FIXED POINTS OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN THE INTERMEDIATE SENSE IN CAT(0) SPACES

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Abstract. The purpose of this paper is to investigate the demiclosed principle, the existence theorems and convergence theorems in CAT(0) spaces for a class of mappings which is essentially wider than that of asymptotically nonexpansive mappings. The structure of fixed point set of such mappings is also studied. Our results generalize, unify and extend several comparable results in the existing literature.

1. Introduction

Let $C$ be a nonempty subset of metric space $(X, d)$. Recall that, a mapping $T : C \to X$ is said to be (i) nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in C$ (ii) asymptotically nonexpansive if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that $d(T^n x, T^n y) \leq k_n d(x, y)$ for all $x, y$ in $C$ and $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers. Class of asymptotically nonexpansive mappings includes a class of nonexpansive mappings as a proper subclass ([14]).

In 1993, Bruck, Kuczumow and Reich [4] introduced a notion of asymptotically nonexpansive mapping in the intermediate sense. A mapping $T : C \to X$ is said to be asymptotically nonexpansive in the intermediate sense if $T$ is uniformly continuous and the following inequality holds:

$$
\limsup_{n \to \infty} \sup_{x, y \in C} \{d(T^n x, T^n y) - d(x, y)\} \leq 0.
$$

The concept of asymptotically nonexpansive mappings in the intermediate sense is more general than that of asymptotically nonexpansive mappings.

One of the fundamental and celebrated results in the theory of nonexpansive mappings is Browder’s demiclosed principle [3] which states that if $C$ is a nonempty closed convex subset of a uniformly convex Banach space $X$ and...
$T : C \to X$ is a nonexpansive mapping, then $I - T$ is demiclosed at each $y \in X$, that is, for any sequence $\{x_n\}$ in $C$ with $x_n \to x$ weakly and $(I - T)x_n \to y$ strongly give $(I - T)x = y$, where $I$ is an identity mapping on $X$.

It is well known that the demiclosed principle plays an important role in studying the asymptotic behavior for nonexpansive mappings (see, for details [1, 13, 16]).

The demiclosed principle for the class of mappings which is essentially wider than that of nonexpansive mappings in the setting of Banach spaces has been studied by several authors (see, for example, [1, 24, 29]). Several papers have been appeared on the iterative approximation of fixed points of nonexpansive, asymptotically nonexpansive and asymptotically nonexpansive mappings in the intermediate sense using Halpern, Mann, Ishikawa iterations in the framework of Hilbert and Banach spaces.

Xu and Ori [30] introduced an implicit iteration process involving finite family $\{T_i : i = 1, 2, \ldots, N\}$ of nonexpansive mappings and proved a weak convergence theorem in Hilbert space and posed the following question:

What conditions on the mappings $\{T_i : i = 1, 2, \ldots, N\}$ and or the control parameter are sufficient to guarantee strong convergence of the sequence.

Several papers deal with an attempt to answer this question in the setting of Hilbert and Banach spaces (see [7, 8, 26, 27] and the references therein).

In recent years, CAT(0) spaces have attracted the attention of many authors as they have played a very important role in different aspects of geometry ([12]). Kirk [18, 19] showed that a nonexpansive mapping defined on a bounded closed convex subset of a complete CAT(0) space has a fixed point. Since then, the fixed point theory in CAT(0) spaces has been rapidly developed. For further details, we refer to [9, 10, 11, 17, 21] and references mentioned therein.

The concept of $\Delta$-convergence in general metric spaces was coined by Lim [22]. Kirk and Panyanak [21] specialized this concept to CAT(0) spaces and showed that many Banach space results involving weak convergence have precise analogs in this setting. Dhompongsa and Panyanak [11] continued to work in this direction. Their results involved Mann and Ishikawa iteration schemes involving one mapping.

Motivated by the work of Dhompongsa and Panyanak [11], the purpose of this paper is two fold: We investigate demiclosedness principle, existence theorem, and the structure of fixed point set of asymptotically nonexpansive mapping in the intermediate sense in the framework of CAT(0) spaces. We also study necessary conditions for $\Delta$ and strong convergence of a sequence generated by finite family of asymptotically nonexpansive mappings in the intermediate sense in CAT(0) spaces.

