3-spheres and \(\mathbb{Z}/2\mathbb{Z}\)-homology 3-spheres obtained from \(S^3\) by Dehn surgery along a knot or link which admits an even net diagram and show that the Casson invariant for \(\mathbb{Z}\)-homology spheres and the \(\mu\)-invariant for \(\mathbb{Z}/2\mathbb{Z}\)-homology spheres can be directly read from the net diagram. We also construct oriented 4-manifolds bounding such homology spheres and find their some properties.

1. Introduction

A framed link in \(S^3\) of \(r\) components is a disjoint collection of \(r\) smoothly imbedded circles \(K_1, \ldots, K_r\) in \(S^3\) with rational numbers \(\frac{p_i}{q_i}\) or \(\infty = \frac{1}{0}\) associated with each imbedded circle \(K_i\). In [8] and [15], Lickorish and Wallace showed that any orientable 3-manifold can be obtained from \(S^3\) by Dehn surgery along a framed link with integral framings. Any presentation of a 3-manifold \(M\) by surgery on a framed link is called a surgery presentation of the 3-manifold \(M\). It is well known that a closed oriented 3-manifold is an integral homology 3-sphere if and only if it can be obtained by Dehn surgery on an algebraically split link in \(S^3\) for which all components are framed by \(\pm 1\) (cf. [4, 9]).

In 1952, Rokhlin introduced a theorem asserting that if \(M\) is a smooth closed oriented spin 4-manifold, then the signature of \(M\) is divisible by 16 [10]. This theorem is now known as the Rokhlin Theorem, which has played a significant role in the study of 4-dimensional topology and also gives rise the \(\mu\)-invariant for \(\mathbb{Z}/2\mathbb{Z}\)-homology, for short, \(\mathbb{Z}/2\)-homology 3-spheres whose properties are related to the most fundamental questions of the manifold theory such as the triangulability of topological manifolds [2, 5].

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In 1984-85, Casson introduced an integral valued invariant \( \lambda : S \to \mathbb{Z} \) from the set \( S \) of oriented \( \mathbb{Z} \)-homology 3-spheres to the set \( \mathbb{Z} \) of integers, which reduces modulo 2 to the \( \mu \)-invariant, by using representations from their fundamental groups into \( SU(2) \) [1].

In this paper, we characterize surgery presentations for \( \mathbb{Z} \)-homology 3-spheres and \( \mathbb{Z}/2 \)-homology 3-spheres obtained from \( S^3 \) by Dehn surgery along a knot or link which admits an even net diagram \( D \) and show that the Casson invariant for \( \mathbb{Z} \)-homology spheres and the \( \mu \)-invariant for \( \mathbb{Z}/2 \)-homology spheres can be directly calculated from the given net diagram \( D \). We also construct oriented 4-manifolds bounding such homology spheres and give some properties.

This paper is organized as follows. In Section 2, we review some fundamental notions for surgery presentations for closed 3-manifolds. In Section 3, we characterize surgery presentations for \( \mathbb{Z} \)- and \( \mathbb{Z}/2 \)-homology 3-spheres obtained by Dehn surgery along even net diagrams. We also construct 4-manifolds bounding the \( \mathbb{Z} \)- or \( \mathbb{Z}/2 \)-homology spheres obtained from an even net diagram and give some properties of their intersection forms. In Section 4, we give the formulas for the Casson invariant for \( \mathbb{Z} \)-homology 3-spheres and the \( \mu \)-invariant for \( \mathbb{Z}/2 \)-homology 3-spheres obtained by Dehn surgery along even net diagrams, which show that the invariants can be directly calculated from the given net diagrams. In Section 5, we give an infinite family of \( \mathbb{Z} \)- and \( \mathbb{Z}/2 \)-homology 3-spheres whose Casson invariants and the \( \mu \)-invariants are vanishing, and are not vanishing.

2. Surgery presentation

Let \( \Sigma \) be an integral homology 3-sphere. A framed link \( L \) in \( \Sigma \) of \( r \) components is a disjoint collection \( K_1 \cup \cdots \cup K_r \) of \( r \) smoothly imbedded circles, \( K_1, \ldots, K_r \), in \( \Sigma \) with rational numbers \( \frac{p_i}{q_i} \), or \( \infty = \frac{1}{0} \), called the framing, associated with each imbedded circle \( K_i \). Throughout this paper, we denote the oriented closed 3-manifold obtained from \( \Sigma \) by a Dehn surgery along a framed link \( L \) with a framing \( \left( \frac{q_1}{p_1}, \ldots, \frac{q_r}{p_r} \right) \) by \( M^3(\bigcup K_r; \frac{q_1}{p_1}, \ldots, \frac{q_r}{p_r}; \Sigma) \), or simply \( M^3(L; \Sigma) \), otherwise specified. When \( \Sigma = S^3 \), the 3-sphere, we shall delete \( S^3 \) from the notations for short. If all the framings \( \frac{p_i}{q_i} \) are integers or \( \infty \), a Dehn surgery on \( \Sigma \) along a framed link \( L \) is called an integral surgery, otherwise it is called a rational surgery. Lickorish [8] and Wallace [15] showed that any closed orientable 3-manifold can be obtained by an integral surgery on \( S^3 \) along a framed link in \( S^3 \).

A framed link \( L \) in \( S^3 \) with integral framings determines an orientable 4-manifold \( M^4(L) \) obtained by adding 2-handles to the 4-ball \( D^4 \) along the circles in \( L \) via the framings. Note that the resulting 4-manifold makes no difference how we orient the circles, and an orientation for \( M^4(L) \), and so \( \partial M^4(L) \), is determined by extending a fixed orientation on \( D^4 \) over \( M^4(L) \).

