A CERTAIN SUBCLASS OF JANOWSKI TYPE FUNCTIONS ASSOCIATED WITH \( k \)-SYMMETRIC POINTS

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Abstract. We introduce a subclass \( S^k(A, B) \ (−1 \leq B < A \leq 1) \) of functions which are analytic in the open unit disk and close-to-convex with respect to \( k \)-symmetric points. We give some coefficient inequalities, integral representations and invariance properties of functions belonging to this class.

1. Introduction

Let \( \mathcal{A} \) denote the class of functions which are analytic in the open unit disk \( U \) and normalized by \( f(0) = 0 \) and \( f'(0) = 1 \). Also let \( \mathcal{S} \) denote the subclass of \( \mathcal{A} \) consisting of all functions which are univalent in \( U \).

Let \( f(z) \) and \( F(z) \) be analytic in \( U \). Then we say that the function \( f(z) \) is subordinate to \( F(z) \) in \( U \), if there exists an analytic function \( w(z) \) in \( U \) such that \(|w(z)| \leq 1 \) and \( f(z) = F(w(z)) \), denote by \( f \prec F \) or \( f(z) \prec F(z) \). If \( F(z) \) is univalent in \( U \), then the subordination is equivalent to \( f(0) = F(0) \) and \( f(U) \subset F(U) \).

Now, we denote by \( \mathcal{S}^*(A, B) \) and \( \mathcal{C}(A, B) \) the subclasses of \( \mathcal{A} \) as follows:

\[ \mathcal{S}^*(A, B) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} < \frac{1 + Az}{1 + Bz}, z \in U \right\} \]

and

\[ \mathcal{C}(A, B) = \left\{ f \in \mathcal{A} : \exists g \in \mathcal{S}^*(A, B) \text{ such that } \frac{zf'(z)}{g(z)} < \frac{1 + Az}{1 + Bz}, z \in U \right\}, \]

respectively. For \( A = 1 - 2\alpha \) and \( B = -1 \) in (1) and (2), we can obtain the classes \( \mathcal{S}^*(1 - 2\alpha, -1) = \mathcal{S}^*(\alpha) \) and \( \mathcal{C}(1 - 2\alpha, -1) = \mathcal{C}(\alpha) \), consisting of functions which are starlike of order \( \alpha \) and close-to-convex of order \( \alpha \), respectively. Especially, we can obtain the classes \( \mathcal{S}^*(1, -1) = \mathcal{S}^* \) and \( \mathcal{C}(1, -1) = \mathcal{C} \) which

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are the classes of starlike functions and close-to-convex functions, respectively, for $A = 1$ and $B = -1$.

Sakaguchi [6] once introduced a classes $S^*_s$ of functions starlike with respect to symmetric points, it consists of functions $f(z) \in S$ satisfying

$$\text{Re} \left\{ \frac{zf'(z)}{f(z) - f(-z)} \right\} > 0 \quad (z \in U).$$

Following him, many authors discussed this class and its subclasses (see [4], [5], [7], [8], [10], [11], [12] and [13]). In the present paper, we introduced the following class of analytic functions with respect to $k$-symmetric points, and obtain some interesting results.

**Definition.** Let $S^{(k)}_s(A, B)$ denote the class of functions in $S$ satisfying the inequality

$$\left| \frac{zf'(z)}{f_k(z)} - 1 \right| < \left| A - B \frac{zf'(z)}{f_k(z)} \right| \quad (z \in U),$$

where $-1 \leq A < B \leq 1$, $k \geq 1$ is a fixed positive integer and $f_k(z)$ is defined by the following equality

$$f_k(z) = \frac{1}{k} \sum_{\mu=0}^{k-1} \varepsilon^{-\mu} f(\varepsilon^\mu z),$$

where $\varepsilon = \exp(\frac{2\pi i}{k})$ with $k \in \mathbb{Z}$.

