EINSTEIN LIGHTLIKE HYPERSURFACES OF A
LORENTZIAN SPACE FORM WITH A SEMI-SYMMETRIC
METRIC CONNECTION

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Abstract. In this paper, we prove a classification theorem for Einstein
lightlike hypersurfaces $M$ of a Lorentzian space form $(\overline{M}(c), \overline{g})$ with a
semi-symmetric metric connection subject such that the second funda-
mental forms of $M$ and its screen distribution $S(TM)$ are conformally
related by some non-zero constant.

1. Introduction

In the classical theory of spacetime, the Riemannian curvature tensor will
affect the rate of change of separation of null and timelike curves (see Sections
4.1 and 4.2 in [8]). Null curves can represent the histories of photons, the
effect of the Riemannian curvature tensor will be to distort or focus small
bundles of light rays. While the rest spaces of timelike curves are spacelike
subspaces of the tangent spaces, the rest spaces of null curves are lightlike
subspaces of the tangent spaces [12]. To investigate this, Hawking and Ellis
introduced the notion of so-called screen spaces in Section 4.2 of their book
[8]. Since for any semi-Riemannian manifold there is a natural existence of
the general theory of degenerate (lightlike) submanifolds to fill a gap in the
study of submanifolds. Since then there has been very active study on lightlike
geometry of submanifolds (see up-to date results in two books [5, 7]).

The classification of Einstein hypersurfaces $M$ in Euclidean spaces $\mathbb{R}^{n+1}$
was first studied by Fialkow [10] and Thomas [13] in the middle of 1930’s. It
was proved that if $M$ is a connected Einstein hypersurface ($n \geq 3$), that is
$Ric = \kappa g$, for some constant $\kappa$, then $\kappa$ is non-negative. Moreover,
• if $\kappa = 0$, then $M$ is locally isometric to $\mathbb{R}^n$ and
• if $\kappa > 0$, then $M$ is contained in an $n$-sphere.

Received July 28, 2011; Revised January 5, 2012.
2010 Mathematics Subject Classification. Primary 53C25, 53C40, 53C50.
Key words and phrases. screen homothetic, Einstein manifold, semi-symmetric metric
connection.
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163
Motivated by the rich existing Riemannian geometry endowed with a semi-symmetric metric connection (see two papers of Hayden [9] and Yano [15]), we study lightlike hypersurfaces \( M \) of a semi-Riemannian manifold \( \bar{M} \) admitting a semi-symmetric metric connection. The objective of this paper is the study of lightlike version of above classical results. We focus on the geometry of Einstein lightlike hypersurfaces \((M, g)\) of a Lorentzian space form \((\bar{M}(c), \bar{g})\) with a semi-symmetric metric connection subject such that whose shape operator is homothetic to the shape operator of its screen distribution. The reason for this geometric condition on \( M \) is due to the fact that such a class admits a canonical integrable screen distribution and a symmetric induced Ricci tensor of \( M \) [1]. These both conditions are required to recover an induced scalar curvature of \( M \) of a Lorentzian manifold [3]. The paper contains several new results which are related to the symmetric Ricci tensor. Calling such a class by screen homothetic lightlike hypersurfaces \((M, g)\), we prove that \( M \) is locally a product manifold \( L \times M^\rho \times M^\sigma \), where \( L \) is a null curve, and \( M^\rho \) and \( M^\sigma \) are leaves of some distributions of \( M \) (Theorem 4.1). Using this theorem we prove a characterization theorem for Einstein screen homothetic lightlike hypersurfaces \( M \) of a Lorentzian space form \( \bar{M}(c) \) with a semi-symmetric metric connection (Theorem 4.2).

2. Semi-symmetric metric connection

Let \((\bar{M}, \bar{g})\) be a semi-Riemannian manifold. A connection \( \bar{\nabla} \) on \( \bar{M} \) is called a semi-symmetric metric connection [9, 14, 15] if it is metric, i.e., \( \bar{\nabla}_X \bar{g} = 0 \) and its torsion tensor \( \bar{T} \) satisfies
\[
\bar{T}(X, Y) = \pi(Y)X - \pi(X)Y,
\]
for any vector fields \( X \) and \( Y \) of \( \bar{M} \), where \( \pi \) is a 1-form on \( \bar{M} \).

Let \((M, g)\) be a lightlike hypersurface of a semi-Riemannian manifold \((\bar{M}, \bar{g})\). It is well known that the normal bundle \( TM^\perp \) of the lightlike hypersurfaces \( M \) is a vector subbundle of \( TM \), of rank 1. A complementary vector bundle \( S(TM) \) of \( TM^\perp \) in \( TM \) is non-degenerate distribution on \( M \), which called a screen distribution on \( M \), such that
\[
TM = TM^\perp \oplus_{\text{orth}} S(TM),
\]
where \( \oplus_{\text{orth}} \) denotes the orthogonal direct sum. We denote such a lightlike hypersurface by \( M = (M, g, S(TM)) \). Denote by \( F(M) \) the algebra of smooth functions on \( M \) and by \( \Gamma(E) \) the \( F(M) \) module of smooth sections of a vector bundle \( E \) over \( M \). It is well-known [4] that, for any null section \( \xi \) of \( TM^\perp \) on a coordinate neighborhood \( \mathcal{U} \subset M \), there exists a unique null section \( N \) of a unique vector bundle \( tr(TM) \) in \( S(TM)^\perp \) satisfying
\[
\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \quad \forall X \in \Gamma(S(TM)).
\]
We call \( tr(TM) \) and \( N \) the transversal vector bundle and the null transversal vector field of \( M \) with respect to the screen distribution respectively. Then the
tangent bundle $\bar{T}\bar{M}$ of $\bar{M}$ is decomposed as follow:

\begin{equation}
\bar{T}\bar{M} = TM \oplus \text{tr}(TM) = \{TM^\perp \oplus \text{tr}(TM)\} \oplus_{\text{orth}} S(TM).
\end{equation}