2. Preliminaries

Let $(X, d)$ be a metric space. A geodesic from $x$ to $y$ in $X$ is a map $c$ from a closed interval $[0, l] \subset \mathbb{R}$ to $X$ such that $c(0) = x$, $c(l) = y$, and
For any \( x, y \in X \) and \( \alpha \in [0, 1] \), there exists a unique point \( z \in [x, y] \) such that

\[
(2.2) \quad d(x, z) = \alpha d(x, y) \quad \text{and} \quad d(y, z) = (1 - \alpha) d(x, y).
\]

Notation \((1 - \alpha)x \oplus ty\) is used for the unique point \( z \) satisfying (2.2).
Lemma 2.2 ([11, Lemma 2.4]). For \( x, y, z \in X \) and \( \alpha \in [0, 1] \), we have

\[
d(z, \alpha x \oplus (1 - \alpha)y) \leq \alpha d(z, x) + (1 - \alpha) d(z, y) \quad \forall z \in X.
\]

Let \( \{x_n\} \) be a bounded sequence in a closed convex subset \( C \) of a CAT(0) space \( X \). For \( x \in X \), set

\[
r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).
\]

The asymptotic radius \( r(\{x_n\}) \) of \( \{x_n\} \) is given by

\[
r(\{x_n\}) = \inf \{ r(x, \{x_n\}) \mid x \in X \}
\]

and the asymptotic center \( A(\{x_n\}) \) of \( \{x_n\} \) is the set

\[
A(\{x_n\}) = \{ x \in X : r(\{x_n\}) = r(x, \{x_n\}) \}.
\]

It is known that, in a CAT(0) space, \( A(\{x_n\}) \) consists of exactly one point [10, Proposition 7].

We now recall the definition of \( \Delta \)-convergence and weak convergence \((\rightharpoonup)\) in CAT(0) space.

**Definition 2.3** ([21]). A sequence \( \{x_n\} \) in a CAT(0) space \( X \) is said to \( \Delta \)-converge to \( x \in X \) if \( x \) is the unique asymptotic center of \( \{u_n\} \) for every subsequence \( \{u_n\} \) of \( \{x_n\} \).

In this case we write \( \Delta \)-lim \( x_n = x \) and call \( x \) the \( \Delta \)-limit of \( \{x_n\} \).

Recall that a bounded sequence \( \{x_n\} \) in \( X \) is said to be regular if \( r(\{x_n\}) = r(\{u_n\}) \) for every subsequence \( \{u_n\} \) of \( \{x_n\} \). In the Banach space it is known that, every bounded sequence has a regular subsequence [15, Lemma 15.2].

Since in a CAT(0) space every regular sequence \( \Delta \)-converges, we see that every bounded sequence in \( X \) has a \( \Delta \)-convergent subsequence, also it is noticed that [21, p. 3690].

**Lemma 2.4.** Given \( \{x_n\} \subset X \) such that \( \{x_n\} \) \( \Delta \)-converges to \( x \) and given \( y \in X \) with \( y \neq x \), then

\[
\limsup_n d(x_n, x) < \limsup_n d(x_n, y).
\]

In Banach space above condition is known as the Opial’s property.

Now, recall the definition of weak convergence in CAT(0) space.

**Definition 2.5** ([17]). Let \( C \) be a closed convex subset of a CAT(0) space \( X \). A bounded sequence \( \{x_n\} \) in \( C \) is said to converge weakly to \( \omega \in C \) if and only if \( \Phi(w) = \inf_{x \in C} \Phi(x) \), where \( \Phi(x) = \limsup_{n \to \infty} d(x_n, x) \).

Note that \( \{x_n\} \rightharpoonup w \) if and only if \( A_C (\{x_n\}) = \{w\} \).

Nanjaras and Panyanak [23] established following relation between \( \Delta \)-convergence and weak convergence in a CAT(0) space:
**Lemma 2.6** ([23, Proposition 3.12]). Let \( \{x_n\} \) be a bounded sequence in a \( \text{CAT}(0) \) space \( E \) and let \( C \) be a closed convex subset of \( X \) which contains \( \{x_n\} \). Then

(i) \( \Delta - \lim_n x_n = x \) implies \( \{x_n\} \to x \),

(ii) The converse of (i) is true if \( \{x_n\} \) is regular.

We now recall some results which play crucial role to prove the main results:

**Lemma 2.7** ([11, Lemma 2.8]). If \( \{x_n\} \) is a bounded sequence in \( X \) with \( A(\{x_n\}) = \{x\} \) and \( \{u_n\} \) is a subsequence of \( \{x_n\} \) with \( A(\{u_n\}) = \{u\} \) and the sequence \( \{d(x_n, u)\} \) converges, then \( x = u \).

**Lemma 2.8** ([9, Proposition 2.1]). If \( C \) be a closed convex subset of \( X \) and if \( \{x_n\} \) is a bounded sequence in \( C \), then the asymptotic center of \( \{x_n\} \) is in \( C \).

