THE SNOWFLAKE CURVE AS AN ATTRACTOR OF AN ITERATED FUNCTION SYSTEM

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Abstract. Although the snowflake curve, the boundary of the Koch snowflake, is one of the well-known fractals, there is no iterated function system (IFS) for it in the literature. In this study, we give an IFS for this familiar closed curve.

1. Introduction

Many fractals can be obtained as attractors of iterated function systems (IFS) ([1], [2], [3]). The Koch curve is a well-known example. It is the attractor of the IFS consisting of the similarity contractions $f_1, f_2, f_3, f_4$ on the plane such that

- $f_1(x, y) = \left( \frac{x}{3}, \frac{y}{3} \right)$
- $f_2(x, y) = \left( \frac{x}{6} - \frac{x}{6} + \frac{1}{3}, \frac{\sqrt{3}y}{6} + \frac{y}{6} \right)$
- $f_3(x, y) = \left( \frac{x}{6} + \frac{\sqrt{3}y}{6} + \frac{1}{3}, \frac{\sqrt{3}x}{6} + \frac{y}{6} + \frac{\sqrt{3}}{6} \right)$
- $f_4(x, y) = \left( \frac{x}{3} + \frac{2}{3}, \frac{y}{3} \right)$

(see Fig. 1) [4]. It is known that there is another IFS for the Koch curve consisting of only two contractions with contractivity factor $\frac{1}{\sqrt{3}}$ which map the triangle $ABC$ to the triangles $DAB$ and $EAC$ respectively as shown in Fig. 2. The Koch snowflake can also be obtained as the attractor of the IFS consisting of seven contractions which can be immediately written from Fig. 3. These contractions are also similarities.

Since the images of an attractor under the contractions of an IFS are usually small copies of the attractor, the notion of self-similarity is geared to the notion of iterated function system. One can easily exhibit some IFS for the Koch curve...
and the Koch snowflake. However, it is clear that there is no IFS consisting only of similarity contractions for the snowflake curve.

In the next section, we give an IFS consisting of 8 contractions each of which is the composition of some similarities, symmetries and foldings.

2. An IFS for the snowflake curve

Consider the contractions $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$
    f_1(x, y) = \left( -\frac{1}{3}|x|, \frac{1}{3}y \right) + \left( -\frac{1}{2}, \frac{\sqrt{3}}{6} \right),
$$

$$
    f_2(x, y) = \left( \frac{1}{\sqrt{3}}|y|, -\frac{1}{\sqrt{3}}|x| \right) + \left( -\frac{1}{2}, \frac{\sqrt{3}}{2} \right).
$$

Let $f_1^1$ be the folding defined by $f_1^1(x, y) = (|x|, y)$ for all $x, y \in \mathbb{R}$. Let $f_1^2$ be the projective similarity with center $C = \left( -\frac{1}{4}, \frac{\sqrt{3}}{4} \right)$ and contractivity factor $\frac{1}{3}$ which maps the polygonal path $ABCDEFG$ onto the path $A'B'C'D'E'F'G'$ as shown in Fig. 4. Then the contraction $f_1$ can be written as the composition of the folding $f_1^1$ and the projective similarity $f_1^2$ as shown in Fig. 5: $f_1 = f_1^2 \circ f_1^1$. 

Figure 3. The Koch snowflake as the union of 7 similar copies of itself.

Figure 4. $f_2^2$ maps the path $ABCDEFG$ onto the path $A'B'C'D'E'F'G'$.

Let $f_1^2$ and $f_2^2$ be the foldings defined on the plane by $f_1^2(x, y) = (x, |y|)$ and $f_2^2(x, y) = (-|x|, y)$ respectively. Let $f_3^2$ be the projective similarity with center $C = \left(0, \frac{\sqrt{3}}{2}\right)$ and contractivity factor $\frac{1}{\sqrt{3}}$ which maps the polygonal path $ABCD$ onto the polygonal path $A'B'C'D'$ as shown in Fig. 6. Let $f_4^2$ be the
Figure 5. The mapping $f_1 = f_1^2 \circ f_1^1$.

Figure 6. The projective similarity $f_1^2$ (left) and the symmetry $f_2^1$ (right).
symmetry with respect to the angle bisector between $AC$ and $AD$ as shown in Fig 6. Then the contraction $f_2$ can be written as the composition of the foldings $f_1, f_2^2$ and the projective similarity $f_3^2$ and the symmetry $f_4^2$ as shown in Fig. 7: $f_2 = f_4^2 \circ f_3^2 \circ f_2^2 \circ f_1$.

Note that $f_1$ and $f_2$ map the snowflake curve, $S$, onto its quadrant (see Fig. 8).

The mappings $f_1$ and $f_2$ may be adapted as below ($f_3$ to $f_8$) to map the snowflake curve onto its parts in the other three quadrants.

\[
f_3(x, y) = \left( -\frac{1}{3} |x|, \frac{1}{3} y \right) + \left( -\frac{1}{2}, -\frac{\sqrt{3}}{6} \right)
\]
\[
f_4(x, y) = \left( \frac{1}{\sqrt{3}} |y|, \frac{1}{\sqrt{3}} |x| \right) + \left( -\frac{1}{2}, -\frac{\sqrt{3}}{2} \right)
\]
\[
f_5(x, y) = \left( \frac{1}{3} |x|, \frac{1}{3} y \right) + \left( \frac{1}{2}, -\frac{\sqrt{3}}{6} \right)
\]
Figure 8. The snowflake curve $S$ as an attractor of the IFS.

$$f_6(x, y) = \left(-\frac{1}{\sqrt{3}}|y|, \frac{1}{\sqrt{3}}|x|\right) + \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

$$f_7(x, y) = \left(\frac{1}{3}|x|, \frac{1}{3}y\right) + \left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right)$$

$$f_8(x, y) = \left(-\frac{1}{\sqrt{3}}|y|, -\frac{1}{\sqrt{3}}|x|\right) + \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

The attractor of the IFS consisting of $\{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8\}$ is the desired fractal, the snowflake curve (see Fig. 8).

There are alternative iterated function systems for the snowflake curve each of which is formed by 6 contractions. Let $g_1$ be the mapping that folds the right part of the line $AG$ onto the left part of it (as done by $f_1$ before) and $g_2$ be the mapping that folds the below part of the line $EH$ onto the above part of it (see Fig. 9 (left)). $g_2 \circ g_1$ maps the snowflake curve to the Koch curve. As done in the previous section (see Fig. 2), the Koch curve can be obtained as the union of the images of itself under two similarities $h_1$ and $h_2$. So that the Koch curve is the union of the images of the snowflake curve under the contractions $f_1 = h_1 \circ g_2 \circ g_1$ and $f_2 = h_2 \circ g_2 \circ g_1$.

One can apply similar procedure to obtain the other parts (the other two Koch curves) of the snowflake curve (see Fig. 9).
Note: In general, a folding map $g$ that maps the half plane $ax + by + c > 0$ onto the other half plane can be defined by

$$g(x, y) = \left( \frac{b^2 x - aby - ac - a|ax + by + c|}{a^2 + b^2}, \frac{a^2 y - abx - bc - b|ax + by + c|}{a^2 + b^2} \right).$$

Note that in our case $a = 1$, $b = -\sqrt{3}$ and $c = 0$.

Acknowledgements. Authors cordially thank the referees for their important contributions and creative reviews to the article.

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