A COUNTEREXAMPLE FOR IMPROVED SOBOLEV INEQUALITIES OVER THE 2-ADIC GROUP

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Abstract. On the framework of the 2-adic group \( \mathbb{Z}_2 \), we study a Sobolev-like inequality where we estimate the \( L^2 \) norm by a geometric mean of the \( BV \) norm and the \( \dot{B}^{-1,\infty} \) norm. We first show, using the special topological properties of the \( p \)-adic groups, that the set of functions of bounded variations \( BV \) can be identified to the Besov space \( \dot{B}^{1,\infty} \). This identification lead us to the construction of a counterexample to the improved Sobolev inequality.

1. Introduction

The general improved Sobolev inequalities were initially introduced by P. Grard, Y. Meyer and F. Oru in [6]. For a function \( f \) such that \( f \in \dot{W}^{s_1,p}(\mathbb{R}^n) \) and \( f \in \dot{B}^{-\beta,\infty}(\mathbb{R}^n) \), these inequalities read as follows:

\[
\|f\|_{\dot{W}^{s,q}} \leq C \|f\|_{\dot{W}^{s_1,q}}^{\theta} \|f\|_{\dot{B}^{-\beta,\infty}}^{1-\theta},
\]

where \( 1 < p < q < +\infty \), \( \theta = p/q \), \( s = \theta s_1 - (1 - \theta)\beta \) and \( -\beta < s < s_1 \). The method used for proving these estimates relies on the Littlewood-Paley decomposition and on a dyadic block manipulation and this explains the fact that the value \( p = 1 \) is forbidden here.

In order to study the case \( p = 1 \), it is necessary to develop other techniques. The case when \( p = 1, s = 0 \) and \( s_1 = 1 \) was treated by M. Ledoux in [11] using a special cut-off function; while the case \( s_1 = 1, p = 1 \) was studied by A. Cohen, W. Dahmen, I. Daubechies and R. De Vore in [5]. In this last article, the authors give a BV-norm weak estimation using wavelet coefficients and isoperimetric inequalities and obtained, for a function \( f \) such that \( f \in BV(\mathbb{R}^n) \) and \( f \in \dot{B}^{-\beta,\infty}(\mathbb{R}^n) \), the estimation below:

\[
\|f\|_{\dot{W}^{s,q}} \leq C \|f\|_{BV}^{1/q} \|f\|_{\dot{B}^{-\beta,\infty}}^{1-1/q},
\]

where \( 1 < q \leq 2, 0 \leq s < 1/q \) and \( \beta = (1 - sq)/(q - 1) \).

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In a previous work (see [3], [4]), we studied the possible generalizations of inequalities of type (1) and (2) to other frameworks than $\mathbb{R}^n$. In particular, we worked over stratified Lie groups and over polynomial volume growth Lie groups and we obtained some new weak-type estimates.

The aim of this paper is to study inequalities of type (1) and (2) in the setting of the 2-adic group $\mathbb{Z}_2$. The main reason for working in the framework of $\mathbb{Z}_2$ is that this group is completely different from $\mathbb{R}^n$ and from stratified or polynomial Lie groups. Indeed, since the 2-adic group is totally discontinuous, it is not absolutely trivial to give a definition for smoothness measuring spaces. Thus, the first step to do, in order to study these Sobolev-like inequalities, is to give an adapted characterization of such functional spaces. In the present article, this will be achieved using the Littlewood-Paley approach and, once this task is done, we will immediately prove -following the classical path exposed in [6]- the inequalities (1) in the setting of the 2-adic group $\mathbb{Z}_2$.

For the estimate (2), we introduce the $BV$ space in the following manner: we will say that $f \in BV(\mathbb{Z}_2)$ if there exists a constant $C > 0$ such that
\[
\int_{\mathbb{Z}_2} |f(x + y) - f(x)| dx \leq C |y|_2 \quad (\forall \, y \in \mathbb{Z}_2).
\]
As a surprising fact, we obtain the following.

**Theorem 1.** We have the following relationship between the space of functions of bounded variation $BV(\mathbb{Z}_2)$ and the Besov space $\dot{B}^{1, \infty}_1(\mathbb{Z}_2)$:
\[
BV(\mathbb{Z}_2) \simeq \dot{B}^{1, \infty}_1(\mathbb{Z}_2).
\]

Of course, this identification is false in $\mathbb{R}^n$ and it is this special relationship in $\mathbb{Z}_2$ that give us our principal theorem which is the 2-adic counterpart of the inequality (2):

**Theorem 2.** The following inequality is false in $\mathbb{Z}_2$. There is not an universal constant $C > 0$ such that we have
\[
\|f\|_{L^2}^2 \leq C \|f\|_{BV} \|f\|_{\dot{B}^{-1, \infty}_\infty}
\]
for all $f \in BV \cap \dot{B}^{-1, \infty}_\infty(\mathbb{Z}_2)$.