**Lemma 2.9** (Tan and Xu [28]). Suppose \( \{a_n\} \) and \( \{b_n\} \) are two sequences on nonnegative numbers such that

\[
a_{n+1} \leq a_n + b_n \quad \text{for all} \quad n \geq 1,
\]

and \( \sum b_n \) converges, then \( \lim_{n \to \infty} a_n \) exists.

We now define a modified implicit iteration process for finite family of asymptotically nonexpansive mapping in intermediate sense as below:

Let \( C \) be a nonempty closed convex subset of a \( \text{CAT}(0) \) space \( X \), and \( \{T_1, T_2, \ldots, T_N\} \) be a finite family of \( N \) asymptotically nonexpansive self mappings on \( C \) in the intermediate sense. We generate the sequence \( \{x_n\} \subset C \) by

\[
x_0 \in K, \quad x_1 = (1 - \alpha_1)x_0 \oplus \alpha_1 T_1 x_1, \quad x_2 = (1 - \alpha_2)x_1 \oplus \alpha_2 T_2 x_2, \\
\vdots \\
x_N = (1 - \alpha_N)x_{N-1} \oplus \alpha_N T_N x_N, \\
x_{N+1} = (1 - \alpha_{N+1})x_N \oplus \alpha_{N+1} T_1^k x_{N+1}, \\
\vdots \\
x_{2N} = (1 - \alpha_{2N})x_{2N-1} \oplus \alpha_{2N} T_N^k x_{2N}, \\
x_{2N+1} = (1 - \alpha_{2N+1})x_{2N} \oplus \alpha_{2N+1} T_1^k x_{2N+1}, \\
\vdots
\]

where \( \{\alpha_n\} \) is an appropriate sequence in \([0, 1]\).

We can write down above table in the following compact form:

\[
x_n = (1 - \alpha_n)x_{n-1} \oplus \alpha_n T_{i(n)}^{k(n)} \quad \text{for all} \quad n \geq 1,
\]

where \( n = (k(n) - 1)N + i(n) \), \( k(n) \geq 1 \) is a positive integer such that \( k(n) \to \infty \) and \( I \) is the index set \( \{1, 2, \ldots, N\} \).
3. Main results

Denote by $Fix(T)$ the set of fixed points of $T$, that is, $Fix(T) = \{x \in C : Tx = x\}$. Put,
\[
C_n = \sup_{x,y \in C} (d(T^n x, T^n y) - d(x, y)) \vee 0.
\]

**Theorem 3.1.** Let $X$ be a complete CAT(0) space, $C$ a nonempty closed convex subset of $X$. If $T : C \to C$ is an asymptotically nonexpansive mapping in the intermediate sense, then has a fixed point.

**Proof.** Let $x \in C$. Define
\[
\Phi(u) = \limsup_{n \to \infty} d(T^n x, u) \text{ for any } u \in C.
\]
Then, we have
\[
d(T^{n+m} x, T^m u) \leq d(T^n x, u) + C_m
\]
for any $n, m \geq 1$. On taking limit as $n \to \infty$, we obtain
\[
(3.1) \quad \Phi(T^m u) \leq \Phi(u) + C_m
\]
for any $m \geq 1$.

Let $w \in C$ be such that $\Phi(w) = \inf\{\Phi(u) : u \in C\} = \Phi_0$. Then by (3.1), for any $n \geq 1$, we have
\[
(3.2) \quad \Phi(T^n w) \leq \Phi(w) + C_n = \Phi_0 + C_n.
\]

Using inequality (2.1), we obtain
\[
d\left(\frac{T^m x + T^h w}{2}, \frac{T^m w + T^h w}{2}\right)^2 \\
\leq \frac{1}{2} d(T^m x, T^m w)^2 + \frac{1}{2} d(T^m x, T^h w)^2 - \frac{1}{4} d(T^m w, T^h w)
\]
which on taking limit as $n \to \infty$ gives
\[
\Phi_0^2 \leq \Phi\left(\frac{T^m w + T^h w}{2}\right)^2 \leq \frac{1}{2} \Phi(T^m w)^2 + \frac{1}{2} \Phi(T^h w)^2 - \frac{1}{4} d(T^m w, T^h w)^2.
\]

Using (3.2), we have
\[
d\left(T^m w, T^h w\right)^2 \leq 2 (\Phi_0 + C_m)^2 + 2 (\Phi_0 + C_h)^2 - 4 \Phi_0^2.
\]
As $T$ is asymptotically nonexpansive mapping in the intermediate sense, so
\[
\limsup_{m,h \to \infty} (T^m w, T^h w) \leq 0,
\]
which implies that $\{T^n w\}$ is a Cauchy sequence. Let $v = \lim_{n \to \infty} T^n w$. By uniform continuity of $T$, we get $Tv = v$. □

