AN IDEAL - BASED ZERO-DIVISOR GRAPH OF POSETS

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Abstract. The structure of a poset \( P \) with smallest element 0 is looked at from two viewpoints. Firstly, with respect to the Zariski topology, it is shown that \( \text{Spec}(P) \), the set of all prime semi-ideals of \( P \), is a compact space and \( \text{Max}(P) \), the set of all maximal semi-ideals of \( P \), is a compact \( T_1 \) subspace. Various other topological properties are derived. Secondly, we study the semi-ideal-based zero-divisor graph structure of poset \( P \), denoted by \( G_I(P) \), and characterize its diameter.

1. Preliminaries

Throughout this paper, \((P, \leq)\) denotes a poset with a least element 0, and all prime and maximal semi-ideals of \( P \) are assumed to be proper. For \( M \subseteq P \), let \((M)^l := \{x \in P : x \leq m \text{ for all } m \in M\}\) denote the lower cone of \( M \) in \( P \), and dually let \((M)^u := \{x \in P : m \leq x \text{ for all } m \in M\}\) be the upper cone of \( M \) in \( P \). For \( A, B \subseteq P \), we write \((A, B)^l\) instead of \((A \cup B)^l\) and dually for the upper cones. If \( M = \{x_1, x_2, \ldots, x_n\} \) is finite, then we use the notation \((x_1, x_2, \ldots, x_n)^l\) instead of \((\{x_1, x_2, \ldots, x_n\})^l\) (and dually). We use \( \text{Spec}(P) \) and \( \text{Max}(P) \) for the spectrum of prime semi-ideals and the maximal semi-ideals of \( P \), respectively.

Following [10], a nonempty subset \( I \) of \( P \), \( I \) is called a semi-ideal of \( P \) if \( b \in I \) and \( a \leq b \), then \( a \in I \). A proper semi-ideal \( I \) of \( P \) is called prime if for any \( a, b \in P \), \((a, b)^l \subseteq I\) implies \( a \in I \) or \( b \in I \). In [5], Radomir Halas, in which he has used the term ideals for the semi-ideals of a poset, defined a class of \( n \)-prime semi-ideals in posets, a semi-ideal \( I \) is called \( n \)-prime if for pairwise distinct elements \( x_1, x_2, \ldots, x_n \in P \), if \((x_1, x_2, \ldots, x_n)^l \subseteq I\), then at least \((n - 1)\) of \( n \)-subsets \((x_2, x_3, \ldots, x_n)^l\), \((x_1, x_3, \ldots, x_n)^l\), \(\ldots\), \((x_1, x_2, \ldots, x_{n-1})^l\) is a subset of \( I \). From Theorem 3 of [5], we can observe that every prime semi-ideal of \( P \) is \( n \)-prime. For any semi-ideal \( J \) of \( P \) and \( a \in P \), we define \( V(a) = \{I \in \text{Spec}(P) : a \in I\}\) and \( D(I) = \text{Spec}(P) \setminus V(I) \). Let \( V(J) = \cap_{a \in J} V(a) \). Then \( F = \{V(J) : J \text{ is an semi-ideal of } P\} \) is closed under finite unions and arbitrary intersections, so that there is a topology on \( \text{Spec}(P) \) for which \( F \) is

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the family of closed sets. This is called the Zariski topology. It is easy to see that, for any subset $A$ of $P$, $(A)^{l}$ is a semi-ideal of $P$. If $A = \{a\}$, for any $a \in P$, then $(a)^{l}$ is the smallest semi-ideal containing $a$, and also $V(a) = V((a)^{l})$. Also $B = \{D(a) : a \in P\}$ form a basis for a topology on $Spec(P)$. It is also clear that $Max(P) \subseteq Spec(P)$.