Let $P$ be the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (2.2). From the decompositions (2.2) and (2.3), the local Gauss and Weingarten formulas of $M$ and $S(TM)$ are given respectively by

\begin{align}
\bar{\nabla}_X Y &= \nabla_X Y + B(X, Y)N, \\
\bar{\nabla}_X N &= -A_N X + \tau(X)N, \\
\nabla_X PY &= \nabla_X PY + C(X, PY)\xi, \\
\nabla_X \xi &= -A^*_\xi X - \tau(X)\xi,
\end{align}

for any $X, Y \in \Gamma(TM)$, where the symbols $\nabla$ and $\bar{\nabla}^*$ are the induced linear connections on $TM$ and $S(TM)$ respectively, $B$ and $C$ are the local second fundamental forms on $TM$ and $S(TM)$ respectively, $A_N$ and $A^*_\xi$ are the shape operators on $TM$ and $S(TM)$ respectively and $\tau$ is a 1-form on $TM$. The induced connection $\bar{\nabla}$ on $\bar{M}$ is not metric and satisfies

\begin{equation}
(\bar{\nabla}_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y),
\end{equation}

for any $X, Y, Z \in \Gamma(TM)$, where $\eta$ is a 1-form on $TM$ such that

\begin{equation}
\eta(X) = \bar{g}(X, N), \forall X \in \Gamma(TM).
\end{equation}

But the connection $\bar{\nabla}^*$ is metric. Using (2.1) and (2.4), we show that

\begin{equation}
T(X, Y) = \pi(Y)X - \pi(X)Y, \quad \forall X, Y \in \Gamma(TM)
\end{equation}

and $B$ is symmetric, where $T$ is the torsion tensor with respect to $\nabla$. From (2.8) and (2.10), we show that the induced connection $\nabla$ of $M$ is a semi-symmetric non-metric connection of $M$. From the fact $B(X, Y) = \bar{g}(\nabla_X Y, \xi)$, we know that $B$ is independent of the choice of a screen distribution and satisfies

\begin{equation}
B(X, \xi) = 0, \quad \forall X \in \Gamma(TM).
\end{equation}

The above second fundamental forms are related to their shape operators by

\begin{align}
g(A^*_\xi X, Y) &= B(X, Y), \\
g(A_N X, N) &= 0, \\
g(A_N X, PY) &= C(X, PY), \\
g(A_N X, N) &= 0,
\end{align}

for all $X, Y \in \Gamma(TM)$. By (2.12), we show that $A^*_\xi$ is $\Gamma(S(TM))$-valued self-adjoint shape operators related to $B$ and satisfies

\begin{equation}
A^*_\xi \xi = 0.
\end{equation}

In general, $S(TM)$ is not necessarily integrable. The following result gives equivalent conditions for the integrability of $S(TM)$:

**Theorem 2.1.** Let $M$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ admitting a semi-symmetric metric connection. The following assertions are equivalent:
(1) The screen distribution $S(TM)$ is an integrable distribution.

(2) $C$ is symmetric, i.e., $C(X, Y) = C(Y, X)$ for all $X, Y \in \Gamma(S(TM))$.

(3) The shape operator $A_N$ is self-adjoint with respect to $g$, i.e.,
$$g(A_N X, Y) = g(X, A_N Y), \quad \forall X, Y \in \Gamma(S(TM)).$$

Proof. First, note that a vector field $X$ on $M$ belongs to $S(TM)$ if and only if we have $\eta(X) = 0$. Next, by using (2.6) and (2.10), we have
$$C(X, Y) - C(Y, X) = \eta([X, Y]), \quad \forall X, Y \in \Gamma(S(TM)),$$
which implies the equivalence of (1) and (2). Finally, the equivalence of (2) and (3) follows from the first equation of (2.13) [denote (2.13) $1\) $ \hfill \square$

Note 1. In case $S(TM)$ is integrable, $M$ is locally a product manifold $L \times M^*$ where $L$ is a null curve tangent to the normal bundle $TM^\perp$ and $M^*$ is a leaf of the screen distribution $S(TM)$ [4, 5].

Denote by $\bar{R}$, $R$ and $R^*$ the curvature tensors of the semi-symmetric metric connection $\bar{\nabla}$ on $\bar{M}$, the induced connection $\nabla$ on $M$ and the induced connection $\nabla^*$ on $S(TM)$ respectively. Using the Gauss-Weingarten equations (2.4)~(2.7) for $M$ and $S(TM)$, for any $X, Y, Z, W \in \Gamma(TM)$, we obtain the Gauss-Codazzi equations for $M$ and $S(TM)$:

\begin{align}
(2.15) \quad \bar{g}(\bar{R}(X, Y)Z, PW) &= g(R(X, Y)Z, PW) + B(X, Z)C(Y, PW) - B(Y, Z)C(X, PW), \\
(2.16) \quad \bar{g}(\bar{R}(X, Y)Z, \xi) &= (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\
&+ [\tau(X) - \pi(X)]B(Y, Z) - [\tau(Y) - \pi(Y)]B(X, Z), \\
(2.17) \quad \bar{g}(\bar{R}(X, Y)Z, N) &= g(R(X, Y)Z, N), \\
(2.18) \quad \bar{g}(\bar{R}(X, Y)\xi, N) &= g(A_N^2 X, A_N Y) - g(A_N^2 Y, A_N X) - 2d\tau(X, Y), \\
&= g(R(X, Y)PZ, PW), \\
(2.19) \quad g(R^*(X, Y)PZ, PW) &= C(X, PZ)B(Y, PW) - C(Y, PZ)B(X, PW), \\
&= (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\
&+ [\tau(Y) + \pi(Y)]C(X, PZ) - [\tau(X) + \pi(X)]C(Y, PZ). 
\end{align}
The Ricci tensor, denoted by $\bar{\text{Ric}}$, of $\bar{M}$ is defined by
\begin{equation}
\bar{\text{Ric}}(X, Y) = \text{trace}\{Z \rightarrow \bar{R}(Z, X)Y\}
\end{equation}
for any $X, Y \in \Gamma(T\bar{M})$. Locally, $\bar{\text{Ric}}$ is given by
\begin{equation}
\bar{\text{Ric}}(X, Y) = \sum_{i=1}^{m+2} \epsilon_i \bar{g}(\bar{R}(E_i, X)Y, E_i),
\end{equation}
where $\{E_1, \ldots, E_{m+2}\}$ is an orthonormal frame field of $TM$. If
\begin{equation}
\bar{\text{Ric}} = \bar{\kappa}g,
\end{equation}
then we say that $\bar{M}$ is an Einstein manifold. If $\dim(\bar{M}) > 2$, then $\bar{\kappa}$ is a constant. For $\dim(\bar{M}) = 2$, any $\bar{M}$ is Einstein but $\bar{\kappa}$ in (2.22) is not necessarily constant. The scalar curvature $\bar{\text{r}}$ is defined by
\begin{equation}
\bar{\text{r}} = \sum_{i=1}^{m+2} \epsilon_i \bar{\text{Ric}}(E_i, E_i).
\end{equation}
Putting (2.22) in (2.23) implies that $\bar{M}$ is Einstein if and only if
\begin{equation}
\bar{\text{Ric}} = \frac{\bar{\text{r}}}{m+2} \bar{g}.
\end{equation}
A semi-Riemannian manifold $\bar{M}$ of constant curvature $c$ is called a space form and denote it by $\bar{M}(c)$. In this case, the curvature $\bar{R}$ of $\bar{M}$ is given by
\begin{equation}
\bar{R}(X, Y)Z = c\{\bar{g}(Y, Z)X - \bar{g}(X, Z)Y\}, \quad \forall X, Y, Z \in \Gamma(T\bar{M}).
\end{equation}

3. Induced Ricci and scalar curvatures

Consider an induced quasi-orthonormal frame field $\{\xi; W_a\}$ on $M$, where $TM^\perp = \text{Span}\{\xi\}$ and $S(TM) = \text{Span}\{W_a\}$ and let $E = \{\xi, N, W_a\}$ be the corresponding frame field on $\bar{M}$. Then, by using (2.21), we obtain
\begin{equation}
\bar{R}(X, Y) = \sum_{a=1}^{m} \epsilon_a \bar{g}(\bar{R}(W_a, X)Y, W_a) + \bar{g}(\bar{R}(\xi, X)Y, N) + \bar{g}(\bar{R}(N, X)Y, \xi),
\end{equation}
where $\epsilon_a$ denotes the causal character ($\pm 1$) of respective vector field $W_a$. Let $\bar{R}^{(0, 2)}$ denote the induced Ricci type tensor of type $(0, 2)$ on $\bar{M}$ given by
\begin{equation}
\bar{R}^{(0, 2)}(X, Y) = \text{trace}\{Z \rightarrow \bar{R}(Z, X)Y\}, \quad \forall X, Y \in \Gamma(TM).
\end{equation}
Using the induced quasi-orthonormal frame field $\{\xi; W_a\}$ on $M$, we obtain
\begin{equation}
\bar{R}^{(0, 2)}(X, Y) = \sum_{a=1}^{m} \epsilon_a \bar{g}(\bar{R}(W_a, X)Y, W_a) + \bar{g}(\bar{R}(\xi, X)Y, N).
Substituting (2.15) and (2.17) in (3.1) an using (2.12) and (2.13), we obtain
\[
R^{(0,2)}(X,Y) = \bar{Ric}(X,Y) + B(X,Y)\text{tr}A_N
- g(A_NX, A_\xi^*Y) - \bar{g}(R(\xi,Y)X, N)
\]
for any \(X, Y \in \Gamma(TM)\). This shows that \(R^{(0,2)}\) is not symmetric. A tensor field \(R^{(0,2)}\) of \(M\) is called its induced Ricci tensor, denoted by \(\bar{Ric}\), if it is symmetric. Using (2.18), (3.3) and the first Bianchi’s identity, we obtain
\[
R^{(0,2)}(X,Y) - R^{(0,2)}(Y,X) = 2d\tau(X,Y), \quad \forall X, Y \in \Gamma(TM).
\]

**Theorem 3.1.** Let \(M\) be a lightlike hypersurface of a semi-Riemannian manifold \(\bar{M}\) admitting a semi-symmetric metric connection. Then the Ricci type tensor \(R^{(0,2)}\) of \(M\) is an induced symmetric Ricci tensor if and only if the 1-form \(\tau\) is closed, i.e., \(d\tau = 0\), on any coordinate neighborhood \(U \subset M\).