By the definition of $f_k(z)$, we can easily obtain the expansion of $f_k(z)$. That is, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then

$$f_k(z) = z + \sum_{n=2}^{\infty} \sigma_k(n) a_n z^n,$$

where $\sigma_k(n) = \begin{cases} 1, n = lk + 1 \\ 0, n \neq lk + 1 \end{cases} \quad (l \in \mathbb{N}_0)$. And we note that $f_1(z) = f(z)$ and $f_2(z) = \frac{1}{2}(f(z) - f(-z))$.

Now the following identities follow directly from the above definition [3]:

$$f_k(\varepsilon^\mu z) = \varepsilon^\mu f_k(z),$$

$$f_k'(\varepsilon^\mu z) = f_k(z) = \frac{1}{k} \sum_{\mu=0}^{k-1} f'(\varepsilon^\mu z).$$

**Remark 1.1.** Using the definition of the subordination, we can easily obtain that the equivalent condition of belonging to the class $S^{(k)}_s(A, B)$ ($-1 \leq B < A \leq 1$) is

$$\frac{zf'(z)}{f_k(z)} \prec \frac{1 + Az}{1 + Bz} \quad (z \in U).$$
It is easy to know that $S_k^{(2)}(1,-1) = S_k^*$ and $S_k^{(1)}(1,-1) = S^*$, so $S_k^{(k)}(A,B)$ has a meaning as the generalization of $S_k^*$ and $S^*$, respectively.

In this paper, we will discuss the coefficient inequalities, integral representations and some invariance properties of functions belonging to the class $S_k^{(k)}(A,B)$.

2. Coefficient inequalities

Theorem 2.1. Let $f(z) \in S_k^{(k)}(A,B)$. Then $f_k(z) \in S^*(A,B) \subset S$.

Proof. For $f(z) \in S_k^{(k)}(A,B)$, we can obtain $\frac{zf'(z)}{f_k(z)} < \frac{1 + Az}{1 + Bz}$. Substituting $z$ by $\varepsilon^\mu z$ respectively ($\mu = 0, 1, 2, \ldots, k - 1$), then

$$\varepsilon^\mu \frac{zf'(\varepsilon^\mu z)}{f_k(\varepsilon^\mu z)} < \frac{1 + Az}{1 + Bz} \quad (z \in U).$$

According to the definition of $f_k(z)$ and $\varepsilon = \exp(\frac{2\pi i}{k})$, we know $\varepsilon^{-\mu} f_k(\varepsilon^\mu z) = f_k(z)$. Then the equation (8) becomes

$$\frac{zf'(\varepsilon^\mu z)}{f_k(z)} < \frac{1 + Az}{1 + Bz} \quad (z \in U).$$

Let $\mu = 0, 1, 2, \ldots, k - 1$ in (9) respectively, and sum them we can get

$$\frac{zf_k'(z)}{f_k(z)} = \frac{1}{k} \sum_{\mu=0}^{k-1} \frac{zf'(\varepsilon^\mu z)}{f_k(z)} < \frac{1 + Az}{1 + Bz} \quad (z \in U).$$

That is, $f_k(z) \in S^*(A,B) \subset S$. \qed

Putting $A = 1, B = -1$ and $k = 2$ in Theorem 2.1, we can obtain the following corollary.

Corollary 2.2. Let $f(z) \in S_k^*$, defined as (3). Then the odd function $\frac{1}{2}(f(z) - f(-z))$ is a starlike function.

Remark 2.3. Let $f(z) \in S_k^{(k)}(A,B)$. Then $f(z)$ is a close-to-convex function.

We need the following lemma to give the coefficient estimate of functions in the class $S_k^{(k)}(A,B)$.

Lemma 2.4. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ and satisfy the inequality

$$\left| \frac{zf'(z)}{g(z)} - 1 \right| \leq \frac{A - Bzf'(z)}{g(z)} \quad (z \in U),$$