In [2], I. Beck introduced the idea of a zero-divisor graph of a commutative ring. Let the zero-divisors of $R$ be the vertices and connect two vertices $a$ and $b$ by an edge in case $ab = 0$. Later in [1], D. F. Anderson and P. S. Livingston have considered only non-zero zero-divisors as vertices of the zero-divisor graph of $R$, denoted by $\Gamma(R)$, is the (undirected) graph with vertices $Z(R)^{*} = Z(R)\setminus\{0\}$, the set of non-zero zero-divisors of $R$, and for distinct $x, y \in Z(R)^{*}$, the vertices $x$ and $y$ are adjacent if and only if $xy = 0$. In [9], S. P. Redmond generalized this notion by replacing elements whose product is zero with elements whose product lies in some ideal $I$ of $R$.

In [6], R. Halaš and M. Jukl have introduced the concept of a graph structure of posets, let $(P, \leq)$ be a poset with 0. Then the zero-divisor graph of $P$, denoted by $G(P)$, is an undirected graph whose vertices are just the elements of $P$ with two distinct vertices $x$ and $y$ are joined by an edge if and only if $(x,y)^{l} = \{0\}$, and proved some interesting results related with clique and chromatic number of this graph structure.

In [7], V. Joshi introduced the zero divisor graph $G_{I}(P)$ of a poset $P$ (with 0) with respect to an ideal $I$, and proved $G_{I}(P)$ is connected with its diameter 3, also and if $G_{I}(P)$ contains a cycle, then the core $K$ of $G_{I}(P)$ is a union of 3-cycles and 4-cycles.

In this paper, we study the zero divisor graph $G_{I}(P)$ of a poset $P$ with respect to a semi-ideal $I$ as semi-ideal need not be an ideal in poset. Let $P$ be a poset and $J$ be a semi-ideal of $P$. Then the graph of $P$ with respect to the semi-ideal $J$, denoted by $G_{J}(P)$, is the graph whose vertices are the set $\{x \in P\setminus J : (x,y)^{l} \subseteq J$ for some $y \in P\setminus J\}$ with distinct vertices $x$ and $y$ are adjacent if and only if $(x,y)^{l} \subseteq J$. If $J = \{0\}$, then $G_{J}(P) = G(P)$, and $J$ is a prime semi-ideal of $P$ if and only if $G_{J}(P) = \phi$. For distinct vertices $x$ and $y$ of a graph $G$, let $d(x, y)$ be the length of the shortest path from $x$ to $y$. The diameter of a connected graph is the supremum of the distances between vertices.

Following [5], let $I$ be a semi-ideal of $P$. Then the extension of $I$ by $x \in P$ is meant the set $(I : x) = \{a \in P : (a,x)^{l} \subseteq I\}$. For any subset $S$ of $P$, we define $I_{S} = \{a \in P : (a,s)^{l} \subseteq I$ for all $s \in S\}$. Note that $I_{S} = \cap_{s \in S}(I : s)$, if $S = \{a\}$, then $I_{S} = (I : s)$. Let $P$ be the intersection of all prime semi-ideals of $P$. Then we set $Supp(a) = \cap_{y \in (P,a)}V(x)$. In this paper the notations of graph theory are from [3], the notations of posets are from [5] and [7], and the notations of topology are from [4] and [8].
2. Topological space of $\text{Spec}(P)$

In this section, we associate the poset properties of $P$ and the topological
properties of $\text{Spec}(P)$. We start this section with the following useful lemma.

Lemma 2.1. Let $P$ be a poset and $A$ a subset of $P$. Then

(i) If $x \in A$, then $V(A) \subseteq V(x)$ and $D((\mathbb{P} : x)) \subseteq V(x)$.

(ii) If $V(A) = \emptyset$, then $A = P$.

(iii) $D(A) = \emptyset$ if and only if $A \subseteq \mathbb{P}$.

(iv) $V(\{0\}) = \text{Spec}(P)$ and $V(P) = \emptyset$.

(v) $V(I) \cup V(J) = V(I \cap J)$ for any semi-ideals $I, J$ of $P$.