**Note 2.** If \(R^{(0,2)}\) is symmetric, then the 1-form \(\tau\) is closed on \(TM\). Therefore there exists a smooth function \(f\) such that \(\tau = df\). Consequently we get \(\tau(X) = X(f)\). If we take \(\xi = \alpha \xi\), it follows that \(\tau(X) = \dot{\tau}(X) + X(\ln \alpha)\).

We call the pair \(\{\xi, N\}\) such that the corresponding 1-form \(\tau\) vanishes the canonical null pair of \(M\). Although \(S(TM)\) is not unique, it is canonically isomorphic to the factor vector bundle \(S(TM)^\sharp = TM/Rad(TM)\) considered by Kupeli [11]. Thus all \(S(TM)\) are mutually isomorphic. For this reason, we consider only lightlike hypersurfaces \(M\) with the canonical null pair \(\{\xi, N\}\) of a semi-Riemannian manifold \(\bar{M}\) admitting a semi-symmetric metric connection.

The scalar curvature \(\bar{r}\) of \(\bar{M}\), defined by (2.23), and the scalar quantity \(r\) of \(M\), obtained from \(R^{(0,2)}\) by the method of (2.23) are given by
\[
\bar{r} = \bar{Ric}(\xi, \xi) + \bar{Ric}(N, N) + \sum_{a=1}^m \epsilon_a \bar{Ric}(W_a, W_a),
\]
\[
r = R^{(0,2)}(\xi, \xi) + \sum_{a=1}^m \epsilon_a R^{(0,2)}(W_a, W_a),
\]
respectively. Using these relations and (3.3), we obtain
\[
R^{(0,2)}(\xi, \xi) = \bar{Ric}(\xi, \xi)
\]
\[
R^{(0,2)}(W_a, W_a) = \bar{Ric}(W_a, W_a) + g(A_N^*W_a, W_a)\text{tr}A_N
- g(A_NW_a, A_\xi^*W_a) - \bar{g}(R(\xi,W_a)W_a, N).
\]

Thus we have
\[
r = \bar{r} + \text{tr}A_\xi^*\text{tr}A_N - \text{tr}(A_N^*A_N) - \sum_{a=1}^m \epsilon_a \{ \bar{g}(R(\xi,W_a)W_a, N) + \bar{g}(\bar{R}(N,W_a)W_a, N) \}.
\]
For any semi-Riemannian space form $\tilde{M}(c)$, we have
\[ \bar{R}(\xi, Y)X = c\bar{g}(X, Y)\xi, \quad \bar{R}ic(X, Y) = (m + 1)c\bar{g}(X, Y) \]
and $\bar{\rho} = cm(m + 1)$, $g(\bar{R}(N, W), W, N) = 0$. Thus
\[ R^{(0, 2)}(X, Y) = mc\bar{g}(X, Y) + B(X, Y)trA_N - g(A_N X, A_N Y); \]
\[ r = m^2c + trA^*_N trA_N - tr(A^*_N A_N). \]

**Definition 1.** A lightlike hypersurface $M$ of a semi-Riemannian manifold $(\tilde{M}, \bar{g})$ is screen homothetic [1] if there exist a non-zero constant $\varphi$ such that the shape operators $A_N$ and $A^*_N$ of $M$ and its screen distribution $S(TM)$ respectively are related by $A_N = \varphi A^*_N$, or equivalently,
\[ C(X, PY) = \varphi B(X, PY), \quad \forall X, Y \in \Gamma(TM). \]

**Theorem 3.2.** Let $M$ be a screen homothetic lightlike hypersurface of a semi-Riemannian space form $\tilde{M}(c)$ admitting a semi-symmetric metric connection. Then the Ricci type tensor $R^{(0, 2)}$ is symmetric, the screen distribution $S(TM)$ is integrable and $M$ is a locally product manifold $L \times M^*$ where $L$ is a null curve tangent to $TM^\perp$ and $M^*$ is a leaf of $S(TM)$.

**Proof.** By using (2.18), (2.24) and the fact $A_N = \varphi A^*_N$, we show that $\tau$ is closed, i.e., $d\tau = 0$ on $TM$. Thus $R^{(0, 2)}$ is symmetric. By using (3.7), we show that $C$ is symmetric on $S(TM)$. Thus, by Theorem 2.1, $S(TM)$ is integrable. From Note 1, $M$ is locally a product manifold $L \times M^*$, where $L$ is a null curve and $M^*$ is a leaf of the screen distribution $S(TM)$. $\square$

**Theorem 3.3.** Let $M$ be a screen homothetic lightlike hypersurface of a semi-Riemannian manifold $\tilde{M}(c)$ admitting a semi-symmetric metric connection. Then we have $c = 0$.

**Proof.** Using (2.16), (2.24) and the facts $\tilde{M} = \tilde{M}(c)$ and $\tau = 0$, we have
\[ \langle \nabla X B \rangle(Y, Z) - \langle \nabla Y B \rangle(X, Z) = \pi(X)B(Y, Z) - \pi(Y)B(X, Z) \]
for all $X, Y, Z \in \Gamma(TM)$. Using (2.17), (2.20), (3.7) and (3.8), we have
\[ (X\varphi)B(Y, PZ) - (Y\varphi)B(X, PZ) = c\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\} \]
for all $X, Y, Z \in \Gamma(TM)$. Replace $Y$ by $\xi$ in this equation, we obtain
\[ (\xi\varphi)B(X, PZ) = cg(X, PZ). \]
Assume that $M$ is screen homothetic. By (3.10), we have $c = 0$. $\square$