Any presentation of a 3-manifold \( M \) by surgery on a framed link is called a surgery presentation of \( M \). There are many surgery presentations for the same
manifold. Any two integral surgery presentations of the same manifold can be related by a finite sequence of Kirby moves [6]. Any two rational surgery presentations of the same manifold can be related by a finite sequence of generalized Kirby moves [3, 11, 12]. For our convenience, we review the Kirby moves on a framed link \( L = K_1 \cup \cdots \cup K_r \) in \( S^3 \) which do not change the 3-manifold \( M^3(L) \) obtained by Dehn surgery on \( S^3 \) along the framed link \( L \):

**Kirby Move K1.** Add or delete an unknotted circle with framing 1 or \(-1\), which belongs to an imbedded 3-ball \( D^3 \) in \( S^3 \) that does not intersect the other components of \( L \), see Figure 1.

\[
L \leftrightarrow L' = L \cup \circ \pm 1
\]

**Figure 1**

**Kirby Move K2.** Add one component of the link \( L \) to another as follows. Let \( K_i \) and \( K_j \) be the two components of \( L \) with framings \( n_i \) and \( n_j \), respectively, and let \( K_j \) be a longitude defining the framing \( n_j \) of the component \( K_j \), that is, \( \text{lk}(K_j, K_j) = n_j \), where \( \text{lk} \) denotes the linking number. Now, replace \( K_i \cup K_j \) with \( K'_i \cup K_j \), where \( K'_i = K_i \sharp_b K_j \), the band connected sum of \( K_i \) and \( K_j \), and \( b \) is any band missing the other components of \( L \). The rest of the link \( L \) remains unchanged, see Figure 2.

\[
\begin{array}{c}
K_i \\
n_i
\end{array}
\leftrightarrow
\begin{array}{c}
K_j \\
n_j
\end{array}
\rightarrow
\begin{array}{c}
K'_i \\
n'_i
\end{array}
\cup
\begin{array}{c}
\text{full twists} \\
\text{b}
\end{array}
\cup
\begin{array}{c}
K_j \\
n_j
\end{array}
\leftrightarrow
\begin{array}{c}
\tilde{K}_j \\
n_j
\end{array}
\]

**Figure 2**

The framings of all components but \( K'_i \) in \( L' \) are preserved, while the framing \( n'_i \) of the new component \( K'_i \) is given by the formula:

\[
n'_i = n_i + n_j + 2\text{lk}(K_i, K_j).
\]

Note that the Kirby move K1 corresponds in the 4-manifold \( M^4(L) \) to taking connected sum with or splitting off a copy of the complex projective plane \( \mathbb{C}P^2 \) with a canonical orientation or \( \overline{\mathbb{C}P^2} \) with reversed orientation, depending on the framing \(+1\) or \(-1\). The Kirby move K2 corresponds in \( M^4(L) \) to sliding the \( i \)-th 2-handle over the \( j \)-th 2-handle via the band \( b \).
Two framed links $L$ and $L'$ in $S^3$ are said to be $\partial$-equivalent if the link $L'$ can be obtained from $L$ by a finite sequence of Kirby moves of types $K1$ and $K2$; denoted by $L \sim_\partial L'$. In [6], Kirby proved that for given two framed links $L$ and $L'$ in $S^3$, $L \sim_\partial L'$ if and only if $\partial M^4(L)$ is diffeomorphic to $\partial M^4(L')$ by an orientation-preserving diffeomorphism.

On the other hand, Fenn and Rourke showed [3] that the Kirby moves $K1$ and $K2$ are equivalent to the $K$-move as shown in Figure 3.

Here, if $n_i$ is the framing of the component $K_i$ in $L$, then the framing $n'_i$ of the corresponding component $K'_i$ in $L'$ is given by the formula:

$$n'_i = n_i \mp lk(K_0, K_i)^2.$$ 

Let $L = K_1 \cup \cdots \cup K_r$ be an oriented framed link in $S^3$ whose $i$-th component $K_i$ is framed by $\frac{p_i}{q_i}$. Then the symmetric rational matrix $\Lambda(L) = (\ell_{ij}), i,j = 1,2,\ldots, r$ with the entries

$$\ell_{ij} = \begin{cases} \frac{p_i}{q_i}, & \text{if } i = j; \\ lk(K_i, K_j), & \text{if } i \neq j, \end{cases}$$

is called the linking matrix for $L$. It is well known that the symmetric integral matrix $\overline{\Lambda}(L) = (q_i, \ell_{ij}), i,j = 1,2,\ldots, r$ is a presentation matrix for the homology $H_1(M^3(L); \mathbb{Z})$. We remark that the 3-manifold $M^3(L)$ is an integral homology 3-sphere if and only if the determinant of $\overline{\Lambda}(L)$ is equal to $\pm 1$, namely, $\det(\overline{\Lambda}(L)) = \pm 1$ or, equivalently, $\det(\Lambda(L)) = \pm \frac{q_1q_2\cdots q_r}{q_1q_2\cdots q_r}$.

Let $M$ be a compact oriented connected and simply connected 4-manifold with $\partial M \neq \emptyset$. The intersection number $a \cdot b$ of 2-cycles $a$ and $b$ induces a symmetric bilinear form

$$(2.2) \quad Q_M : H_2(M; \mathbb{Z}) \otimes H_2(M; \mathbb{Z}) \to \mathbb{Z},$$

which is called the intersection form of $M$. Its signature, that is the number of positive eigenvalues minus the number of negative eigenvalues, is called the signature of $M$ and is denoted by $\text{sign}(M)$. It is well known that the intersection...
form $Q_M$ in (2.2) is unimodular if and only if $\partial M$ is an integral homology 3-sphere. Furthermore, let $L$ be an integral framed link in $S^3$ and let $M = M^4(L)$ be the 4-manifold with boundary $\partial M^4(L) = M^3(L)$ obtained by Dehn surgery on $S^3$ along $L$. Then $Q_M$ is isomorphic to the linking matrix $\Lambda(L)$ for $L$. Hence the 3-manifold $\partial M^4(L) = M^3(L)$ is an integral homology 3-sphere if and only if the determinant of $\Lambda(L)$ is equal to $\pm 1$, namely, $\det(\Lambda(L)) = \pm 1$, and $\text{sign}(M^4(L)) = \text{sign}(\Lambda(L))$. For more details, see [13, 14].

3. Dehn surgery on even net diagrams

In [7], Seo and the author showed that any (oriented) link $L$ in $S^3$ can be represented by a link diagram of the form as shown in Figure 4 or Figure 5. In the figures, each tangle labeled $a_{ij}^i$ $(1 \leq i \leq m, 1 \leq j \leq n)$ denotes a 2-tangle as shown in Figure 6. Given a link $L$ in $S^3$, such a link diagram $D$ in Figure 4 or Figure 5 representing $L$ according as $m$ is even or $m$ is odd is called a net presentation of $L$ or a net diagram of $L$, and is denoted by $D = (a_{ij}^i)_{1 \leq i \leq m, 1 \leq j \leq n}$ in a matrix notation.