(vi) $\bigcap_{i \in A} V(I_i) = V(\bigcup_{i \in A} I_i)$, $I_i$ is a semi-ideal of $P$ for each $i \in A$.

Lemma 2.2. Let $P$ be a poset. If $A$ is a subset of $\text{Spec}(P)$, then there exists
a semi-ideal $J = \cap A$ of $P$ with $cl(A) = V(J)$. In particular, if $A$ is a closed
subset of $\text{Spec}(P)$, then $A = V(J)$ for some semi-ideal $J$ of $P$.

Proof. Let $A$ be a subset of $\text{Spec}(P)$ and $J = \cap A$. Then it is easy to verify
that $cl(A) \subseteq V(J)$ as $A \subseteq V(J)$. Let $P_1 \in V(J)$ and let $D(x)$ be any arbitrary
element in $B$ such that $P_1 \in D(x)$. Suppose that $D(x) \cap A = \emptyset$. Then $x \in J,$
and so $P_1 \in V(x)$, a contradiction. Thus $D(x) \cap A \neq \emptyset$, and hence, the result
follows from Theorem 17.5 of [8].

With the help of Lemma 2.2, we have the following remark and some
important characterizations of $\text{Spec}(P)$.

Remark 2.3. Let $P$ be a poset. Then

(i) The closure of $I \in \text{Spec}(P)$ is $V(I)$.

(ii) A point $I \in \text{Spec}(P)$ is closed if and only if $I \in \text{Max}(P)$.

(iii) If $I, J \in \text{Spec}(P)$ with $cl(I) = cl(J)$, then $I = J$.

Theorem 2.4. Let $S$ be a subset of $P$. Then $\mathbb{P}_S = \cap V(\mathbb{P}_S)$.

Proof. Clearly, $\mathbb{P}_S \subseteq \cap V(\mathbb{P}_S)$. Let $a \in \cap V(\mathbb{P}_S)$. Suppose on the contrary that
$a \in P \setminus \mathbb{P}$. Then $(a, s)^l \notin I$ for some $I \in \text{Spec}(P)$ and some $s \in S$ which
implies $a \notin I$ and $s \notin I$. So we can get $\mathbb{P}_S \subseteq I$. Thus $a \notin I \in V(\mathbb{P}_S)$, a
contradiction.

Theorem 2.5. Let $P$ be a poset and $a, b \in P$. Then $\text{int}(V(a)) \subseteq \text{int}(V(b))$ if
and only if $(\mathbb{P} : a) \subseteq (\mathbb{P} : b)$.

Proof. Let $\text{int}(V(a)) \subseteq \text{int}(V(b))$ for any $a, b \in P$ and $x \in (\mathbb{P} : a)$. Then $\text{Spec}(P) \setminus V(x) \subseteq \text{int}(V(a)) \subseteq \text{int}(V(b)) \subseteq V(b)$, which gives $(b, x)^l \subseteq \mathbb{P}$, so
$x \in (\mathbb{P} : b)$.

Conversely, let $(\mathbb{P} : a) \subseteq (\mathbb{P} : b)$ and let $I \in \text{int}(V(a))$. Suppose
$I \notin V(b)$. By Lemma 2.2, since $I \notin \text{Spec}(P) \setminus V(a)$, then there is $0 \neq c \in P$
with $\text{Spec}(P) \setminus V(a) \subseteq V(c)$ and $c \notin I$ which imply $(a, c)^l \subseteq \mathbb{P}$. Clearly
$(b, c)^l \notin I$. Then $c \in (\mathbb{P} : a)$ and $c \notin (\mathbb{P} : b)$, a contradiction. Thus $I \in V(b)$
and hence $\text{int}(V(a)) \subseteq V(b)$ which implies $\text{int}(V(a)) \subseteq \text{int}(V(b))$. 

\hfill \Box
Theorem 2.6. Let $P$ be a poset. Then \( \text{cl}(D(a)) = V((\mathbb{P} : a)) = \text{Supp}(a) = \text{Spec}(P) \cap \text{int} V(a) \) for every \( a \in P \).