**4. Einstein lightlike hypersurfaces**

In this section, let $M$ be a screen homothetic Einstein lightlike hypersurface equipped with the canonical null pair $\{\xi, N\}$ of a Lorentzian space form.
(M^{m+2}(c), \bar{g}) admitting a semi-symmetric metric connection. Under this hypothesis, we show that S(TM) is integrable by Theorem 3.2 and \( c = 0 \) by Theorem 3.3. Using (3.5), (3.6) and \( M \)

Thus we have

Thus we have

\[
Ric(X, Y) = (r/m)g(X, Y)
\]

which provides a geometric interpretation of lightlike Einstein hypersurfaces (same as in Riemannian case) as we have shown that the constant \( \kappa = r/m \).

\[M\]

Since \( \xi \) is an eigenvector field of \( A^*_\xi \) corresponding to the eigenvalue 0 due to (2.14) and \( A^*_\xi \) is \( S(TM) \)-valued real self-adjoint operator, \( A^*_\xi \) have \( m \) real orthonormal eigenvector fields in \( S(TM) \) and is diagonalizable. Consider a frame field of eigenvectors \( \{ \xi, E_1, \ldots, E_m \} \) of \( A^*_\xi \) such that \( \{ E_1, \ldots, E_m \} \) is an orthonormal frame field of \( S(TM) \). Then we have

\[A^*_\xi E_i = \lambda_i E_i, \quad 1 \leq i \leq m.
\]

Since \( M \) is screen homothetic and \( Ric = \kappa g \), the equation (3.5) reduce to

\[
g(A^*_\xi X, A^*_\xi Y) - sg(A^*_\xi X, Y) + \varphi^{-1}\kappa g(X, Y) = 0,
\]

where \( s = tr A^*_\xi \). Put \( X = Y = E_1 \) in (4.1), each \( \lambda_i \) is a solution of the equation

\[
x^2 - sx + \varphi^{-1}\kappa = 0.
\]

The equation (4.2) has at most two distinct solutions which are smooth real valued functions on \( U \). Assume that there exists \( p \in \{ 0, 1, \ldots, m \} \) such that

\[
\lambda_1 = \cdots = \lambda_p = \rho \quad \text{and} \quad \lambda_{p+1} = \cdots = \lambda_m = \sigma,
\]

by renumbering if necessary. From (4.2), we have

\[
s = \rho + \sigma = pp + (m - p)\sigma, \quad \rho\sigma = \varphi^{-1}\kappa.
\]

Since \( \varphi \) and \( \kappa \) are constants, \( \rho\sigma \) is a constant. From this result and the equation

\[
(p - 1)\rho = -(m - p - 1)\sigma,
\]

we show that the functions \( \rho \) and \( \sigma \) are constants.

**Theorem 4.1.** Let \( M \) be a screen homothetic Einstein lightlike hypersurface of a Lorentzian space form \((M(c), \bar{g})\) admitting a semi-symmetric metric connection. Then \( M \) is locally a product manifold \( L \times M_\rho \times M_\sigma \), where \( L \) is a null curve tangent to \( TM^\perp \), and \( M_\rho \) and \( M_\sigma \) are totally umbilical leaves of some integrable distributions of \( M \).

**Proof.** If (4.2) has only one solution \( \rho \), by Theorem 3.2, we show that \( M = L \times M^* \cong L \times M^* \times \{ x \} \) for any \( x \in M \), where \( M^* = M_\rho \). Since \( B(X, Y) = g(A^*_\xi X, Y) = pg(X, Y) \) for all \( X, Y \in \Gamma(TM) \), \( M \) is totally umbilical. By (3.7), we get \( C(X, PY) = \varphi p g(X, PY) \) for all \( X, Y \in \Gamma(TM) \). Thus \( M^* \) is also totally umbilical. Thus this theorem is true. \( \square \)
Assume that (4.2) has exactly two distinct solutions $\rho$ and $\sigma$. In case $p = 0$ or $p = m$: We also show that \( M = L \times M^* \cong L \times M^* \times \{x\} \) for any \( x \in M \) and \( M^* = M_0 \) or \( M_\sigma \). \( M \) and \( M^* \) are also totally umbilical. In this case, this theorem is also true. In case \( 0 < p < m \): Consider the following four distributions \( D_\rho, D_\sigma, D^*_\rho \) and \( D^*_\sigma \) on \( M \):

\[
D_\rho = \{ X \in \Gamma(TM) \mid A_\rho X = \rho PX \}, \quad D^*_\rho = PD_\rho, \\
D_\sigma = \{ U \in \Gamma(TM) \mid A_\sigma U = \sigma PU \}, \quad D^*_\sigma = PD_\sigma.
\]

Clearly we show that \( D_\rho \cap D_\sigma = TM^\perp \) and \( D^*_\rho \cap D^*_\sigma = \{0\} \).

**Step 1.** If \( D_\rho \neq D_\sigma \), then \( S(TM) = D^*_\rho \oplus_{\text{orth}} D^*_\sigma \).

For any \( X \in \Gamma(D_\rho) \) and \( U \in \Gamma(D_\sigma) \), we get \( A_\rho^* PX = A_\rho^* X = \rho PX \) and \( A_\rho^* PU = A_\rho^* U = \sigma PU \). This implies that the projection morphism \( P \) maps \( \Gamma(D_\rho) \) onto \( \Gamma(D^*_\rho) \) and \( \Gamma(D_\sigma) \) onto \( \Gamma(D^*_\sigma) \). Since \( PX \) and \( PU \) are eigenvector fields of the real self-adjoint operator \( A_\rho^* \) corresponding to the different eigenvalues \( \rho \) and \( \sigma \) respectively, we have \( g(PX, PU) = 0 \). From the facts \( g(X, U) = g(PX, PU) = 0 \) and \( B(X, U) = g(A_\rho^* X, U) = \rho g(PX, PU) = 0 \), we show that \( D_\rho \perp D_\sigma \) and \( D^*_\rho \perp_{\text{ortho}} D^*_\sigma \) respectively.