This striking fact says that the improved Sobolev inequalities of type (2) depend on the group’s structure and that they are no longer true for the 2-adic group $\mathbb{Z}_2$.

The plan of the article is the following: in Section 2 we recall some well known properties about $p$-adic groups, in Section 3 we define Sobolev and Besov spaces, in Section 4 we prove Theorem 1 and, finally, we prove Theorem 2 in Section 5.

Before finishing these preliminary remarks, it is important to say that the inequality (1) was generalized in [13] where the Besov space $\dot{B}^{-\beta, \infty}_\infty$ is replaced by the $BMO$ space. Thus, with the study of $p$-adic $BMO$ functions given in
[7] and [8] it would be interesting to investigate if such generalization is still valid in a $p$-adic setting.

2. $p$-adic groups

We write $a|b$ when $a$ divide $b$ or, equivalently, when $b$ is a multiple of $a$. Let $p$ be any prime number, for $0 \neq x \in \mathbb{Z}$, we define the $p$-adic valuation of $x$ by $\gamma(x) = \max\{r : p^r|x| \geq 0\}$ and, for any rational number $x = \frac{a}{b} \in \mathbb{Q}$, we write $\gamma(x) = \gamma(a) - \gamma(b)$. Furthermore if $x = 0$, we agree to write $\gamma(0) = +\infty$.

Let $x \in \mathbb{Q}$ and $p$ be any prime number, with the $p$-adic valuation of $x$ we can construct a norm by writing

$$|x|_p = \begin{cases} p^{-\gamma} & \text{if } x \neq 0 \\ p^{-\infty} & \text{if } x = 0. \end{cases}$$

This expression satisfy the following properties

a) $|x|_p \geq 0$, and $|x|_p = 0 \iff x = 0$;

b) $|xy|_p = |x|_p|y|_p$;

c) $|x + y|_p \leq \max\{|x|_p, |y|_p\}$, with equality when $|x|_p \neq |y|_p$.

When a norm satisfy c) it is called a non-Archimedean norm and an interesting fact is that over $\mathbb{Q}$ all the possible norms are equivalent to $|\cdot|_p$ for some $p$: this is the so-called Ostrowski theorem (see [1] for a proof).

**Definition 2.1.** Let $p$ be any prime number. We define the field of $p$-adic numbers $\mathbb{Q}_p$ as the completion of $\mathbb{Q}$ when using the norm $|\cdot|_p$.

We present in the following lines the algebraic structure of the set $\mathbb{Q}_p$. Every $p$-adic number $x \neq 0$ can be represented in a unique manner by the formula

$$x = p^\gamma(x_0 + x_1p + x_2p^2 + \cdots),$$

where $\gamma = \gamma(x)$ is the $p$-adic valuation of $x$ and $x_j$ are integers such that $x_0 > 0$ and $0 \leq x_j \leq p - 1$ for $j = 1, 2, \ldots$. Remark that this canonical representation implies the identity $|x|_p = p^{-\gamma}$.

Let $x, y \in \mathbb{Q}_p$, using the formula (4) we define the sum of $x$ and $y$ by $x + y = p^\gamma(x + y)(c_0 + c_1p + c_2p^2 + \cdots)$ with $0 \leq c_j \leq p - 1$ and $c_0 > 0$, where $\gamma(x + y)$ and $c_j$ are the unique solution of the equation

$$p^\gamma(x_0 + x_1p + x_2p^2 + \cdots) + p^\gamma(y_0 + y_1p + y_2p^2 + \cdots) = p^\gamma(x_0 + y_0 + x_1p + c_2p^2 + \cdots).$$

Furthermore, for $a, x \in \mathbb{Q}_p$, the equation $a + x = 0$ has a unique solution in $\mathbb{Q}_p$ given by $x = -a$. In the same way, the equation $ax = 1$ has a unique solution in $\mathbb{Q}_p$: $x = 1/a$.

We take now a closer look at the topological structure of $\mathbb{Q}_p$. With the norm $|\cdot|_p$ we construct a distance over $\mathbb{Q}_p$ by writing

$$d(x, y) = |x - y|_p.$$
and we define the balls $B_{\gamma}(x) = \{ y \in \mathbb{Q}_p : d(x, y) \leq p^\gamma \}$ with $\gamma \in \mathbb{Z}$. Remark that, from the properties of the $p$-adic valuation, this distance has the ultrametric property (i.e., $d(x, y) \leq \max\{d(x, z), d(z, y)\} \leq |x|_p + |y|_p$).