**Definition 3.1.** Let $D = (a_{ij}^i)_{1 \leq i \leq m, 1 \leq j \leq n}$ be a net diagram of a link in $S^3$. Then $D$ is said to be even if all $a_{ij}^i$’s are even, and $D$ is said to be odd if all $a_{ij}^i$’s are odd.

**Remark 3.2.** (1) Any link $L$ in $S^3$ can always be represented by an odd net diagram $D = (a_{ij}^i)_{1 \leq i \leq m, 1 \leq j \leq n}$ with $a_{ij}^i = \pm 1$ [7]. It is worth mentioning that a link in $S^3$ may be represented by a net diagram which is of the even type.

(2) Let $D = (a_{ij}^i)_{1 \leq i \leq m, 1 \leq j \leq n}$ be an even net diagram of a link $L$ in $S^3$ with $r$ components. Then it follows at once from the diagrams in Figures 4 and 5
that $m$ is an even integer if and only if $r = 1$, that is, $D$ is a knot diagram, and that $m$ is an odd integer if and only if $r = n$.

(3) Let $D = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ be an odd net diagram of a link $L$ in $S^3$ with $r$ components. Then it is not difficult to see from the diagrams in Figures 4 and 5 that $\gcd(m + 1, n) = r$.

Let $D = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n} = D_1 \cup \cdots \cup D_r (r \geq 1)$ be any oriented even net diagram of an integral framed link $L$ in $S^3$ with framings $p_1, \ldots, p_r$. We define $\hat{D}$ to be a framed link diagram obtained from $D$ by replacing each 2-tangle labeled $a_{ij} (\neq 0)$ with a new 2-tangle $E_{ij}$ as shown in Figure 7. The framing $\hat{p}_k$ on $\hat{D}_k$ ($k = 1, \ldots, r$) in $\hat{D}$ corresponding to $D_k$ in $D$ is defined by the following
formula. For $k = 1, \ldots, r$,

$$\hat{p}_k = p_k - \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{a_{ij}}{|a_{ij}|^2} |k(\hat{D}_1, \hat{D}_1^{ij})|^2.$$
Figure 9. It follows that this orientation for $\mathcal{D}$ induces an orientation for $\hat{\mathcal{D}}$ which is the same on the components $\hat{D}_{ij}^u (1 \leq u \leq s) \subset \mathcal{D}$ in Figure 7, after the transformations as illustrated in Figure 10.

Using Figure 9 and (2.1), it is not difficult to see that the framing $\bar{p}_k$ on $\mathcal{D}_k$ is given by the right hand side of the formula in (3.3). Since the linking numbers of all pairs of the unknotted circles $\mathcal{D}_{ij}^u (1 \leq u \leq s)$ in the tangle $b_{ij}$ are all zero, the transformation in Figure 10 gives the framing $\frac{a_j}{|a_j|}2$ on $\hat{D}_{ij}^u$. Also, the transformation does not change the framing on $\mathcal{D}_k$ for each $k = 1, \ldots, r$ and hence $\bar{p}_k = \bar{p}_k$ for $k = 1, \ldots, r$. This shows that $L \sim \hat{L}$, completing the proof. □

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure9.png}
\caption{Figure 9}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure10.png}
\caption{Figure 10}
\end{figure}

Definition 3.4. Let $D = (a_{ij}^j)_{1 \leq i \leq m, 1 \leq j \leq n}$ be an even net diagram of an integral framed link in $\mathbb{S}^3$. Then the framed link diagram $\hat{D}$ is called the blow-up of $D$. 
Proof. First, if \( D \) be the linking matrix for \( \hat{D} \) as shown in Figure 4. Then it is obvious from Figure 9 that \( \hat{D} \) is an algebraically split link with \( d = (\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|) + 1 \) components and so it is straightforward that \( \Lambda(\hat{D}) \) is the \( d \times d \) diagonal matrix whose diagonal entries are the framings of the components. For each pair \((i,j)\), let \( I_{ij} \) denote the \((|a_{ij}| \times |a_{ij}|)\) identity matrix. Then for some unimodular integral matrices \( U_1 \) and \( U_2 \), we have

\[
U_2 \Lambda(\hat{D}) U_2^T = (p_1) \oplus \left( \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} a_{ij} I_{ij} \right),
\]

\[
U_1 U_2 \Lambda(\hat{D}) U_2^T U_1^T = (p_1) \oplus I_{\sigma_+(D)} \oplus (-1) I_{\sigma_-(D)}.
\]

Lemma 3.5. Let \( L = K_1 \cup \cdots \cup K_r \) be a framed link in \( S^3 \) with integral framings \( p_1, \ldots, p_r \) which admits an even net diagram \( D = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n} \). Let \( \sigma \) be any oriented even net diagram. Then \( \Lambda(\sigma) \) be the linking matrix for \( \hat{L} \).

(1) If \( m \) is even, then there exists a unimodular integral matrix \( U \) such that

\[
U \Lambda(\hat{L}) U^T = (p_1) \oplus I_{\sigma_+(\hat{D})} \oplus (-1) I_{\sigma_-(\hat{D})}.
\]

(2) If \( m \) is odd, then there exists a unimodular integral matrix \( U \) such that

\[
U \Lambda(\hat{L}) U^T = \begin{pmatrix}
\hat{p}_1 & 0 & \cdots & 0 \\
0 & \hat{p}_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \hat{p}_n
\end{pmatrix} \oplus I_{\sigma_+(\hat{D})} \oplus (-1) I_{\sigma_-(\hat{D})},
\]

where \( I_{\sigma_+(\hat{D})} \) denotes the \((\sigma_+(\hat{D}) \times \sigma_+(\hat{D}))\) identity matrix, \( U^T \) denotes the transpose matrix of \( U \), and

\[
\hat{p}_k = p_k - \frac{1}{2} \sum_{\ell=1}^{m+1} \left( a_{k+1}^{2\ell-1} + a_{k+1}^{2\ell-1} \right), k = 1, \ldots, n.
\]
Next, suppose that $m$ is odd. Then we know that $r = n$, that is, $D$ is a framed link diagram with exactly $n$ components $D_1, \ldots, D_n$ with framings $p_1, \ldots, p_n$. We choose an orientation for $D$ as shown in Figure 11. Let