Proof. It is easy to verify that \( D(a) \subseteq V((\mathbb{P} : a)) \) which implies \( \text{cl}(D(a)) \subseteq V((\mathbb{P} : a)) \). Let \( I \in V((\mathbb{P} : a)) \) and \( D(x) \) be any arbitrary element in \( B \) such that \( I \in D(x) \). We now claim that \( D(x) \cap D(a) \neq \phi \). If \( I \notin D(a) \) and suppose \( D(x) \cap D(a) = \phi \), then \( D(x, a)^{\phi} \subseteq D(x) \cap D(a) = \phi \) which implies \( (x, a)^{\phi} \subseteq \mathbb{P} \).

Then \( x \in I \), a contradiction to \( I \in D(x) \). Thus \( D(x) \cap D(a) \neq \phi \) and hence \( V((\mathbb{P} : a)) \subseteq \text{cl}(D(a)) \). By the definition, we have \( V((\mathbb{P} : a)) = \text{Supp}(a) \). It remains to prove that \( \text{cl}(D(a)) = \text{Spec}(P) \cap \text{int} V(a) \).

Let \( I_1 \in \text{cl}(D(a)) \) and suppose that \( I_1 \in \text{int} V(a) \). Then there exists an open set \( U \) of \( \text{Spec}(P) \) with \( I_1 \in U \subseteq V(a) \), and so \( I_1 \notin \text{Spec}(P) \cup U \), a contradiction as \( \text{Spec}(P) \cup U \) is a closed set containing \( D(a) \). So \( \text{cl}(D(a)) \subseteq \text{Spec}(P) \cap \text{int} V(a) \).

Let \( I_1 \in \text{Spec}(P) \cap \text{int} V(a) \) and let \( D(x) \) be any arbitrary element in \( B \) with \( I_1 \in D(x) \). Suppose that \( D(x) \cap D(a) = \phi \). Then \( I_1 \in D((\mathbb{P} : a)) \subseteq V(a) \), a contradiction. \( \square \)

Lemma 2.7. Let \( P \) be a poset with greatest element \( e \). Then \( \text{Spec}(P) \) does not contain any clopen subset.

Proof. Suppose that \( A \) is a clopen subset of \( \text{Spec}(P) \) and let \( J = \cap A \) and \( J_1 = \cap A^C \). Then by Lemma 2.2 \( A = \text{cl}(A) = V(J) \) and \( A^C = V(J_1) \), and so \( V(J) \cap V(J_1) = \phi \) which gives \( e \in P = J \cup J_1 \), a contradiction. \( \square \)

Lemma 2.8. Let \( P \) be a poset with greatest element \( e \). If \( F \subseteq \text{Spec}(P) \) is a closed set and \( D(K) \) is an open set in \( \text{Spec}(P) \) satisfying \( F \cap \text{Max}(P) \subseteq D(K) \), then \( F \subseteq D(K) \).

Proof. Suppose that there is \( I \in F \) with \( I \notin D(K) \). Then \( K \cup L \subseteq I \), since \( F = V(L) \) for some semi-ideal \( L \) of \( P \). Hence, each maximal semi-ideal \( M \) containing \( I \) is also in \( F \). Then \( M \in F \cap \text{Max}(P) \), and so \( M \in D(K) \), a contradiction. \( \square \)

Theorem 2.9. Let \( P \) be a poset with greatest element \( e \). Then

(i) \( \text{Max}(P) \) is a compact \( T_1 \) subspace.

(ii) If \( \text{Spec}(P) \) is normal, then \( \text{Max}(P) \) is a Hausdorff space.