**Step 2.** If \( D_\rho \neq D_\sigma \), then \( S(TM) = D^*_\rho \oplus_{\text{orth}} D^*_\sigma \).

Since \( \{E_i\}_{1 \leq i \leq p} \) and \( \{E_a\}_{p+1 \leq a \leq m} \) are vector fields of \( D^*_\rho \) and \( D^*_\sigma \) respectively and \( D^*_\rho \) and \( D^*_\sigma \) are mutually orthogonal vector subbundle of \( S(TM) \), \( D^*_\rho \) and \( D^*_\sigma \) are non-degenerate distributions of rank \( p \) and rank \( (m - p) \) respectively. Thus \( S(TM) = D^*_\rho \oplus_{\text{orth}} D^*_\sigma \).

**Step 3.** \( \text{Im}(A_\rho^* - \rho P) \subset \Gamma(D^*_\rho) \) and \( \text{Im}(A_\sigma^* - \sigma P) \subset \Gamma(D^*_\sigma) \).

From (4.1), we show that \((A^*_\rho - \rho P)A^*_\sigma + \rho_\sigma \rho P = 0 \). Let \( Y \in \text{Im}(A_\rho^* - \rho P) \), then there exists \( X \in \Gamma(TM) \) such that \( Y = (A_\rho^* - \rho P)X \). Then \((A_\rho^* - \sigma P)Y = 0 \) and \( Y \in \Gamma(D_\rho) \). Thus \( \text{Im}(A_\rho^* - \rho P) \subset \Gamma(D_\rho) \). Since the morphism \( A_\rho^* - \rho P \) maps \( \Gamma(TM) \) onto \( \Gamma(S(TM)) \), we have \( \text{Im}(A_\rho^* - \rho P) \subset \Gamma(D^*_\rho) \). By duality, we also have \( \text{Im}(A_\sigma^* - \sigma P) \subset \Gamma(D^*_\sigma) \).

**Step 4.** \( D^*_\rho \) and \( D^*_\sigma \) are integrable distributions.

For \( X, Y \in \Gamma(D_\rho) \) and \( U \in \Gamma(D_\sigma) \), using (2.8), (2.12) and Step 1, we have

\[
(\nabla_X B)(Y, U) = -g((A_\rho^* - \rho P)\nabla_X Y, U) + \rho B(X, U)\eta(U).
\]

Replacing \( Z \) by \( U \) to (3.8) and using Step 1, we have \( (\nabla_X B)(Y, U) = (\nabla_Y B)(X, U) \). From this results, (2.10) and Step 1, we have \( g((A_\rho^* - \rho P)[X, Y], U) = 0 \). Since \( D^*_\rho \) is non-degenerate and \( \text{Im}(A_\rho^* - \rho P) \subset \Gamma(D^*_\rho) \), we have \( (A_\rho^* - \rho P)[X, Y] = 0 \). Thus \( [X, Y] \in \Gamma(D_\rho) \) and \( D_\rho \) is integrable. By duality, \( D_\sigma \) is also integrable. For any \( X, Y \in \Gamma(D^*_\rho) \), since \( D^*_\rho \) is integrable, we have \( [X, Y] \in \Gamma(D^*_\rho) \). Also since \( S(TM) \) is integrable, \( [X, Y] \in \Gamma(S(TM)) \). This results imply \( [X, Y] \in \Gamma(D^*_\rho) \). Thus \( D^*_\rho \) is integrable. By duality, so is \( D^*_\sigma \).
Step 5. $\rho\pi(X) = \sigma\pi(X) = 0$ for all $X \in \Gamma(TM)$.

For $X, Y \in \Gamma(D^*_\rho)$, using (2.8), (2.12) and the fact $\rho$ is a constant, we have

$$(\nabla_X B)(Y, Z) = -g((A^*_\rho - \rho P)\nabla_X Y, Z) + \rho B(X, Y)\eta(Z)$$

for any $Z \in \Gamma(TM)$. Using this equation, (2.10), (3.8) and the facts $(A^*_\rho - \rho P)[X, Y] = 0$ and $(A^*_\rho - \rho P)X = 0$ for any $X \in \Gamma(D_\rho)$, we obtain

$$\rho\pi(X)g(Y, Z) = \rho\pi(Y)g(X, Z).$$

Taking $Z \in \Gamma(TM)$ and using $S(TM)$ is non-degenerate, we have $\rho\pi(X)Y = \rho\pi(Y)X$. Suppose there exists a vector field $X_0 \in \Gamma(D^*_\rho)$ such that $\rho\pi(X_0)_{x} \neq 0$ at each point $x \in M$, then $X = fX_0$ for any $X \in \Gamma(D^*_\rho)$, where $f$ is a smooth function. It follows that all vectors from the fiber $(D^*_\rho)_x$ are colinear with $(X_0)_x$. It is a contradiction as dim $(D^*_\rho)_x = p > 1$. Thus we have $\rho\pi|_{D^*_\rho} = 0$. Using this result and (4.4), we have $\sigma\pi|_{D^*_\rho} = 0$. By duality, we also have $\sigma\pi|_{D^*_\sigma} = 0$. By using (4.4), we have $\rho\pi|_{D^*_\sigma} = 0$. Thus we have our assertion.

Step 6. $D^*_\rho$ and $D^*_\sigma$ are auto-parallel distributions.