We gather with the next proposition some important facts concerning the balls in $\mathbb{Q}_p$.

**Proposition 2.1.** Let $\gamma$ be an integer, then we have

1) the ball $B_{\gamma}(x)$ is an open and a closed set for the distance (5).
2) every point of $B_{\gamma}(x)$ is its center.
3) $\mathbb{Q}_p$ endowed with this distance is a complete Hausdorff metric space.
4) $\mathbb{Q}_p$ is a locally compact set.
5) the $p$-adic group $\mathbb{Q}_p$ is a totally discontinuous space.

For a proof of this proposition and more details see the books [1], [10] or [14].

### 3. Functional spaces

In this article, we will work with the subset $\mathbb{Z}_2$ of $\mathbb{Q}_2$ which is defined by $\mathbb{Z}_2 = \{ x \in \mathbb{Q}_2 : |x|_2 \leq 1 \}$, and we will focus on real-valued functions over $\mathbb{Z}_2$. Since $\mathbb{Z}_2$ is a locally compact commutative group, there exists a Haar measure $dx$ which is translation invariant i.e., $d(x + a) = dx$, furthermore we have the identity $d(xa) = |a|_2^2 dx$ for $a \in \mathbb{Z}_2^\times$. We will normalize the measure $dx$ by setting

$$\int_{\{ |x|_2 \leq 1 \}} dx = 1.$$  

This measure is then unique and we will note $|E|$ the measure for any subset $E$ of $\mathbb{Z}_2$.

Another type of measures can be considered on the $p$-adic setting (see for example [9]). However, in this article we will only work with the previous one.

Lebesgue spaces $\mathbb{L}^p(\mathbb{Z}_2)$ are thus defined in a natural way:

$$\|f\|_{L^p} = (\int_{\mathbb{Z}_2} |f(x)|^p dx)^{1/p} \text{ for } 1 \leq p < +\infty,$$

with the usual modifications when $p = +\infty$.

Let us now introduce the Littlewood-Paley decomposition in $\mathbb{Z}_2$. We note $F_j$ the Boole algebra formed by the equivalence classes $E \subset \mathbb{Z}_2$ modulo the subgroup $2^j\mathbb{Z}_2$. Then, for any function $f \in \mathbb{L}^1(\mathbb{Z}_2)$, we call $S_j(f)$ the conditionnal expectation of $f$ with respect to $F_j$:

$$S_j(f)(x) = \frac{1}{|B_j(x)|} \int_{B_j(x)} f(y) dy.$$

The dyadic blocks are thus defined by the formula $\Delta_j(f) = S_{j+1}(f) - S_j(f)$ and the Littlewood-Paley decomposition of a function $f : \mathbb{Z}_2 \rightarrow \mathbb{R}$ is given by

$$f = S_0(f) + \sum_{j=0}^{+\infty} \Delta_j(f) \quad \text{where } S_0(f) = \int_{\mathbb{Z}_2} f(x) dx.$$  

Here $\mathbb{Z}_2 = \{ x \in \mathbb{Q}_2 : |x|_2 \leq 1 \}$ and we define the balls $B_{\gamma}(x) = \{ y \in \mathbb{Q}_p : d(x, y) \leq p^\gamma \}$ with $\gamma \in \mathbb{Z}$. Remark that, from the properties of the $p$-adic valuation, this distance has the ultrametric property (i.e., $d(x, y) \leq \max\{d(x, z), d(z, y)\} \leq |x|_p + |y|_p$).

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$$f = S_0(f) + \sum_{j=0}^{+\infty} \Delta_j(f) \quad \text{where } S_0(f) = \int_{\mathbb{Z}_2} f(x) dx.$$
We will need in the sequel some very special sets noted $Q_{j,k}$. Here is the definition and some properties:

**Proposition 3.1.** Let $j \in \mathbb{N}$ and $k = \{0, 1, \ldots, 2^j - 1\}$. Define the subset $Q_{j,k}$ of $\mathbb{Z}_2$ by

\[ Q_{j,k} = \{ k + 2^j \mathbb{Z}_2 \} \quad (7) \]

Then

1) We have the identity $F_j = \bigcup_{0 \leq k < 2^j} Q_{j,k},$

2) For $k = \{0, 1, \ldots, 2^j - 1\}$ the sets $Q_{j,k}$ are mutually disjoint,

3) $|Q_{j,k}| = 2^{-j}$ for all $k$,

4) the 2-adic valuation is constant over $Q_{j,k}$.

The verifications are easy and left to the reader.