where $-1 \leq B < A \leq 1$. Then for $n \geq 2$, we have

$$|na_n - b_n|^2 \leq 2(1 + |AB|) \sum_{j=1}^{n-1} \sum_{j=1}^{n-1} |a_j||b_j|.$$
Proof. Let \( f(z) \) and \( g(z) \) satisfy the inequality
\[
\left| \frac{zf'(z)}{g(z)} - 1 \right| \leq \left| A - B \frac{zf'(z)}{g(z)} \right| \quad (z \in U).
\]
Then (12) is equivalent to
\[
\frac{zf'(z)}{g(z)} \prec 1 + \frac{Az}{1 + Bz}.
\]
By the definition of subordination, there exists a Schwarz function \( w(z) \) which satisfies \( w(0) = 0 \), \( |w(z)| < |z| \) and
\[
zf'(z) = g(z) = 1 + Aw(z) \quad (z \in U)
\]
or
\[
g(z) - zf'(z) = (Bzf'(z) - Ag(z))w(z) \quad (z \in U).
\]
Now if \( w(z) = \sum_{n=1}^{\infty} c_n z^n \), then
\[
(13) \quad \sum_{n=2}^{\infty} (b_n - na_n)z^n = \left( (B - A)z + \sum_{n=2}^{\infty} (Bna_n - Ab_n)z^n \right) \left( \sum_{n=1}^{\infty} c_n z^n \right).
\]
Comparing the coefficient of \( z^n \) in (13), we have
\[
(14) \quad b_n - na_n = (B - A)c_{n-1} + (2Ba_2 - Ab_2)c_{n-2} + \cdots + ((n - 1)Ba_{n-1} - Ab_{n-1})c_1.
\]
Thus the coefficient combination on the right-hand side of (14) depends only on the coefficients combination \( Ba_1, \ldots, (n - 1)Ba_{n-1} - Ab_{n-1} \) on the left-hand side. Hence, for \( n \geq 2 \), we can write
\[
(15) \quad \sum_{j=2}^{n} (b_j - ja_j)z^j + \sum_{j=n+1}^{\infty} d_j z^j = \left( \sum_{j=1}^{n-1} (jBa_j - Ab_j)z^j \right) w(z),
\]
with \( a_1 = b_1 = 1 \). Squaring the modulus of the both sides of (15) and integrating along \( |z| = r < 1 \), and using the fact that \( |w(z)| < 1 \), we obtain
\[
\sum_{j=2}^{n} |b_j - ja_j|^2 r^{2j} + \sum_{j=n+1}^{\infty} |d_j|^2 r^{2j} < \sum_{j=1}^{n-1} |jBa_j - Ab_j|^2 r^{2j}.
\]
Letting \( r \to 1 \) on the left-hand side of this inequality, we obtain
\[
\sum_{j=2}^{n} |b_j - ja_j|^2 \leq \sum_{j=1}^{n-1} |jBa_j - Ab_j|^2.
\]
This implies that
\[
|na_n - b_n|^2 \leq \sum_{j=1}^{n-1} \left( |jB a_j - Ab_j|^2 - |b_j - ja_j|^2 \right) \\
\leq \sum_{j=1}^{n-1} (B^2 - 1) j^2 |a_j|^2 + (A^2 - 1) |b_j|^2 + 2j(1 + |AB|)|a_j||b_j| \\
\leq 2(1 + |AB|) \sum_{j=1}^{n-1} j|a_j||b_j|,
\]

since \(-1 \leq B < A \leq 1\), hence the proof of Lemma 2.4 is complete. \(\square\)

Applying the above Lemma 2.4, we give the following theorem about the coefficient estimate of functions in \(S^k_s(A, B)\).

**Theorem 2.5.** Let \(f(z) \in S^k_s(A, B)\). Then we have

(i) For \(n = lk + 1\) (\(l \in \mathbb{N}_0\)),
\begin{align*}
(n - 1)^2 |a_n|^2 \leq 2(1 + |AB|) \sum_{j=0}^{l-1} (jk + 1) |a_{jk+1}|^2.
\end{align*}

(ii) For \(n \neq lk + 1\) (\(l \in \mathbb{N}_0\)),
\begin{align*}
n^2 |a_n|^2 \leq 2(1 + |AB|) \sum_{j=0}^{\lfloor \frac{n-1}{k} \rfloor} (jk + 1) |a_{jk+1}|^2,
\end{align*}

where \(\lfloor \frac{n-1}{k} \rfloor\) denotes the biggest integer among the integers smaller than \(\frac{n-1}{k}\).