**Theorem 3.2.** Let $X$ be a complete CAT(0) space and $C$ be a nonempty closed convex subset of $X$. If $T : C \to C$ is an asymptotically nonexpansive mapping in the intermediate sense, then $Fix(T)$ is closed and convex.
Proof. As $T$ is continuous, so $\text{Fix}(T)$ is closed. In order to prove $\text{Fix}(T)$ is convex, it is enough to prove that $\frac{x+y}{2} \in \text{Fix}(T)$ whenever $x,y \in \text{Fix}(T)$. Set $w = \frac{x+y}{2}$. For any $n \geq 1$, we have
\[
d(T^n w, w)^2 = d \left( T^n w, \frac{x+y}{2} \right)^2 \leq \frac{1}{2} d(x, T^n w)^2 + \frac{1}{2} d(y, T^n w)^2 - \frac{1}{4} d(x, y)^2.
\]
(3.3)
Using (2.3), we obtain
\[
d(x, T^n w)^2 = d \left( T^n x, T^n w \right)^2 \leq \left\{ d(w, x) + C_n \right\}^2 \leq \left\{ \frac{1}{2} d(x, y) + C_n \right\}^2.
\]
(3.4)
Similarly,
\[
d(y, T^n w)^2 \leq \left\{ \frac{1}{2} d(x, y) + C_n \right\}^2.
\]
(3.5)
From (3.3)-(3.5), we get
\[
d(T^n w, w)^2 \leq C_n (C_n + d(x, y))
\]
for any $n \geq 1$. Hence $\lim_{n \to \infty} T^n w = w$, and $T w = w \in \text{Fix}(T)$. \qed

We now prove demiclosed principle for asymptotically nonexpansive mapping in intermediate sense.

**Proposition 3.3.** Let $C$ be a closed convex subset of a complete CAT(0) space $X$ and $T : C \to C$ be an asymptotically nonexpansive mapping in the intermediate sense. If $\{x_n\}$ is a bounded sequence in $C$ such that $\lim_{n \to \infty} d(x_n, Tx_n) = 0$ and $\{x_n\} \to w$, then $T w = w$.

**Proof.** Define
\[
\Phi(x) = \limsup_{n \to \infty} d(T^m x_n, x) \text{ for all } x \in C \text{ and } m \geq 1.
\]
Then as observed in (3.1), we have
\[
\Phi(T^m w) \leq \Phi(w) + C_m \text{ for all } m \geq 1 \text{ and } w \in C.
\]
Hence
\[
\limsup_{m \to \infty} \Phi(T^m w) \leq \Phi(w).
\]
(3.6)
Using inequality (2.1), we have
\[ d\left(\frac{T^m x_n + w \oplus T^m w}{2}\right)^2 \leq \frac{1}{2} d(T^m x_n, w)^2 + \frac{1}{2} d(T^m x_n, T^m w)^2 - \frac{1}{4} d(w, T^m w)^2 \]
for all \( n, m \geq 1 \). By taking limit as \( n \to \infty \), we get
\[ \Phi\left(\frac{w \oplus T^m w}{2}\right)^2 \leq \frac{1}{2} \Phi(w)^2 + \frac{1}{2} \Phi(T^m w)^2 - \frac{1}{4} d(w, T^m w)^2 \]
for any \( m \geq 1 \). As \( \{x_n\} \rightharpoonup w \), so \( A(\{x_n\}) = \{w\} \) and we have
\[ \Phi(w)^2 \leq \Phi\left(\frac{w \oplus T^m w}{2}\right)^2 \leq \frac{1}{2} \Phi(w)^2 + \frac{1}{2} \Phi(T^m w)^2 - \frac{1}{4} d(w, T^m w)^2. \]
That is,
\[ 4\Phi(w)^2 \leq 2\Phi(w)^2 + 2\Phi(T^m w)^2 - d(w, T^m w)^2 \]
for any \( m \geq 1 \), which implies that
\[ d(w, T^m w)^2 \leq 2\Phi(T^m w)^2 - 2\Phi(w)^2. \]
By (3.6) and (3.7), we have
\[ \lim_{m \to \infty} d(w, T^m w) = 0 \]
and \( T w = w \). \( \square \)

In the light of Lemma 2.6, we get following result from Proposition 3.3.

**Corollary 3.4.** Let \( C \) be a closed convex subset of a \( \text{CAT}(0) \) space \( X \), and \( T : C \to C \) be any asymptotically nonexpansive mapping in the intermediate sense. If bounded sequence \( \{x_n\} \) in \( C \) \( \Delta \)-converges to \( x \) and \( d(x_n, T x_n) \to 0 \), then \( x \in C \) and \( T x = x \).