$$D_1, \ldots, D_n$$

be the components of $\overline{D}$ with the induced orientation by $D$, corresponding to $D_1, \ldots, D_n$ in $D$, and let $D_1^{ij}, \ldots, D_s^{ij}$ ($s_{ij} = \frac{|a_{ij}|}{2}$) be the components of $b^{ij}_1 \subset \overline{D}$ as shown in Figure 9. We observe that for each pair $(i, j)$ and $u = 1, \ldots, s_{ij}$,

$$lk(D_1^{ij}u, D_k) = \begin{cases} (-1)^{u+1}, & \text{if } i \text{ is odd and } j = k; \\ (-1)^u, & \text{if } i \text{ is odd and } j = k + 1; \\ 0, & \text{otherwise.} \end{cases}$$

This gives

$$lk(D_1^{ij}u, D_k)^2 = \begin{cases} 1, & \text{if } i \text{ is odd and } j = k, k + 1; \\ 0, & \text{otherwise.} \end{cases}$$

Let $\bar{p}_1, \ldots, \bar{p}_n$ denote the framings of $\overline{D}_1, \ldots, \overline{D}_n$, respectively. By (2.1), we obtain that for each $k = 1, \ldots, n$,

$$\bar{p}_k = p_k - \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{u=1}^{s_{ij}} \frac{a_{ij}^u}{|a_{ij}^u|} lk(D_1^{ij}u, D_k)^2$$

$$= p_k - \sum_{\ell=1}^{m+1} \left( \sum_{k=1}^{n} \frac{a_k^{2\ell-1}}{|a_k^{2\ell-1}|} + \sum_{\ell=1}^{n} \frac{a_{k+1}^{2\ell-1}}{|a_{k+1}^{2\ell-1}|} \right)$$
\[ = p_k - \sum_{\ell=1}^{m+1} a_{k \ell}^2 - a_{k+1}^{2 \ell-1} \cdot \frac{a_{k+1}^{2 \ell-1}}{2}. \]

Since the transformation in Figure 10 does not change the framing on \( D_k \) for all \( k = 1, \ldots, n \), we get \( \hat{p}_k = \bar{p}_k \) and hence

\[ \hat{p}_k = p_k - \sum_{\ell=1}^{m+1} a_{k \ell}^2 - a_{k+1}^{2 \ell-1}, \quad k = 1, \ldots, n. \]

Now, for two pairs \((i, j)\) and \((k, l)\) with \( 1 \leq i, k \leq m, 1 \leq j, l \leq n \), let

\[ \Lambda_{ij} = \left( \gamma_{uv} \right)_{1 \leq u \leq \left| a_i^j \right|, 1 \leq v \leq \left| a_k^l \right|} \]

be the \(( \left| a_i^j \right| \times \left| a_k^l \right| )\) integral matrix defined by

\[ \gamma_{uv} = \begin{cases} a_i^j, & \text{if } (i, j) = (k, l) \text{ and } u = v; \\ lk(D_u, D_i^j), & \text{otherwise}. \end{cases} \]

Observe that if either \((i, j) \neq (k, l)\) or \( u \neq v \), then \( lk(D_u, D_i^j) = 0 \). This gives that for all pairs \((i, j)\) and \((k, l)\) with \((i, j) \neq (k, l)\), the matrix \( \Lambda_{ij} \) is the \(( \left| a_i^j \right| \times \left| a_k^l \right| )\) zero matrix, and for \((i, j) = (k, l)\), \( \Lambda_{ij} = \Lambda_{ij}^{ij} = \left( \frac{\left| a_i^j \right| \times \left| a_k^l \right|}{2} \right) \) diagonal matrix given by

\[ \Lambda_{ij}^{ij} = \begin{pmatrix} a_i^j & 0 & \cdots & 0 \\ 0 & a_i^j & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_i^j \end{pmatrix} = a_i^j \cdot I_{ij}. \]

Let

\[ \Lambda_0 = \begin{pmatrix} \bar{p}_1 & 0 & \cdots & 0 \\ 0 & \bar{p}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{p}_n \end{pmatrix}. \]

For each pair \((i, j)\) with \( 1 \leq i \leq m, 1 \leq j \leq n \), let

\[ \Lambda_j = \left( \gamma_{uv} \right)_{1 \leq u \leq n, 1 \leq v \leq \left| a_i^j \right|} \]

be the \((n \times \left| a_i^j \right| )\) matrix defined by

\[ \gamma_{uv} = lk(D_u, D_i^j). \]
Then it is straightforward from Figure 9 that for each \( j = 1, \ldots, n \) and \( \ell = 1, \ldots, \frac{m+1}{2} \), \( \Lambda_j^{2k-1} \) is the zero matrix. For \( k = 1, \ldots, \frac{m+1}{2} \), we have

\[
\Lambda_k^{2k-1} = \begin{pmatrix}
\frac{a_{2k-1}}{|a_1^k|} & -\frac{a_{2k-1}}{|a_1^k|} & \frac{a_{2k-1}}{|a_1^k|} & \cdots & (-1)^{s_{2k-1}} & -\frac{a_{2k-1}}{|a_1^k|} \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{a_{2k-1}}{|a_1^k|} & \frac{a_{2k-1}}{|a_1^k|} & -\frac{a_{2k-1}}{|a_1^k|} & \cdots & (-1)^{s_{2k-1}+1} \times \\
\end{pmatrix}
\]

Then the linking matrix \( \Lambda(D) \) of \( D \) with respect to the order for the pairs

\( (1, 1), \ldots, (1, n), (2, 1), \ldots, (2, n), \ldots, (m, 1), \ldots, (m, n) \).
is given by the \((mn + 1) \times (mn + 1)\) block matrix:

\[
\Lambda(\mathcal{D}) = \begin{pmatrix}
\Lambda_0 & \Lambda_1^1 & \Lambda_1^2 & \cdots & \Lambda_1^m \\
(\Lambda_1^1)^T & \Lambda_{11} & O & O & \cdots & O \\
(\Lambda_1^2)^T & O & \Lambda_{12} & O & \cdots & O \\
(\Lambda_1^3)^T & O & O & \Lambda_{13} & \cdots & O \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
(\Lambda_m^m)^T & O & O & O & \cdots & \Lambda_{mm}^{mn}
\end{pmatrix}.
\]