**Proof.** We note that \(zf'(z)\) and \(f_k(z)\) satisfy the condition of Lemma 2.4. And, at the same time, by the definition of \(f_k(z)\) we have
\[
f_k(z) = z + \sum_{n=2}^{\infty} \sigma_k(n) a_n z^n = z + \sum_{l=1}^{\infty} a_{lk+1} z^{lk+1}.
\]

Using Lemma 2.4, let \(n = lk + 1\) in (11), we can get (16). And if \(n \neq lk + 1\), from (11), we can get (17). \(\square\)

Next, we give that sufficient condition for functions belonging to the class \(S^k_s(A, B)\).

**Theorem 2.6.** Let \(f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be analytic in \(U\). If for \(-1 \leq B < A \leq 1\), we have
\[
\sum_{n=2, n \neq lk+1}^{\infty} (1 + |B|)|a_n| + \sum_{l=1}^{\infty} (lk + (A - B)(lk + 1)) |a_{lk+1}| \leq A - B.
\]
Then \( f(z) \in S^{(k)}(A, B) \).

Proof. At first, we note that \( f_k(z) = z + \sum_{n=2}^{\infty} \sigma_k(n) a_n z^n \) for \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \). For the proof of Theorem 2.6, it suffices to show that the values for \( z f'/f_k \) satisfy
\[
\left| \frac{z f'(z) - f_k(z)}{A f_k(z) - B z f'(z)} \right| \leq 1.
\]
And we have
\[
\left| \frac{z f'(z) - f_k(z)}{A f_k(z) - B z f'(z)} \right| = \left| \frac{\sum_{n=2}^{\infty} (n - \sigma_k(n)) a_n z^n}{(A - B) z + \sum_{n=2}^{\infty} (A \sigma_k(n) - B n) a_n z^n} \right|
\leq \frac{\sum_{n=2}^{\infty} (n - \sigma_k(n)) |a_n| |z|^n}{(A - B) - \sum_{n=2}^{\infty} |A \sigma_k(n) - B n| |a_n| |z|^n}.
\]
This last expression is bounded above by 1 if
\[
\sum_{n=2}^{\infty} (n - \sigma_k(n)) |a_n| \leq (A - B) - \sum_{n=2}^{\infty} |A \sigma_k(n) - B n| |a_n|,
\]
which is equivalent to
\[
(18) \quad \sum_{n=2}^{\infty} (n - \sigma_k(n)) + |A \sigma_k(n) - B n| |a_n| \leq A - B.
\]
Hence \( \left| \frac{z f'(z) - f_k(z)}{A f_k(z) - B z f'(z)} \right| \leq 1 \), and Theorem 2.6 is proved. \( \square \)

Corollary 2.7. For \( k = 2 \), \( A = 1 - 2\alpha \) and \( B = -1 \) in Theorem 2.6, we can obtain the result in [1].

3. Integral representations and invariance properties

We give the integral representation of functions in the class \( S^{(k)}(A, B) \) and investigate the invariance properties of the following operators:
\[
F(z) = \frac{m+1}{z^m} \int_0^z t^{m-1} f(t) dt
\]
and
\[
f_\lambda(z) = (1 - \lambda) z + \lambda f(z),
\]
where \( m \in \mathbb{N} \) and \( 0 \leq \lambda \leq 1 \). And we introduce some lemmas we need.

Lemma 3.1 ([6]). Let \( N(z) \) be regular and \( D(z) \) starlike in \( \mathbb{U} \) and \( N(0) = D(0) = 0 \). Then for \( -1 \leq B < A \leq 1 \),
\[
\frac{N'(z)}{D'(z)} \leq \frac{1 + Az}{1 + Bz}.
\]
implies that
\[
\frac{N(z)}{D(z)} < \frac{1 + Az}{1 + Bz}.
\]

Lemma 3.2 ([2]). If \( g(z) \in S^*(A, B) \), then
\[
G(z) = \frac{m + 1}{z^m} \int_0^z t^{m-1} g(t) dt \in S^*(A, B).
\]

In Theorems 3.3 and 3.4, we give the integral representations of functions in \( S_s^{(k)}(A, B) \).