Now we prove the following lemmas needed in the sequel.

**Lemma 3.5.** Let \( C \) be a nonempty closed convex subset of a complete \( \text{CAT}(0) \) space \( X \) and \( \{T_1, T_2, \ldots, T_N\} : C \to C \) be \( N \) asymptotically nonexpansive mappings in the intermediate sense with
\[ D_n = \max \left\{ \max_{1 \leq j \leq N} \sup_{x,y \in C} \left( d(T_j^m x, T_j^m y) - d(x, y) \right), 0 \right\} \]
such that \( \sum_{n=1}^\infty D_n < \infty \). Suppose that \( x_0 \in C \), \( \{\alpha_n\} \) is a real sequence in \([a, b]\) for some \( a, b \in [0, 1] \), and \( \{x_n\} \) be the sequence defined by (2.5). If \( F = \bigcap_{i=1}^N \text{Fix}(T_i) \neq \emptyset \), then
(i) \( \lim_{n \to \infty} d(x_n, p) \) exists for all \( p \in F \).
(ii) \( \lim_{n \to \infty} d(x_n, F) \) exists.
Proof. Take $p \in Fix(T)$ and applying inequality (2.3), we have
\[
d(x_n, p) = d((1 - \alpha_n)x_{n-1} \oplus \alpha_nT^k_{i(n)}x_n, p) \\
\leq (1 - \alpha_n)d(x_{n-1}, p) + \alpha_n d(T^k_{i(n)}x_n, p) \\
\leq (1 - \alpha_n)d(x_{n-1}, p) + \alpha_n (D_n + d(x_n, p)) \\
= d(x_{n-1}, p) + \frac{\alpha_n}{1 - \alpha_n}D_n \\
\leq d(x_{n-1}, p) + \frac{b}{1 - b}D_n.
\]
(3.8)

Taking infimum over all $p \in F$, we have
\[
d(x_n, F) \leq d(x_{n-1}, F) + \frac{b}{1 - b}D_n.
\]
(3.9)

It follow from Lemma 2.9, and (3.8) and (3.9) that $\lim_{n \to \infty} d(x_n, p)$ and $\lim_{n \to \infty} d(x_n, F)$ exists. \(\square\)

Lemma 3.6. Let $C$ be a nonempty closed convex subset of a complete CAT(0) space $X$ and $\{T_1, T_2, \ldots, T_N\} : C \to C$ be $N$ asymptotically nonexpansive mappings in the intermediate sense with
\[
D_n = \max \left\{ \max_{1 \leq i \leq N} \sup_{x, y \in C} \left( d\left(T^n_j x, T^n_j y\right) - d(x, y)\right), 0 \right\}
\]
such that $\sum_{n=1}^{\infty} D_n < \infty$. Suppose that $x_0 \in C$, $\{\alpha_n\}$ is a real sequence in $[a, b]$ for some $a, b \in (0, 1)$, and $\{x_n\}$ be the sequence defined by (2.5). If $F = \bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$, then $\lim_{n \to \infty} d(x_n, T_i x_n) = 0$ for all $i \in \{1, 2, \ldots, N\}$.

Proof. Using (2.5) and (2.1), we have
\[
d^2(x_n, p) = d^2\left( (1 - \alpha_n)x_{n-1} \oplus \alpha_n T^k_{i(n)}x_n, p \right) \\
\leq \alpha_n d^2\left(T^k_{i(n)}x_n, p\right) + (1 - \alpha_n) d^2(x_{n-1}, p) \\
- \alpha_n(1 - \alpha_n) d^2\left(T^k_{i(n)}x_n, x_{n-1}\right) \\
\leq \alpha_n \left[D_k(n) + d(x_n, p)\right]^2 + (1 - \alpha_n) d^2(x_{n-1}, p) \\
- \alpha_n(1 - \alpha_n) d^2\left(T^k_{i(n)}x_n, x_{n-1}\right) \\
= \frac{\alpha_n}{1 - \alpha_n} \left[D^2_k(n) + 2D_k(n)d(x_n, p)\right] + d^2(x_{n-1}, p) \\
- \alpha_n d^2\left(T^k_{i(n)}x_n, x_{n-1}\right)
\]
which on simplification implies that
\[
d^2 \left( T^{(n)}_{i(n)} x_n, x_{n-1} \right)
\leq \frac{1}{1 - \alpha_n} \left[ \sigma^2 + 2D_{k(n)} d(x_n, p) \right] + \frac{1}{\alpha_n} \left[ d^2(x_{n-1}, p) - d^2(x_n, p) \right]
\leq \frac{1}{1 - \alpha_n} \left[ \sigma^2 + 2D_{k(n)} d(x_n, p) \right] + \frac{1}{\alpha_n} \left[ d^2(x_{n-1}, p) - d^2(x_n, p) \right].
\]