Proof. (i) Let \( B = \{ D(s_i) : s_i \in J \} \) be the basis of \( P \) for any subset \( J \) of \( P \), and suppose that \( \text{Max}(P) = \bigcup_{s_i \in J} D(s_i) \cap \text{Max}(P) \). Then \( \phi = \cap_{s_i \in J} (\text{Max}(P) \cap D(s_i)) = \cap_{s_i \in J} V(s_i) \cap \text{Max}(P) = V(\cup_{s_i \in J} \{ s_i \} \cap \text{Max}(P) \) which implies \( e \in (s_i)^{\phi} \) and \( e = s_i \) for some \( s_i \in J \). So \( \text{Max}(P) = D(s_i) \).

Let \( M_1 \) and \( M_2 \) be two distinct elements in \( \text{Max}(P) \). Then \( M_1 \in D(M_2) \) and \( M_2 \in D(M_1) \), and so \( \text{Max}(P) \) is a \( T_1 \) space.

(ii) Let \( M_1 \) and \( M_2 \) be distinct elements in \( \text{Max}(P) \). Then \( \{ M_1 \} \) and \( \{ M_2 \} \) are closed subsets in both \( \text{Spec}(P) \) and \( \text{Max}(P) \). If \( \text{Spec}(P) \) is normal, then there exist disjoint open sets \( D(I) \) and \( D(J) \) such that \( \{ M_1 \} \subseteq D(I) \) and \( \{ M_2 \} \subseteq D(J) \) for some semi-ideals \( I \) and \( J \) of \( P \), respectively. So, \( M_1 \in \)}
Theorem 3.2, $I$ be a semi-ideal of $P$. Then $G_I(P)$ is connected and $\text{diam}(G_I(P)) \leq 3$.

Lemma 3.5. Let $I$ be a semi-ideal of $P$. Then a pentagon or hexagon can not be a $G_I(P)$.

Proof. Suppose that $G_I(P)$ is $a - b - c - d - e - a$, a pentagon. Then by Theorem 3.2, $I \cup \{a\}$ is a semi-ideal of $P$. Then in the pentagon, $(a, b)^l \subseteq I$ and $(a, e)^l \subseteq I$. Since $I \cup \{a\}$ is a semi-ideal, and $(a, c)^l \not\subseteq I$, we have $a \leq c$. Similarly, we can show that $a \leq d$. Thus $a \in (c, d)^l \subseteq I$, a contradiction to $a \notin I$. The proof for the hexagon is the same.

Theorem 3.6. If $I \cup \{x\}$ is not a semi-ideal of $P$ for any $x \in P \setminus I$ and $|G_I(P)| \geq 3$, then every pair of vertices in $G_I(P)$ is contained in a cycle of length $\leq 6$.

Proof. Let $a, b \in G_I(P)$. If $(a, b)^l \subseteq I$, then $a - b$ is an edge of triangles or rectangles by Corollary 3.3. If $a - x - b$ is a path in $G_I(P)$, then it is contained in a cycle of length $\leq 4$. If $a - x - y - b$ is a path in $G_I(P)$, then we find cycles $a - x - y - c - a$ and $b - y - x - d - b$ where $c \neq x$ and $d \neq y$. This gives cycle $a - x - d - b - y - c - a$ of length 6.

Lemma 3.7. Let $P$ be a poset and let $a, b \in G_P(P)$. Then

(i) $\text{Supp}(a) \cup \text{Supp}(b) \neq \text{Spec}(P)$ if and only if $\text{Supp}(a) \cup \text{Supp}(b) \subseteq V(c)$ for some $c \in G_P(P)$. 

3. Properties of semi-ideal-based zero-divisor graphs

In this section, we associate the poset properties of $P$ and the graph properties of semi-ideal-based zero-divisor graphs of poset. Although the proof of the following three theorems are just similar of that for Theorem 2.4, Lemma 2.12 and Theorem 2.13 given in [7] to semi-ideal $I$ of $P$.

Theorem 3.1 ([7]). Let $I$ be a semi-ideal of $P$. Then $G_I(P)$ is connected and $\text{diam}(G_I(P)) \leq 3$.