It is not difficult to see that for unimodular integral matrices \(U_1\) and \(U_2\),

\[
U_2\Lambda(\mathcal{D})U_2^T = \Lambda_0 \oplus \left( \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} \Lambda_{ij} \right) = \Lambda_0 \oplus \left( \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} \frac{a_{ij}^2}{|a_{ij}|} I_{ij} \right),
\]

(3.7)

\[
U_1U_2\Lambda(\mathcal{D})U_2^T U_1^T = \Lambda_0 \oplus I_{\sigma_1(\mathcal{D})} \oplus (-1)^{I_{\sigma_2(\mathcal{D})}}.
\]

On the other hand, the effect of the Kirby move \(K_1\) replaces the linking matrix \(A\) to \(A \oplus (\pm 1)\), and the Kirby move \(K_2\) that slides \(K_i\) over \(K_j\) replaces the linking matrix \(A\) to \(A'\) obtained from \(A\) by adding (or subtracting) the \(j\)-th row to (from) the \(i\)-th row and the \(j\)-th column to (from) the \(i\)-th column. This facts, together with (3.6) and (3.7), implies the desired assertions (1) and (2).

This completes the proof of Lemma 3.5. \(\square\)

For a given integer \(p\), we define

\[
\epsilon(p) = \begin{cases} 
\frac{p}{|p|}, & \text{if } p \neq 0; \\
0, & \text{if } p = 0.
\end{cases}
\]

**Theorem 3.6.** Let \(L = K_1 \cup \cdots \cup K_r (r \geq 1)\) be a framed link in \(S^3\) with integer framings \(p_1, \ldots, p_r\) which admits an even net diagram \(D = (a_{ij}^1)_{1 \leq i \leq m, 1 \leq j \leq n}\) and let \(\hat{L}\) be the framed link in \(S^3\) represented by the blow-up \(\hat{D}\) of \(D\). Let \(\Lambda(L)\) and \(\Lambda(\hat{L})\) be the linking matrices for \(L\) and \(\hat{L}\), respectively.

1. If \(m\) is even, then \(r = 1\) and

\[
|\det(\Lambda(L))| = |\det(\Lambda(\hat{L}))| = |p_1|,
\]

\[
\text{sign}(\Lambda(\hat{L})) = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{a_{ij}^1}{2} + \epsilon(p_1).
\]

2. If \(m\) is odd, then \(r = n\) and

\[
|\det(\Lambda(L))| = |\det(\Lambda(\hat{L}))| = |\prod_{k=1}^{n} \hat{p}_k|,
\]

\[
\text{sign}(\Lambda(\hat{L})) = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{a_{ij}^1}{2} + \sum_{k=1}^{n} \epsilon(\hat{p}_k),
\]

where \(\hat{p}_1, \ldots, \hat{p}_n\) are the integers in (3.5).
Proof. (1) Suppose that \( m \) is even. By Lemma 3.5(1), we obtain that \( r = 1 \) and
\[
|\det(\Lambda(\hat{L}))| = |p_1|,
\]
\[
\text{sign}(\Lambda(\hat{L})) = \epsilon(p_1) + \frac{\sigma_+(D) - \sigma_-(D)}{2} = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{a_{ij}}{2} + \epsilon(p_1).
\]

(2) Suppose that \( m \) is odd. By Lemma 3.5(2), we have that \( r = n \) and
\[
|\det(\Lambda(\hat{L}))| = |\prod_{k=1}^{n} \hat{p}_k|,
\]
\[
\text{sign}(\Lambda(\hat{L})) = \sum_{k=1}^{n} \epsilon(\hat{p}_k) + \frac{\sigma_+(D) - \sigma_-(D)}{2} = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{a_{ij}}{2} + \sum_{k=1}^{n} \epsilon(\hat{p}_k).
\]

Finally, it follows from Theorem 3.3 that \( L \) and \( \hat{L} \) are \( \partial \)-equivalent and hence \( |\det(\Lambda(\hat{L}))| = |\det(\Lambda(L))| \). This completes the proof. \( \square \)

**Theorem 3.7.** Let \( D = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n} \) be an even net diagram of a link \( L \) in \( S^3 \) of components \( K_1, \ldots, K_r \) with integer framings \( p_1, \ldots, p_r \). Then \( M^3(L) \) is a \( \mathbb{Z} \)-homology 3-sphere if and only if either \( r = 1 \) and \( p_1 = \pm 1 \) or \( r = n \) and
\[
(3.8) \quad \prod_{k=1}^{n} \left( p_k - \frac{1}{2} \sum_{\ell=1}^{m+1} (a_{k\ell}^{2\ell-1} + a_{k\ell+1}^{2\ell-1}) \right) = \pm 1.
\]

Proof. Let \( D = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n} \) be an even net diagram of a link \( L \) in \( S^3 \) of components \( K_1, \ldots, K_r \) with integer framings \( p_1, \ldots, p_r \), and let \( \Lambda(L) \) and \( \Lambda(\hat{L}) \) be the linking matrices for \( L \) and \( \hat{L} \), respectively.

Suppose that \( M^3(L) \) is a \( \mathbb{Z} \)-homology 3-sphere. Then \( \det(\Lambda(L)) = \pm 1 \). Note that \( m \) is either even or odd. If \( m \) is even, then it follows from Theorem 3.6 (1) that \( r = 1 \) and \( \Lambda(L) = (p_1) \). This gives \( p_1 = \pm 1 \). Now if \( m \) is odd, then it follows from Theorem 3.6(2) that \( r = n \) and \( |\det(\Lambda(L))| = |\prod_{k=1}^{n} \hat{p}_k| = \pm 1 \). This gives the equality (3.8).