Since \( D_n \to 0 \) as \( n \to \infty \) and \( d(x_n, p) \) is convergent, therefore on taking limit as \( n \to \infty \), we get
\[
\lim_{n \to \infty} d \left( T^{(n)}_{i(n)} x_n, x_{n-1} \right) = 0.
\]

Moreover
\[
d(x_n, x_{n-1}) \leq d \left( (1 - \alpha_n) x_{n-1} + \alpha_n T^{(n)}_{i(n)} x_n, x_{n-1} \right)
\leq (1 - \alpha_n) d(x_{n-1}, x_{n-1}) + \alpha_n d \left( T^{(n)}_{i(n)} x_n, x_{n-1} \right)
\]
implies
\[
\lim_{n \to \infty} d(x_n, x_{n-1}) = 0.
\]

From (3.10) and (3.11), we have
\[
\frac{1}{n-\infty} d \left( T^{(n)}_{i(n)} x_n, x_{n-1} \right) \leq \lim_{n \to \infty} \left\{ d \left( T^{(n)}_{i(n)} x_n, x_{n-1} \right) + d(x_n, x_{n-1}) \right\} = 0
\]
and
\[
\lim_{n \to \infty} d(x_n, x_{n+j}) = 0 \quad \forall j = 1, 2, \ldots.
\]

Letting \( \sigma_n = d \left( T^{(n)}_{i(n)} x_n, x_{n} \right) \), by (3.12), we have \( \sigma_n \to 0 \).

Since for each \( n > N \), we have \( n = (k(n) - 1) + i(n) \) where \( i(n) \in \{1, 2, \ldots, N\} \), also \( i(n) = i(n + N) \) and \( k(n) + 1 = k(n + N) \), so we have
\[
d(x_n, T_n x_n) \leq d(x_n, x_{n+N}) + d \left( x_{n+N}, T^{(n+1)}_{i(n)} x_{n+1} \right)
+ d \left( T^{(n+1)}_{i(n)} x_{n+1}, T^{(n+1)}_{i(n)} x_{n+N} \right) + d \left( T^{(n+1)}_{i(n)} x_{n+N}, T_{i(n)} x_{n+N} \right)

= d(x_n, x_{n+N}) + d \left( x_{n+N}, T^{(n+1)}_{i(n+N)} x_{n+N} \right)
+ d \left( T^{(n+1)}_{i(n+N)} x_{n+N}, T_{i(n+N)} x_{n+N} \right) + d \left( T^{(n+1)}_{i(n+N)} x_{n+N}, T_{i(n)} x_{n+N} \right)
\]
\[
\leq \sigma_{n+N} + D_{k(n+N)} + 2d(x_n, x_{n+N}) + d \left( T^{(n+1)}_{i(n+N)} x_{n+N}, T_{i(n+N)} x_{n+N} \right)
\]
\[
\to 0 \quad \text{as} \quad n \to \infty
\]
by (3.12), (3.13) and uniform continuity of \( T_{i(n)} \).
Consequently, for any $j = 1, 2, \ldots, N$, from (3.13), (3.14) and uniform continuity of $T_{i(n)}$, we have
\[
d(x_n, T_{n+j} x_n) \leq d(x_n, x_{n+j}) + d(x_{n+j}, T_{n+j} x_{n+j}) + d(T_{n+j} x_{n+j}, T_{n+j} x_n)
\rightarrow 0 \quad \text{as } n \rightarrow \infty.
\]
This implies that the sequence
\[
\bigcup_{j=1}^{N} \{ d(x_n, T_{n+j} x_n) \}_{n=1}^{\infty} \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\]
Since for each $l = 1, 2, \ldots, N$, \( \{ d(x_n, T_{l} x_n) \}_{n=1}^{\infty} \) is a subsequence of \( \bigcup_{j=1}^{N} \{ d(x_n, T_{n+j} x_n) \}_{n=1}^{\infty} \), therefore we have
\[
\lim_{n \rightarrow \infty} d(x_n, T_{l} x_n) = 0 \quad \text{for all } l = 1, 2, \ldots, N. \quad \square
\]
Now we prove the $\Delta$-convergence and strong convergence results.