Theorem 3.2 ([7]). Let $I$ be a semi-ideal of $P$ and if $a - x - b$ is a path in $G_I(P)$, then either $I \cup \{x\}$ is a semi-ideal of $P$ or $a - x - b$ is contained in a cycle of length $\leq 4$.

In view of above theorem, we have the following corollary.

Corollary 3.3. Let $|G_I(P)| \geq 3$ and $I \cup \{x\}$ be not a semi-ideal of $P$ for any $x \notin I$. Then any edge in $G_I(P)$ is contained in a cycle of length $\leq 4$, and therefore $G_I(P)$ is a union of triangles and squares.

Theorem 3.4 ([7]). Let $I$ be a semi-ideal of $P$. If $G_I(P)$ contains a cycle, then the core $K$ of $G_I(P)$ is a union of triangles and rectangles. Moreover, any vertex in $G_I(P)$ is either a vertex of the core $K$ of $G_I(P)$ or else is an end vertex of $G_I(P)$.

Lemma 3.5. Let $I$ be a semi-ideal of $P$. Then a pentagon or hexagon can not be a $G_I(P)$.

Proof. Suppose that $G_I(P)$ is $a - b - c - d - e - a$, a pentagon. Then by Theorem 3.2, $I \cup \{a\}$ is a semi-ideal of $P$. Then in the pentagon, $(a, b)^l \subseteq I$ and $(a, e)^l \subseteq I$. Since $I \cup \{a\}$ is a semi-ideal, and $(a, c)^l \not\subseteq I$, we have $a \leq c$. Similarly, we can show that $a \leq d$. Thus $a \in (c, d)^l \subseteq I$, a contradiction to $a \notin I$. The proof for the hexagon is the same.

Theorem 3.6. If $I \cup \{x\}$ is not a semi-ideal of $P$ for any $x \in P \setminus I$ and $|G_I(P)| \geq 3$, then every pair of vertices in $G_I(P)$ is contained in a cycle of length $\leq 6$.

Proof. Let $a, b \in G_I(P)$. If $(a, b)^l \subseteq I$, then $a - b$ is an edge of triangles or rectangles by Corollary 3.3. If $a - x - b$ is a path in $G_I(P)$, then it is contained in a cycle of length $\leq 4$. If $a - x - y - b$ is a path in $G_I(P)$, then we find cycles $a - x - y - c - a$ and $b - y - x - d - b$ where $c \neq x$ and $d \neq y$. This gives cycle $a - x - d - b - y - c - a$ of length 6.

Lemma 3.7. Let $P$ be a poset and let $a, b \in G_P(P)$. Then

(i) $\text{Supp}(a) \cup \text{Supp}(b) \neq \text{Spec}(P)$ if and only if $\text{Supp}(a) \cup \text{Supp}(b) \subseteq V(c)$ for some $c \in G_P(P)$. 

(ii) \( D(a) \cap D(b) \neq \phi \) if and only if there exists \( c \in G_T(P) \) such that \( \phi \neq D(a) \cap D(b) \subseteq V(c) \).

**Proof.** (i) Suppose \( \text{Supp}(a) \cup \text{Supp}(b) \neq \text{Spec}(P) \). Then there exists an element \( P \in \text{Spec}(P) \) with \( x, y \notin P \) for some \( x \in (P : a) \) and \( y \in (P : b) \). So \( (x, y)^I \nsubseteq P \). So there exists \( t \in (x, y)^I \) with \( t \notin P \). It is easy to verify that \( t \in G_T(P) \) and \( \text{Supp}(a) \cup \text{Supp}(b) \subseteq V(t) \). Conversely, let \( \text{Supp}(a) \cup \text{Supp}(b) \subseteq V(c) \) for some \( c \in G_T(P) \) and suppose that \( \text{Supp}(a) \cup \text{Supp}(b) = \text{Spec}(P) \). Then \( c \in P \), a contradiction. Hence, \( \text{Supp}(a) \cup \text{Supp}(b) \neq \text{Spec}(P) \).