Theorem 3.3. Let \( f(z) \in S_s^{(k)}(A, B) \). Then we have
\[
f_k(z) = z \cdot \exp \left\{ (A - B) \frac{1}{k} \sum_{\mu=0}^{k-1} \int_0^z \frac{e^{\mu \zeta} w(\zeta)}{\zeta (1 + B w(\zeta))} d\zeta \right\},
\]
where \( f_k(z) \) is defined by equality (5), \( w(z) \) is analytic in \( U \) and \( w(0) = 0, |w(z)| < 1 \).

Proof. Let \( f(z) \in S_s^{(k)}(A, B) \), from the definition of the subordination, we have
\[
\frac{zf'(z)}{f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)},
\]
where \( w(z) \) is analytic in \( U \) and \( w(0) = 0, |w(z)| < 1 \). Substituting \( z \) by \( \varepsilon^\mu z \) respectively (\( \mu = 0, 1, 2, \ldots, k - 1 \)), we have
\[
\frac{zf'(\varepsilon^\mu z)}{\varepsilon^\mu f_k(z)} = \frac{1 + Aw(\varepsilon^\mu z)}{1 + Bw(\varepsilon^\mu z)}
\]
for \( \mu = 0, 1, 2, \ldots, k - 1 \), and \( z \in U \). Using the equality (6) and (7), sum (21) we can obtain
\[
\frac{zf'_k(z)}{f_k(z)} = \frac{1}{k} \sum_{\mu=0}^{k-1} \frac{1 + Aw(\varepsilon^\mu z)}{1 + Bw(\varepsilon^\mu z)},
\]
and equivalently,
\[
\frac{f'_k(z)}{f_k(z)} - \frac{1}{z} = (A - B) \frac{1}{k} \sum_{\mu=0}^{k-1} \frac{w(\varepsilon^\mu z)}{z(1 + B w(\varepsilon^\mu z))}.
\]

Integrating equality (22), we have
\[
\log \frac{f_k(z)}{z} = (A - B) \frac{1}{k} \sum_{\mu=0}^{k-1} \int_0^z \frac{w(\varepsilon^\mu \zeta)}{\zeta (1 + B w(\varepsilon^\mu \zeta))} d\zeta
\]
\[
= (A - B) \frac{1}{k} \sum_{\mu=0}^{k-1} \int_0^z \frac{w(\zeta)}{\zeta (1 + B w(\zeta))} d\zeta.
\]
Therefore, arrange the above equality for $f_k(z)$, we can obtain

$$f_k(z) = z \cdot \exp \left\{ (A - B) \frac{1}{k} \sum_{\mu=0}^{k-1} \int_0^{e^{\mu z}} \frac{w(\zeta)}{\zeta(1 + Bw(\zeta))} \, d\zeta \right\},$$

and so the proof of Theorem 3.3 is complete. \( \square \)

**Theorem 3.4.** Let $f(z) \in S_0^{(k)}(A, B)$. Then we have

$$f(z) = \int_0^z \exp \left\{ (A - B) \frac{1}{k} \sum_{\mu=0}^{k-1} \int_0^{e^{\mu t}} \frac{w(t)}{t(1 + Bw(t))} \, dt \right\} \cdot \left( \frac{1 + Aw(\zeta)}{1 + Bw(\zeta)} \right) \, d\zeta,$$

where $w(z)$ is analytic in $U$, $w(0) = 0$ and $|w(z)| < 1$.

**Proof.** Let $f(z) \in S_0^{(k)}(A, B)$, from equalities (19) and (20) we have

$$f'(z) = \frac{f_k(z)}{z} \cdot \left( \frac{1 + Aw(z)}{1 + Bw(z)} \right)$$

$$= \exp \left\{ (A - B) \frac{1}{k} \sum_{\mu=0}^{k-1} \int_0^{e^{\mu z}} \frac{w(\zeta)}{\zeta(1 + Bw(\zeta))} \, d\zeta \right\} \cdot \left( \frac{1 + Aw(z)}{1 + Bw(z)} \right).$$

Integrating the above equality, we can obtain

$$f(z) = \int_0^z \exp \left\{ (A - B) \frac{1}{k} \sum_{\mu=0}^{k-1} \int_0^{e^{\mu t}} \frac{w(t)}{t(1 + Bw(t))} \, dt \right\} \cdot \left( \frac{1 + Aw(\zeta)}{1 + Bw(\zeta)} \right) \, d\zeta. \quad \square$$

Next, we investigate two invariance properties for the functions in $S_0^{(k)}(A, B)$.