**Theorem 3.7.** Let $C$ be a nonempty closed convex subset of a complete CAT(0) space $X$ and \( \{ T_{1}, T_{2}, \ldots, T_{N} \} : C \rightarrow C \) be $N$ asymptotically nonexpansive mappings in the intermediate sense with
\[
D_n = \max \left\{ \max_{1 \leq j \leq N} \sup_{x, y \in C} (d(T_{j}^{n} x, T_{j}^{n} y) - d(x, y)), 0 \right\}
\]
such that \( \sum_{n=1}^{\infty} D_n < \infty \). Suppose that $x_0 \in C$, \( \{ \alpha_n \} \) is a real sequence in $[a,b]$ for some $a, b \in (0,1)$, and \( \{ x_n \} \) be the sequence defined by (2.5). If \( F = \bigcap_{n=1}^{N} \text{Fix}(T_{i}) \neq \emptyset \), then \( \{ x_n \} \) is \( \Delta \)-convergent to an element of $F$.

**Proof.** We first show that \( w_{w}(\{ x_n \}) \subseteq F \). Let \( u \in w_{w}(\{ x_n \}) \), then there exists a subsequence \( \{ u_n \} \) of \( \{ x_n \} \) such that \( A(\{ x_n \}) = \{ u \} \). By Lemma 2.8, there exists a subsequence \( \{ v_n \} \) of \( \{ u_n \} \) such that \( \Delta \text{-lim} v_n = v \in C \). By Corollary 3.4, \( v \in F(T_{l}) \). By arbitrariness of $l \in \{ 1, 2, \ldots, N \}$, we have \( v \in F \). By Lemma 3.5 \( \lim_{n \rightarrow \infty} d(x_n, v) \) exists so by Lemma 2.7, we have \( u = v \), i.e., \( w_{w}(\{ x_n \}) \subseteq F \).

To show that \( \{ x_n \} \) \( \Delta \)-converges to a point in $F$, it is enough to show that \( w_{w}(\{ x_n \}) \) consists of exactly one point.

Let \( \{ u_n \} \) be a subsequence of \( \{ x_n \} \) with \( A(\{ u_n \}) = \{ u \} \) and let \( A(\{ x_n \}) = \{ x \} \) for some \( u \in w_{w}(\{ x_n \}) \subseteq F \) and \( \{ d(x_n, v) \} \) converges. By Lemma 2.7, we have \( x = v \in F \). Thus \( w_{w}(\{ x_n \}) = \{ x \} \).

This completes the proof. \( \square \)

**Theorem 3.8.** Let $C$ be a nonempty closed convex subset of a complete CAT(0) space $X$ and \( \{ T_{1}, T_{2}, \ldots, T_{N} \} : C \rightarrow C \) be $N$ asymptotically nonexpansive mappings in the intermediate sense with
\[
D_n = \max \left\{ \max_{1 \leq j \leq N} \sup_{x, y \in C} (d(T_{j}^{n} x, T_{j}^{n} y) - d(x, y)), 0 \right\}
\]
such that $\sum_{n=1}^{\infty} D_n < \infty$. Suppose that $x_0 \in C$, $\{\alpha_n\}$ is a real sequence in $[a, b]$ for some $a, b \in (0, 1)$, and $\{x_n\}$ be the sequence defined by (2.5). Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i : i = 1, 2, \ldots, N\}$ if and only if $\lim \inf_{n \to \infty} d(x_n, F) = 0$.

**Proof.** Necessity is obvious.

To prove the sufficiency, suppose that $\lim \inf_{n \to \infty} d(x_n, F) = 0$, also $\lim_{n \to \infty} d(x_n, F)$ exists by Lemma 3.5(ii). Hence by hypothesis $\lim_{n \to \infty} d(x_n, F) = 0$.

Thus for arbitrary $\varepsilon > 0$, there exists a positive integer $n_1$ such that for all $n \geq n_1$, $d(x_n, F) < \frac{\varepsilon}{8}$.

In particular, $\inf \{d(x_{n1}, p) : p \in F\} < \frac{\varepsilon}{8}$.

Thus there must exist $p^* \in F$ such that $d(x_{n1}, p^*) < \frac{\varepsilon}{4}$.

Furthermore $\sum_{n=1}^{\infty} D_n < \infty$ implies that there exists a positive integer $n_2$ such that $\sum_{n=1}^{\infty} D_n < \frac{\varepsilon}{4M}$ for some positive integer $M$. Let $n_0 = \max\{n_1, n_2\}$.