(ii) Obvious. \( \square \)

Now by Theorem 3.1, and Lemma 3.7, we have the following characterizations of the diameter of \( G_T(P) \).

**Theorem 3.8.** Let \( P \) be a poset and let \( a, b \in G_T(P) \) be distinct elements. Then

(i) For any \( c \in G_T(P) \), we have \( c \) is adjacent to both \( a \) and \( b \) if and only if \( \text{Supp}(a) \cup \text{Supp}(b) \subseteq V(c) \).

(ii) \( d(a, b) = 1 \) if and only if \( D(a) \cap D(b) = \phi \).

(iii) \( d(a, b) = 2 \) if and only if \( D(a) \cap D(b) \neq \phi \) and \( \text{Supp}(a) \cup \text{Supp}(b) \neq \text{Spec}(P) \).

(iv) \( d(a, b) = 3 \) if and only if \( D(a) \cap D(b) \neq \phi \) and \( \text{Supp}(a) \cup \text{Supp}(b) = \text{Spec}(P) \).

**Proof.** (i) and (ii) are trivial.

(iii) Let \( a, b \in G_T(P) \). Then \( d(a, b) = 2 \) if and only if \( (a, b)^I \nsubseteq P \) and there exists \( c \in G_T(P) \) such that \( c \) is adjacent to both \( a \) and \( b \) if and only if \( D(a) \cap D(b) \neq \phi \) and \( \text{Supp}(a) \cup \text{Supp}(b) \subseteq V(c) \) if and only if \( D(a) \cap D(b) \neq \phi \) and \( \text{Supp}(a) \cup \text{Supp}(b) \neq \text{Spec}(P) \) by Lemma 3.7.

(iv) By Theorem 3.1, \( d(a, b) = 3 \) if and only if \( d(a, b) \neq 1, 2 \) if and only if \( D(a) \cap D(b) \neq \phi \) and \( \text{Supp}(a) \cup \text{Supp}(b) = \text{Spec}(P) \) by (i) and (ii). \( \square \)

**Theorem 3.9.** Let \( I \) be a semi-ideal of \( P \) and let \( a \in G_1(P) \). If \( a \) is adjacent to every vertex in \( G_1(P) \), then \( (I : a) \) is a prime semi-ideal of \( P \).

**Proof.** Let \( (x, y)^I \subseteq (I : a) \) for \( x \in P \). Then \( (a, x, y)^I \subseteq I \) and so \( x \in (I : t) \) for all \( t \in (y, a)^I \). Suppose that \( y \notin (I : a) \). Then there exists \( t_1 \in (y, a)^I \) such that \( t_1 \notin I \). We now claim that \( I_{t_1} = I_a \). Clearly \( (I : a) \subseteq (I : t_1) \). Now let \( p \in (I : t_1) \). If \( p \in I \), then \( p \in (I : a) \). Otherwise \( p \notin I \). It is clear that \( p \in G_2(P) \). Since \( a \) is adjacent to every vertex, therefore \( (p, a)^I \subseteq I \). So \( (I : a) = (I : t_1) \). Since \( x \in (I : t_1) \), we have \( x \in (I : a) \). \( \square \)

**Lemma 3.10.** Let \( P \) be a poset. If \( x \in P \) and \( (I : x) \) is maximal among \( (I : a) = \{ y \in P : (a, y)^I \subseteq I \} \), then \( (I : x) \) is a prime semi-ideal of \( P \).

**Proof.** Suppose that \( (a, b)^I \subseteq (I : x) \) and \( a \notin (I : x) \). Then \( (a, b, x)^I \subseteq I \). Let \( z \in (a, x)^I \setminus I \). Then \( (b, z)^I \subseteq (a, b, x)^I \subseteq I \), thus \( b \in (I : z) \). Since \( (I : x) \subseteq (I : z) \)
and $z \notin I$, we have $(I : z) \neq P$. By the maximality of $(I : x)$, we have $(I : x) = (I : z)$, hence $b \in (I : z) = (I : x)$. □

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References


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