For all $X \in \Gamma(D^*_\rho)$ and $U \in \Gamma(D^*_\sigma)$, using (2.8), (2.12), the definition of $D_\rho$ and $D_\sigma$ and the fact that $\rho$ and $\sigma$ are constants, we have

$$(\nabla_X B)(U, Z) = -g((A^*_\rho - \rho P)\nabla_X U, Z),$$

$$(\nabla_U B)(X, Z) = -g((A^*_\sigma - \rho P)\nabla_U X, Z),$$

for all $Z \in \Gamma(S(TM))$. Using these results, (3.8) and Step 5, we obtain

$$g([A^*_\rho - \rho P]\nabla_X U - [A^*_\sigma - \rho P]\nabla_U X, Z) = 0.$$  

Using Step 3 and the fact $S(TM)$ is non-degenerate, we get

$$(A^*_\rho - \rho P)\nabla_X U = (A^*_\sigma - \rho P)\nabla_U X.$$

As the left term of this equation is in $\Gamma(D^*_\rho)$ and the right term is in $\Gamma(D^*_\sigma)$, and $D^*_\rho \cap D^*_\sigma = \{0\}$, we have $(A^*_\rho - \rho P)\nabla_X U = 0$ and $(A^*_\sigma - \rho P)\nabla_U X = 0$. This implies $\nabla_X U \in \Gamma(D_\rho)$ and $\nabla_U X \in \Gamma(D_\sigma)$. From the facts $\nabla_X U = \nabla_X U$ and $\nabla_U X = \nabla_U X$ by Step 1, we have

$$\nabla_X U \in \Gamma(D^*_\rho), \quad \nabla_U X \in \Gamma(D^*_\sigma), \quad \forall X \in \Gamma(D^*_\rho), \quad \forall U \in \Gamma(D^*_\sigma).$$

For $X, Y \in \Gamma(D^*_\rho)$ and $U, V \in \Gamma(D^*_\sigma)$, since $g(X, U) = 0$, we have

$$g(\nabla_Y X, U) + g(X, \nabla_Y U) = 0, \quad g(\nabla_V X, U) + g(U, \nabla_V X) = 0.$$  

Using (4.5) and Step 1, we have $g(X, \nabla_Y U) = g(U, \nabla_V X) = 0$. Thus we get

$$g(\nabla_Y X, U) = 0, \quad g(X, \nabla_V U) = 0.$$  

This result implies that $D^*_\rho$ and $D^*_\sigma$ are auto-parallel distributions.

Since the leaf $M^*$ of $STM$ is a Riemannian manifold and $STM = D^*_\rho \oplus_{orth} D^*_\sigma$, where $D^*_\rho$ and $D^*_\sigma$ are auto-parallel distributions with respect to the induced connection $\nabla^*$ on $M^*$ by Step 6, by the decomposition theorem
of de Rham [2], we have $M^\ast = M_\rho \times M_\sigma$, where $M_\rho$ and $M_\sigma$ are leaves of $D^*_{\rho}$ and $D^*_{\sigma}$ respectively. Thus we have our theorem.

**Theorem 4.2.** Let $M$ be a screen homothetic Einstein lightlike hypersurface of a Lorentzian space form $(\bar{M}(c), \bar{g})$ admitting a semi-symmetric metric connection. Then $c = 0$ and $M$ is locally a product manifold $L \times M_\rho \times M_\sigma$, where $L$ is a null curve tangent to $TM^\perp$, and $M_\rho$ and $M_\sigma$ are leaves of some integrable distributions of $M$ such that

1. If $\kappa \neq 0$, $M_\rho$ or $M_\sigma$ is an $m$-dimensional totally umbilical Einstein Riemannian space form which is isometric to a sphere or a hyperbolic space according to the sign of $\kappa$, and the other is a point.
2. If $\kappa = 0$, $M_\rho$ is an $(m - 1)$-dimensional or an $m$-dimensional totally geodesic Euclidean space and $M_\sigma$ is a spacelike curve or a point.

**Proof.** First of all, we prove if $0 < p < m$, then $\kappa = 0$. Moreover $\rho \sigma = 0$. For $X \in \Gamma(D^\rho_\rho)$ and $U \in \Gamma(D^\sigma_\rho)$, using (2.10), (4.5), (4.6) and Step 5, we have

$$g(R(X, U)U, X) = g(\nabla_X \nabla_U U, X).$$

From the second equation of (4.6), we know that $\nabla_U U$ has no component of $D_\rho$. Since $P$ maps $\Gamma(D_\sigma)$ onto $\Gamma(D^\sigma_\rho)$ and $S(TM) = D^\rho_\rho \oplus_{orth} D^\sigma_\rho$, we have

$$\nabla_U U = P(\nabla_U U) + \eta(\nabla_U U)\xi, \quad P(\nabla_U U) \in \Gamma(D^\sigma_\rho).$$

Using (2.6), (2.12), (3.7) and the facts $A^\rho_\xi X = \rho X$ and $A^\sigma_\xi U = \sigma U$, we have

$$\nabla_X \nabla_U U = \nabla_X P(\nabla_U U) + X(\eta(\nabla_U U))\xi - \varphi \rho \sigma g(U, U)PX.$$

Due to (4.5), we show that $g(\nabla_X P(\nabla_U U), X) = 0$. From the above results we deduce

$$g(R(X, U)U, X) = -\varphi \rho \sigma g(X, X)g(U, U).$$

On the other hand, from the Gauss equation (2.15), we have

$$g(R(X, U)U, X) = \varphi \rho \sigma g(X, X)g(U, U).$$

From the last two equations, we show that if $0 < p < m$, then $\kappa = \varphi \rho \sigma = 0$.