Conversely, we first suppose that \( m \) is even and \( p_1 = \pm 1 \). By Lemma 3.5, the matrix
\[
\Lambda(\hat{L}) = (\pm 1) \oplus I_{\sigma_+(D)} \oplus (-1) I_{\sigma_-(D)}
\]
is a presentation matrix for \( H_1(\partial M^4(\hat{L}); \mathbb{Z}) \). This implies that \( H_1(\partial M^4(\hat{L}); \mathbb{Z}) = 0 \). By Theorem 3.3, \( L \) is \( \partial \)-equivalent to \( \hat{L} \), and \( \partial M^4(\hat{L}) \) and \( \partial M^4(L) \) are diffeomorphic, and hence \( H_1(M^3(\hat{L}); \mathbb{Z}) = H_1(\partial M^4(\hat{L}); \mathbb{Z}) = H_1(\partial M^4(L); \mathbb{Z}) = H_1(\partial M^4(L); \mathbb{Z}) = 0 \).

Now we suppose that \( m \) is odd and the equality (3.8) holds. Then for \( k = 1, \ldots, n \), we have
\[
\hat{p}_k = p_k - \frac{1}{2} \sum_{\ell=1}^{m+1} (a_{k\ell}^{2\ell-1} + a_{k\ell+1}^{2\ell-1}) = \pm 1.
\]
By Lemma 3.5, the matrix

$$\Lambda(\hat{L}) = (\pm 1)I_n \oplus I_{\sigma_+(\rho)} \oplus (-1) I_{\sigma_-(\rho)}$$

is a presentation matrix for $H_1(\partial M^4(\hat{L}); \mathbb{Z})$. This implies that $H_1(\partial M^4(\hat{L}); \mathbb{Z}) = 0$. By a similar argument as above, we obtain $H_1(M^3(L); \mathbb{Z}) = 0$. This completes the proof. □

**Theorem 3.8.** Let $D = (a^j_i)_{1 \leq i \leq m, 1 \leq j \leq n}$ be an even net diagram of a link $L$ in $S^3$ of components $K_1, \ldots, K_r$ with integer framings $p_1, \ldots, p_r$. Then $M^3(L)$ is a $\mathbb{Z}/2$-homology 3-sphere if and only if either $r = 1$ and $p_1$ is odd or $r = n$ and

$$\prod_{k=1}^{n} \left( p_k - \frac{1}{2} \sum_{\ell=1}^{m+1} (a_k^{2\ell-1} + a_{k+1}^{2\ell-1}) \right)$$

is an odd integer.

**Proof.** Let $D = (a^j_i)_{1 \leq i \leq m, 1 \leq j \leq n}$ be an even net diagram of a link $L$ in $S^3$ of components $K_1, \ldots, K_r$ with integer framings $p_1, \ldots, p_r$ and let $\Lambda(L)$ and $\Lambda(\hat{L})$ be the linking matrices for $L$ and $\hat{L}$, respectively.

Suppose that $M^3(L)$ is a $\mathbb{Z}/2$-homology 3-sphere. Then $\det(\Lambda(L))$ must be an odd integer. Note that $m$ is either even or odd. If $m$ is even, then it follows from Theorem 3.6(1) that $r = 1$ and $\Lambda(L) = (p_1)$. This gives $p_1$ is odd. Now if $m$ is odd, then it follows from Theorem 3.6(2) that $r = n$ and $|\det(\Lambda(L))| = |\prod_{k=1}^{n} \hat{p}_k|$, is odd.

Conversely, we first suppose that $m$ is even and $p_1$ is odd. By Lemma 3.5, the matrix

$$\Lambda(\hat{L}) = (p_1) \oplus I_{\sigma_+(\rho)} \oplus (-1) I_{\sigma_-(\rho)}$$

is a presentation matrix for $H_1(\partial M^4(\hat{L}); \mathbb{Z})$. This implies that $H_1(\partial M^4(\hat{L}); \mathbb{Z}/2)$ vanishes. By Theorem 3.3, $L$ is $\partial$-equivalent to $\hat{L}$, $\partial M^4(\hat{L})$ and $\partial M^4(L)$ are diffeomorphic, and hence

$$H_1(M^3(L); \mathbb{Z}/2) = H_1(\partial M^4(L); \mathbb{Z}/2) = H_1(\partial M^4(\hat{L}); \mathbb{Z}/2) = 0.$$

Now we suppose that $m$ is odd and the integer in (3.9) is odd. Then for $k = 1, \ldots, n$, the integer

$$\hat{p}_k = p_k - \frac{1}{2} \sum_{\ell=1}^{m+1} (a_k^{2\ell-1} + a_{k+1}^{2\ell-1})$$

must be odd. The matrix (3.4) in Lemma 3.5 is a presentation matrix for $H_1(\partial M^4(\hat{L}); \mathbb{Z})$. Hence $H_1(\partial M^4(\hat{L}); \mathbb{Z}/2) = 0$, and thus $H_1(M^3(L); \mathbb{Z}/2) = 0$, completing the proof. □
Corollary 3.9. Let $L = K_1 \cup \cdots \cup K_r (r \geq 1)$ be a framed link in $S^3$ with integer framings $p_1, \ldots, p_r$ which admits an even net diagram $D = (a^j_i)_{1 \leq i \leq m, 1 \leq j \leq n}$ and let $\hat{L}$ be the framed link in $S^3$ represented by the blow-up $\hat{D}$ of $D$. Then

$$\text{sign}(M^4(\hat{L})) = \begin{cases} \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{a^j_i}{2} + \epsilon(p_1), & \text{if } m \text{ is even;} \\ \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{a^j_i}{2} + \sum_{k=1}^{r} \epsilon(\hat{p}_k), & \text{if } m \text{ is odd,} \end{cases}$$

where $\hat{p}_1, \ldots, \hat{p}_r$ are the integers in (3.5).