For each $m, n \geq n_0$, it follows from (3.8), that

$$d(x_n, x_m) \leq d(x_n, p^*) + d(x_m, p^*)$$

$$\leq d(x_{n0}, p^*) + M \sum_{n_0+1}^{n} D_j + d(x_{n0}, p^*) + M \sum_{n_0+1}^{m} D_j$$

$$\leq 2d(x_{n0}, p^*) + 2M \sum_{n_0+1}^{n} D_j$$

$$< \varepsilon.$$  

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset $C$ of a complete CAT(0) space and so converges to some $q \in C$. Since $\lim_{n \to \infty} d(x_n, F) = 0$ we get $d(q, F) = 0$, closedness of $F$ gives that $q \in F$.

A mapping $T : C \to C$ is called semi-compact if any sequence $\{x_n\}$ in $C$ satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$ has a convergent subsequence.

A mapping $T : C \to C$ is said to satisfy Condition-I if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > r$ for all $r \in (0, \infty)$ such that for all $x \in C$

$$\|x - Tx\| \geq f(d(x, F(T))),$$

where $d(x, F(T)) = \inf\{\|x - x^*\| : x^* \in F(T) \neq \emptyset\}$. Condition-I was introduced by Senter and Dotson [25].

We now give strong convergence result employing Condition-I.
**Theorem 3.9.** Let $C$ be a nonempty closed convex subset of a complete CAT(0) space $X$ and $\{T_1, T_2, \ldots, T_N\} : C \to C$ be $N$ asymptotically nonexpansive mappings in the intermediate sense with

$$D_n = \max \left\{ \max_{1 \leq j \leq N} \sup_{x, y \in C} \left( d(T_j^n x, T_j^n y) - d(x, y) \right), 0 \right\}$$

such that $\sum_{n=1}^{\infty} D_n < \infty$. Suppose that $x_0 \in C$, $\{\alpha_n\}$ is a real sequence in $[a, b]$ for some $a, b \in (0, 1)$, and $\{x_n\}$ be the sequence defined by (2.5). If at least one of the mappings in $\{T_1, T_2, \ldots, T_N\}$ satisfies Condition-I, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i : i = 1, 2, \ldots, N\}$.

**Proof.** By Lemma 3.6, we see for $x^* \in F$ that

$$\lim_{n \to \infty} d(x_n, x^*) \text{ and } \lim_{n \to \infty} d(x_n, F) \text{ exist.}$$

Let one of $T_i$’s, say $T_s$, $s \in \{1, 2, \ldots, N\}$ satisfy Condition-I. By Lemma 3.6, $\lim_{n \to \infty} d(x_n, T_s x_n) = 0$, so we have $\lim_{n \to \infty} d(x_n, F) = 0$. By the nature of $F$ and the fact that $\lim_{n \to \infty} d(x_n, F)$ exists, we have $\lim_{n \to \infty} d(x_n, F) = 0$ and the result follows from Theorem 3.8. \[\square\]

**Theorem 3.10.** Let $C$ be a nonempty closed convex subset of a complete CAT(0) space $X$ and let $\{T_1, T_2, \ldots, T_N\} : C \to C$ be $N$ asymptotically nonexpansive mappings in the intermediate sense with

$$D_n = \max \left\{ \max_{1 \leq j \leq N} \sup_{x, y \in C} \left( d(T_j^n x, T_j^n y) - d(x, y) \right), 0 \right\}$$

such that $\sum_{n=1}^{\infty} D_n < \infty$. Suppose that $x_0 \in C$, $\{\alpha_n\}$ is a real sequence in $[a, b]$ for some $a, b \in (0, 1)$, and $\{x_n\}$ be the sequence defined by (2.5). If any one of the mappings in $\{T_1, T_2, \ldots, T_N\}$ is semicompact, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i : i = 1, 2, \ldots, N\}$.

**Proof.** Suppose that $T_{i_0}$ is semi-compact for some $i_0 \in \{1, 2, \ldots, N\}$. By Lemma 3.6, we have $\lim_{n \to \infty} d(x_n, T_{i_0} x_n) = 0$. So there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j \to \infty} x_{n_j} \to p \in C$. Now Lemma 3.6 guarantees that $\lim_{n \to \infty} d(x_{n_j}, T_{i_0} x_{n_j}) = 0$ for all $l \in \{1, 2, \ldots, N\}$ and so $d(p, T_{i_0} p) = 0$ for all $l \in \{1, 2, \ldots, N\}$. This implies that $p \in F$. Since $\lim_{n \to \infty} d(x_n, F) = 0$, if follows, as in the proof of Theorem 3.8, that $\{x_n\}$ converges strongly to some common fixed point in $F$.

This completes the proof. \[\square\]

**Remark 3.11.** Any CAT($\kappa$) space is a CAT($\kappa'$) space for every $\kappa' > \kappa$ [2, page 165], therefore the results in this paper can be applied to any CAT($\kappa$) space with $\kappa < 0$. 
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