**Theorem 3.5.** If $f(z) \in S_0^{(k)}(A, B)$, then so does

$$F(z) = \frac{m+1}{z^m} \int_0^z t^{m-1} f(t) \, dt$$

for $m = 1, 2, \ldots$.

**Proof.** By using the equation (23), we have

$$F_k(z) = \frac{m+1}{z^m} \int_0^z t^{m-1} f_k(t) \, dt$$

and

$$\frac{zF'(z)}{F(z)} = -m + \frac{z^m f(z)}{\int_0^z t^{m-1} f(t) \, dt}. $$
Hence
\[
\frac{zF'(z)}{F_k(z)} = \left( -m + \frac{z^m f(z)}{\int_0^z t^{m-1} f(t) dt} \right) \frac{F(z)}{F_k(z)} = \frac{z^m f(z) - m \int_0^z t^{m-1} f(t) dt}{\int_0^z t^{m-1} f(t) dt} := \frac{N(z)}{D(z)}.
\]

Since \( f_k \in S^*(A, B) \), by Lemma 3.2, we note that \( F_k(z) \in S^*(A, B) \). Differentiating (24), we have
\[
\frac{N'(z)}{D'(z)} = \frac{zf'(z)}{f_k(z)} < 1 + Az + Bz.
\]

By Lemma 3.1, we conclude that
\[
\frac{N(z)}{D(z)} < 1 + Az + Bz.
\]

Hence we have \( F(z) \in S^s_k(A, B) \). \( \square \)

**Theorem 3.6.** If \( f(z) \in S^s_k(A, B) \) and \( f_\lambda(z) = (1 - \lambda)z + \lambda f(z), 0 \leq \lambda \leq 1 \), then

(i) for \( B = 0 \), \( f_\lambda(z) \in S^s_k(A, 0) \).

(ii) for \( |z| < \frac{1}{B} \sin(\frac{B}{2A} \pi) \), \( B > 0 \), \( f_\lambda(z) \in S^s_k(A, B) \).

(iii) for \( |z| < \frac{1}{B} \sin(\frac{B}{2B-A} \pi) \), \( B < 0 \), \( f_\lambda(z) \in S^s_k(A, B) \).  

**Proof.** Since \( f(z) \in S^s_k(A, B) \),
\[
\frac{zf'(z)}{f_k(z)} < \frac{1 + Az}{1 + Bz}.
\]

Put
\[
f_{\lambda, k}(z) = \frac{1}{k} \sum_{\mu = 0}^{k-1} \varepsilon^{-\mu} f_\lambda(\varepsilon^\mu z).
\]

Then \( f_{\lambda, k}(z) = (1 - \lambda)z + \lambda f_k(z) \) and \( zf_{\lambda, k}'(z) = (1 - \lambda)z + \lambda zf'(z) \). Hence
\[
\frac{zf_{\lambda, k}'(z)}{f_{\lambda, k}(z)} = \frac{(1 - \lambda)\frac{zf'(z)}{f_k(z)} + \lambda zf'(z)}{(1 - \lambda)f_k(z) + \lambda}.
\]

Since \( f_k \in S^*(A, B) \),
\[
\frac{tf_k(sz)}{sf_k(tz)} \prec \begin{cases} \frac{1 + Bsz}{1 + Btz}, B \neq 0, \\ \exp(A(s - t)z), B = 0. \end{cases}
\]
Put $s = 1$ and $t = 0$ into (25), then we can obtain

$$\frac{f_k(z)}{z} < \begin{cases} (1 + Bz)^{\frac{A}{\lambda} + B}, B \neq 0, \\ \exp(Az), B = 0. \end{cases}$$