In particular, if $M^3(L)$ is a $\mathbb{Z}$-homology or a $\mathbb{Z}/2$-homology 3-sphere, then the rank of the intersection form $Q_L = Q_{M^4(\hat{L})}$ of $M^4(\hat{L})$ is given by

$$\text{rank } Q_L = r + \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{|a^j_i|}{2}.$$

Proof. By Theorems 3.6, 3.7 and 3.8, the assertion follows immediately. \qed

4. The $\mu$-invariant and the Casson invariant

Let $M$ be an oriented $\mathbb{Z}/2$-homology 3-sphere. Then it has a unique spin structure. Choose a smooth compact oriented spin 4-manifold $X$ such that $\partial X = M$. The $\mu$-invariant, $\mu(M)$, of $M$ is a rational residue modulo 2 defined by

$$\mu(M) = \frac{\text{sign}(X)}{8} \mod 2.$$

We note that if $M$ is a $\mathbb{Z}$-homology sphere, then sign$(X)$ is divisible by 8 so that $\mu(M) = 0$ or 1 modulo 2. For more details, see [13, 14]. For a given knot $K$ in $S^3$ and a reduced fraction $\frac{p}{q}$, it is well known that the manifold $M^3(K; \frac{p}{q})$ is a $\mathbb{Z}/2$-homology 3-sphere if and only if $p$ is odd. Let $\Delta_K(t) = \Delta_K(t)$ be the Alexander polynomial of $K$ normalized so that $\Delta_K(1) = 1$ and $\Delta_K(t) = \Delta_K(t^{-1})$. Then it follows from [14, Theorem 2.13] that for every odd $p$,

$$\mu(M^3(K; \frac{p}{q})) = -\mu(L(p, q)) + \frac{q}{2p} \Delta'_K(1) \mod 2,$$

where $L(p, q)$ is the lens space obtained by $\frac{q}{2p}$-surgery on an unknot in $S^3$ and $\Delta'_K(1)$ is the second derivative of $\Delta_K(t)$ evaluated at $t = 1$. For the $\mu$-invariant $\mu(L(p, q))$ of the lens space $L(p, q)$, we refer to [14, Corollary 2.24].

Theorem 4.1. Let $D = (a^j_i)_{1 \leq i \leq 2r, 1 \leq j \leq n}$ be an even net diagram of a knot $K$ in $S^3$. For any odd $p$,

$$\mu(M^3(K; \frac{p}{q})) = -\mu(L(p, q)) + \frac{q}{4p} \sum_{j=1}^{n} \sum_{k=1}^{r} a^j_k (a^{2k-1}_j + a^{2k+1}_j) \mod 2.$$
where $a_{n+1}^{2k-1} = a_1^{2k-1}$ for each $k = 1, 2, \ldots, \ell$. In particular,

\[
(4.11) \quad \mu(M^3(K; \frac{1}{q})) = \frac{q}{4} \sum_{j=1}^{n} \sum_{\ell=1}^{r} \sum_{k=1}^{\ell} a_j^{2\ell} \left(a_j^{2k-1} + a_{j+1}^{2k-1}\right) \pmod{2}.
\]

**Proof.** Since $p$ is odd, $M^3(K; \frac{p}{q})$ is a $\mathbb{Z}/2$-homology 3-sphere. Hence we obtain from (4.10) that

\[
(4.12) \quad \mu(M^3(K; \frac{1}{q})) = \frac{q}{2} \Delta''_K(1) \pmod{2},
\]

because $\mu(L(1, q)) = 0$. But it follows from [7, Theorem 3.5] that

\[
(4.13) \quad \Delta''_K(1) = \frac{1}{2} \sum_{j=1}^{n} \sum_{\ell=1}^{r} \sum_{k=1}^{\ell} a_j^{2\ell} \left(a_j^{2k-1} + a_{j+1}^{2k-1}\right),
\]

where $a_{n+1}^{2k-1} = a_1^{2k-1}, k = 1, 2, \ldots, \ell$. Substituting (4.13) to (4.12), we obtain the desired formula. \qed

Let $\mathcal{S}$ be the class of oriented $\mathbb{Z}$-homology 3-spheres modulo orientation preserving homeomorphism. A **Casson invariant** [1] is a map $\lambda_c : \mathcal{S} \to \mathbb{Z}$ such that

1. $\lambda_c(S^3) = 0$,
2. For any $\mathbb{Z}$-homology 3-sphere $\Sigma$, a knot $K \subset \Sigma$, and an integer $m = 0, \pm 1, \pm 2, \ldots$,

\[
\lambda_c \left(M^3(K; \frac{1}{m+1}; \Sigma)\right) - \lambda_c \left(M^3(K; \frac{1}{m}; \Sigma)\right)
\]

is independent of $m$.
3. For any boundary link $K \cup K'$ in a $\mathbb{Z}$-homology 3-sphere $\Sigma$, that is, $K$ and $K'$ bound disjoint Seifert surfaces in $\Sigma$, and for any two integers $m$ and $n$,

\[
\lambda_c \left(M^3(K \cup K'; \frac{1}{m+1}; \frac{1}{n+1}; \Sigma)\right) - \lambda_c \left(M^3((K \cup K'; \frac{1}{m}; \frac{1}{n}; \Sigma)\right) + \lambda_c \left(M^3(K \cup K'; \frac{1}{m+1}; \frac{1}{n}; \Sigma)\right) = 0.
\]

Note that for any integral homology sphere $\Sigma$, it is well known [1, 13, 14] that

\[
(4.14) \quad \lambda_c(\Sigma) = \mu(\Sigma) \pmod{2}.
\]

In fact, $\lambda_c$ is unique up to sign $\pm 1$. Normalizing $\lambda_c$ by setting $\lambda_c(\text{trefoil}) = +1$, Seo and the author gave the following Theorems 4.2 and 4.3 in [7]:
Theorem 4.2. Let \( D = (a_{i,j}^1)_{1 \leq i \leq 2r, 1 \leq j \leq n} \) be an even net diagram \((r \geq 1, n \geq 2)\) of a knot \( K \) in \( S^3 \). Then

\[
\lambda_c(M^3(K; \frac{1}{q})) = \frac{q}{4} \sum_{j=1}^{n} \sum_{\ell=1}^{r} \sum_{k=1}^{\ell} a_{j}^{2k-1} + a_{j+1}^{2k-1},
\]

where \( a_{n+1}^{2k-1} = a_1^{2k-1} \) for each \( k = 1, 2, \ldots, \ell \). \( \square \)

Theorem 4.3. Let \( D = (a_{i,j}^1)_{1 \leq i \leq 2r-1, 1 \leq j \leq n} \) be an even net diagram \((r, n \geq 2)\) of a link \( L = K_1 \cup K_2 \cup \cdots \cup K_n \) in \( S^3 \) such that for each \( j = 1, 2, \ldots, n, \)

\[
\sum_{k=1}^{r} a_{j}^{2k-1} = 0.
\]

Then for any nonzero integers \( q_1, q_2, \ldots, q_n, \)

\[
\lambda_c \left( M^3 \left( L; \frac{1}{q_1}, \frac{1}{q_2}, \ldots, \frac{1}{q_n} \right) \right)
\]

\[
= -\frac{1}{8} \sum_{k=1}^{n} \sum_{\ell=1}^{r-1} \sum_{i=1}^{\ell} \sum_{j=\ell+1}^{r} q_k a_k^{2\ell} \left( q_{k-1} a_k^{2i-1} a_k^{2j-1} + q_{k+1} a_{k+1}^{2i-1} a_{k+1}^{2j-1} \right),
\]

where \( q_0 = q_n, q_{n+1} = q_1 \) and \( a_{n+1}^{2k-1} = a_1^{2k-1} \) for all \( k = 1, 2, \ldots, r \). \( \square \)

We remark that (4.14) and (4.15) imply the formula in (4.11). Furthermore, we obtain the following immediate corollary from (4.16).