1. Let $\kappa \neq 0$: In case $(tr A^\rho_\xi)^2 \neq 4\varphi^{-1} \kappa$. The equation (4.2) has two non-vanishing distinct solutions $\rho$ and $\sigma$. If $0 < p < m$, then we have $\kappa = 0$. Thus $p = 0$ or $p = m$. If $p = 0$, then $D^\rho_\rho = \{0\}$ and $D^\sigma_\rho = S(TM)$. If $p = m$, then $D^\rho_\rho = S(TM)$ and $D^\sigma_\rho = \{0\}$. From (2.15) and (2.19), we have

$$R^*(X, Y)Z = 2\varphi \rho^2 \{g(Y, Z)X - g(X, Z)Y\}, \quad \forall X, Y, Z \in \Gamma(D^\rho_\rho);$$

$$R^*(U, V)W = 2\varphi \sigma^2 \{g(V, W)U - g(U, W)V\}, \quad \forall U, V, W \in \Gamma(D^\sigma_\rho).$$

Thus either $M_\rho$ or $M_\sigma$ (which are leaves of $D_\rho$ and $D_\sigma$ respectively) is a Riemannian manifold $M^\ast$ of constant curvature either $2\varphi \rho^2$ or $2\varphi \sigma^2$, and the other leaf is a point $\{x\}$. If $M^\ast = M_\rho$, for all $X, Y \in \Gamma(S(TM))$, since $B(X, Y) = \rho g(X, Y)$, we have $C(X, Y) = \varphi \rho g(X, Y)$ and $Ric^*(X, Y) = 2(m - 1) \varphi \rho g(X, Y)$. If $M^\ast = M_\sigma$, for all $U, V \in \Gamma(S(TM))$, since $B(U, V) = \sigma g(U, V)$, we have $C(U, V) = \varphi \sigma g(U, V)$ and $Ric^*(U, V) = 2(m - 1) \varphi \sigma g(U, V)$. Thus $M^\ast$ is
a totally umbilical and \( M \) is locally a product manifold \( L \times M^* \times \{x\} \) or \( L \times \{x\} \times M^* \), where \( M^* \) is an \( m \)-dimensional totally umbilical Einstein Riemannian manifold and the leaf \( M \) of \( \mathcal{S} \) is a totally umbilical and \( \mathcal{N} \)-null curve. Thus the leaf \( M \) is a product manifold \( L \times M^* \times \{x\} \) where \( L \) is a null curve and \( M^* \) is a Riemannian 2-surface of constant curvature \( 2\kappa \) which is isometric to a 2-sphere or a 2-hyperbolic space.

(2) Let \( \kappa = 0 \): The equation (4.2) reduces to \( x(x-s) = 0 \). In case \( trA^*_{\rho} \neq 0 \), let \( \rho = 0 \) and \( \sigma = s \). From (4.4), we have \( (m-p-1)s = 0 \). So \( \rho = m-1 \).

Thus the leaf \( M_{\rho} \) of \( D^p_{\sigma} \) is totally geodesic \( (m-1) \)-dimensional Riemannian manifold and the leaf \( M_{\rho} \) of \( D^p_{\sigma} \) is a spacelike curve. In the sequel, let \( X, Y, Z \in \Gamma(D^p_{\sigma}) \) and \( U \in \Gamma(D^p_{\sigma}) \). From (2.15), (2.19) and \( c = 0 \), we have \( R^*(X,Y)Z = R(X,Y)Z = R(X,Y)Z = R(Y,X)Z = 0 \). Using (4.6) and the fact \( \nabla^*_\nu \) is metric, we have

\[
\frac{g(\nabla_X Y, U)}{g(\nabla_X U, Y)} = -g(Y, \nabla_X U) = -g(Y, \nabla_X U) = 0.
\]

Thus \( \nabla_X Y \in \Gamma(D^p_{\sigma}) \). From this result, (2.6), (4.5) and the integrable property of \( D^p_{\sigma} \), we have \( g(R^*(X,Y)Z, U) = 0 \). This implies \( Q_\rho R^*(X,Y)Z = R^*(X,Y)Z = 0 \), where \( Q_\rho \) is the projection morphism of \( \Gamma(S(TM)) \) on \( \Gamma(D^p_{\sigma}) \) and \( Q_\rho R^* \) is the curvature tensor of \( D^p_{\sigma} \). Thus \( M_{\rho} \) is a Euclidean manifold and \( M \) is locally a product \( L \times M_{\rho} \times M_{\sigma} \), where \( M_{\rho} \) is an \( (m-1) \)-dimensional totally geodesic Euclidean space and \( M_{\sigma} \) is a spacelike curve in \( M \).

In case \( trA^*_{\rho} = 0 \). Then we have \( \rho = \sigma = 0 \) and \( A^*_{\rho} = 0 \) or equivalently \( B = 0 \) and \( D^p_{\rho} = D^p_{\sigma} = S(TM) \). Thus \( M \) is a totally geodesic in \( M \).

Since \( M \) is screen homothetic, we also have \( C = A_N = 0 \). Thus the leaf \( M^* \) of \( S(TM) \) is also totally geodesic. Thus we have \( \nabla_X Y = \nabla_X Y \) for any tangent vector fields \( X \) and \( Y \) to the leaf \( M^* \). This implies that \( M^* \) is a Euclidean \( m \)-space. Thus \( M \) is locally a product \( L \times M^* \times \{x\} \) where \( L \) is a null curve and \( \{x\} \) is a point.

\[\square\]

References


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