(26)

For the case $B = 0$, it suffices to show that

$$\left| \frac{(1 - \lambda)}{f_k(z)} + \lambda \frac{zf'(z)}{f_k(z)} - 1 \right| < A.$$  

(27)

Since $\frac{zf'(z)}{f_k(z)} < 1 + Az$, $\left| \frac{zf'(z)}{f_k(z)} - 1 \right| < A$. Since $\frac{f_k(z)}{z} \sim \exp(Az)$, there exists a Schwarz function $w_2$ which satisfies $w_2(0) = 0$ and $|w_2| < 1$ in $U$ such that

$$\frac{f_k(z)}{z} = \exp(Aw_2(z)).$$

Hence

$$\left| \frac{(1 - \lambda)}{f_k(z)} + \lambda \frac{zf'(z)}{f_k(z)} - 1 \right| = \lambda \left| \frac{zf'(z)}{f_k(z)} - 1 \right| \frac{1}{(1 - \lambda) - \frac{\lambda}{f_k(z)} + \lambda} < \frac{A\lambda}{\left| (1 - \lambda) \exp(-Aw_2(z)) + \lambda \right|}.$$  

Using the fact that $|w_2| < 1$ in $U$, we can obtain

$$|(1 - \lambda) \exp(-Aw_2(z)) + \lambda| > \lambda,$$

by simple calculations. And this implies that

$$\frac{zf'(z)}{f_k(z)} < 1 + Az$$

in $U$. For the case $B \neq 0$, we need to show that

$$\left| \frac{(1 - \lambda)}{f_k(z)} + \lambda \frac{zf'(z)}{f_k(z)} - 1 \right| < \left| A - B \frac{(1 - \lambda)}{f_k(z)} + \lambda \frac{zf'(z)}{f_k(z)} \right|.$$  

(28)

And (28) is equivalent to

$$\left| \frac{zf'(z)}{f_k(z)} - 1 \right| < \left| (A - B)(\frac{1}{\lambda} - 1) \frac{z}{f_k(z)} + A - B \frac{zf'(z)}{f_k(z)} \right|.$$  

Since $\frac{zf'(z)}{f_k(z)} < \frac{A + B}{1 + B}$,

$$\left| \frac{zf'(z)}{f_k(z)} - 1 \right| < \left| A - B \frac{zf'(z)}{f_k(z)} \right|.$$  

We note that

$$\left| \arg \left( \frac{z}{f_k(z)} \right) - \arg \left( A - B \frac{zf'(z)}{f_k(z)} \right) \right| < \frac{\pi}{2}$$  

(29)
implies that
\[ \left| A - B \frac{zf'(z)}{f_k(z)} \right| < \left| (A - B) \left( \frac{1}{\lambda} - 1 \right) \frac{z}{f_k(z)} + A - B \frac{zf'(z)}{f_k(z)} \right|. \]

Hence it suffices to show that (29) holds. Since \( \frac{zf'(z)}{f_k(z)} < \frac{1 + A}{1 + Bz} \),
\[ \left| \arg \left( A - B \frac{zf'(z)}{f_k(z)} \right) \right| \leq \arcsin(|B|r). \tag{30} \]
and
\[ \left| \arg \left( \frac{z}{f_k(z)} \right) \right| = \left| \arg \left( \frac{f_k(z)}{z} \right) \right| \leq \frac{A - B}{B} \arcsin(Br). \tag{31} \]

Hence, by (30), (31) and the hypotheses of Theorem 3.6, we can easily show that
\[ \left| \arg \left( \frac{z}{f_k(z)} \right) - \arg \left( A - B \frac{zf'(z)}{f_k(z)} \right) \right| \]
\[ \leq \left| \arg \left( \frac{z}{f_k(z)} \right) \right| + \left| \arg \left( A - B \frac{zf'(z)}{f_k(z)} \right) \right| \]
\[ \leq \arcsin(|B|r) + \frac{A - B}{B} \arcsin(Br) \]
\[ < \frac{\pi}{2} \]
and this completes the proof of Theorem 3.6. \( \square \)

References


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