Corollary 4.4. Under the same assumption as in Theorem 4.3, if follows that for any nonzero integers \( q_1, q_2, \ldots, q_n, \)

\[
\mu(M^3(L; \frac{1}{q_1}, \frac{1}{q_2}, \ldots, \frac{1}{q_n}))
\]

\[
= -\frac{1}{8} \sum_{k=1}^{n} \sum_{\ell=1}^{r-1} \sum_{i=1}^{\ell} \sum_{j=\ell+1}^{r} q_k a_k^{2\ell} \left( q_{k-1} a_k^{2i-1} a_k^{2j-1} + q_{k+1} a_{k+1}^{2i-1} a_{k+1}^{2j-1} \right) \pmod{2}.
\]

\( \square \)

5. Examples

In this section, we give an infinite family of \( \mathbb{Z} \)- and \( \mathbb{Z}/2 \)-homology 3-spheres whose Casson invariants and the \( \mu \)-invariants vanish. We also give an infinite family of homology 3-spheres whose invariants do not vanish.
By Theorem 4.1, we obtain that for any odd integer \( p \),
\[
\text{sign}(D) = \text{sign}(\lambda_c(M^3(K; \frac{p}{q}))) = 0.
\]

Example 5.1. Let \( K \) be a knot in \( S^3 \) which has an even net diagram \( D = (a_i^j)_{1 \leq i \leq 2r, 1 \leq j \leq 2n} \) given by
\[
D = \begin{pmatrix}
    a_1 & -a_1 & a_1 & -a_1 & \cdots & a_1 & -a_1 \\
    * & * & * & * & \cdots & * & * \\
    a_2 & -a_2 & a_2 & -a_2 & \cdots & a_2 & -a_2 \\
    * & * & * & * & \cdots & * & * \\
    \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
    a_r & -a_r & a_r & -a_r & \cdots & a_r & -a_r \\
    * & * & * & * & \cdots & * & *
\end{pmatrix},
\]
where \( a_i \in 2\mathbb{Z}(1 \leq i \leq r) \) and each * denotes an arbitrary even (possibly distinct) integer. By Theorem 4.1, \( \mu(M^3(K; \frac{p}{q})) = -\mu(L(p, q)) \) for any odd integer \( p \). By Theorem 4.2, \( \lambda_c(M^3(K; \frac{1}{q})) = 0 \). In particular, if the sum of all entries labeled * are equal to \( \pm 1 \), then it follows from Theorem 3.6(1) that \( \lambda_c(M^3(K; \pm 1)) = \text{sign}(M^4(K)) = 0 \).

Example 5.2. Let \( K \) be a knot in \( S^3 \) which has an even net diagram \( D = (a_i^j)_{1 \leq i \leq 2r, 1 \leq j \leq 2n} \) given by
\[
D = \begin{pmatrix}
    2m_1 & 2m_1 & \cdots & 2m_1 \\
    2m_2 & 2m_2 & \cdots & 2m_2 \\
    \vdots & \vdots & \cdots & \vdots \\
    2m_1 & 2m_1 & \cdots & 2m_1 \\
    2m_2 & 2m_2 & \cdots & 2m_2
\end{pmatrix},\ m_1, m_2 \in \mathbb{Z}.
\]
By Theorem 4.1, we obtain that for any odd integer \( p \),
\[
\mu(M^3(K; \frac{p}{q})) = -\mu(L(p, q)) + \frac{qnr(r + 1)m_1m_2}{p} \pmod{2}.
\]
By Theorem 4.2, \( \lambda_c(M^3(K; \frac{1}{q})) = qnr(r + 1)m_1m_2 \). Let \( M^4(\hat{L}; \pm 1) \) be the 4-manifold with \( \partial M^4(\hat{L}; \pm 1) = M^3(L; \pm 1) \). Then, by Theorem 3.6(1), we have \( \text{sign}(M^4(\hat{L}; \pm 1)) = \pm rn(m_1 + m_2) \pm 1 \).

Example 5.3. Let \( L \) be a link in \( S^3 \) which has an even net diagram \( D = (a_i^j)_{1 \leq i \leq 2m-1, 1 \leq j \leq 4n} \) given by
\[
D = \begin{pmatrix}
    a_1 & a_1 & a_1 & a_1 & \cdots & a_1 \\
    * & * & * & * & \cdots & * \\
    a_2 & a_2 & a_2 & a_2 & \cdots & a_2 \\
    * & * & * & * & \cdots & * \\
    \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
    a_r & a_r & a_r & a_r & \cdots & a_r \\
    * & * & * & * & \cdots & *
\end{pmatrix},
\]
where $a_i \in 2\mathbb{Z} (1 \leq i \leq r)$ such that $\sum_{i=1}^{r} a_i = 0$, and each $*$ denotes an arbitrary even (possibly distinct) integer. Let $q_i (1 \leq i \leq 4n)$ be $4n$ integers with $q_1 = q_{4i+1} = -q_{4i+3}$ and $q_2 = q_{4i+2} = -q_{4(i+1)}$ for all $i = 0, 1, 2, \ldots, n-1$. By Theorem 4.3, $\lambda_c (M^3 (L; \frac{1}{q_1}, \ldots, \frac{1}{q_{4n}})) = 0$, and so $\mu (M^3 (L; \frac{1}{q_1}, \ldots, \frac{1}{q_{4n}})) = 0$.

References


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