With the Littlewood-Paley decomposition given in (6), we obtain the following equivalence for the Lebesgue spaces $L^p(\mathbb{Z}_2)$ with $1 < p < +\infty$:

\[
\|f\|_{L^p} \simeq \|S_0(f)\|_{L^p} + \left\| \left( \sum_{j \in \mathbb{N}} |\Delta_j f|^2 \right)^{1/2} \right\|_{L^p}.
\]

See the book [12], Chapter IV, for a general proof.

Let us turn now to smoothness measuring spaces. As said in the introduction, it is not absolutely trivial to define Sobolev and Besov spaces over $\mathbb{Z}_2$ since we are working in a totally discontinuous setting. Here is an example of this situation with the Sobolev space $W^{1,2}(\mathbb{Z}_2)$: one could try to define the quantity $|\nabla f|$ by the formula

\[
|\nabla f| = \lim_{\delta \to 0} \sup_{d(x,y) < \delta} \frac{|f(x) - f(y)|}{d(x,y)}
\]

and define the Sobolev space $W^{1,2}(\mathbb{Z}_2)$ by the norm

\[
\|f\|_* = \|f\|_{L^2} + \left( \int_{\mathbb{Z}_2} |\nabla f|^2 \, dx \right)^{1/2}. \quad (8)
\]

Now, using the Littlewood-Paley decomposition we can also write

\[
\|f\|_{**} = \|S_0 f\|_{L^2} + \left\| \left( \sum_{j \in \mathbb{N}} 2^{2j} |\Delta_j f|^2 \right)^{1/2} \right\|_{2}.
\]

However, the quantities $\| \cdot \|_*$ and $\| \cdot \|_{**}$ are not equivalent: in the case of (8) consider a function $f = c_k$ constant over each $Q_{j,k} = \{ k + 2^j \mathbb{Z}_2 \}$ for some fixed $j$. Then we have $|\nabla f| \equiv 0$ and for these functions the norm $\| \cdot \|_*$ would be equal to the $L^2$ norm.
This is the reason why we will use in this article the Littlewood-Paley approach to characterize Sobolev spaces:

\[
\|f\|_{W^{s,p}} \simeq \|S_0 f\|_{L^p} + \left( \sum_{j \in \mathbb{N}} 2^{j s} |\Delta_j f|^2 \right)^{1/2}
\]

with \(1 < p < +\infty\) and \(s > 0\). For Besov spaces we will define them by the norm

\[
\|f\|_{B^{s,q}_p} \simeq \|S_0 f\|_{L^p} + \left( \sum_{j \in \mathbb{N}} 2^{j s q} \|\Delta_j f\|_{L^q}^q \right)^{1/q},
\]

where \(s \in \mathbb{R}\), \(1 \leq p, q < +\infty\) with the necessary modifications when \(p, q = +\infty\).

Remark 1. For homogeneous functional spaces \(\dot{W}^{s,p}\) and \(\dot{B}^{s,q}_p\), we drop out the term \(\|S_0 f\|_{L^p}\) in (9) and (10).

Let us give some simple examples of function belonging to these functional spaces.

1) The function \(f(x) = \log_2 |x|_2\) is in \(\dot{B}^{1,\infty}_1(\mathbb{Z}_2)\). First note that \(|x|_2 = 2^{-\gamma(x)}\) and thus \(f(x) = -\gamma(x)\). Recall (cf. Proposition 3.1) that over each set \(Q_{j,k}\), the quantity \(\gamma(x)\) is constant, so the dyadic block \(\Delta_j f\) is given by

\[
\Delta_j f(x) = \begin{cases} -1 & \text{over } Q_{j+1,0} \\ 0 & \text{elsewhere.} \end{cases}
\]

Hence, taking the \(L^1\) norm, we have \(\|\Delta_j f\|_{L^1} = \frac{1}{2} 2^{-j}\) and then \(f \in \dot{B}^{1,\infty}_1(\mathbb{Z}_2)\).

2) Set \(h(x) = 1/|x|_2\), we have \(h \in \dot{B}^{-1,\infty}_1\). For this, we must verify \(\sup_{j \geq 0} 2^{-j} \|\Delta_j h\|_{L^\infty} < +\infty\). By definition we obtain \(h(x) = 2^\gamma(x)\) and then

\[
\Delta_j h(x) = \begin{cases} 2^j & \text{over } Q_{j+1,0} \\ 0 & \text{elsewhere.} \end{cases}
\]

We finally obtain \(\|\Delta_j h\|_{L^\infty} = 2^j\) and hence \(2^{-j} \|\Delta_j h\|_{L^\infty} = 1\) for all \(j\), so we write \(h \in \dot{B}^{-1,\infty}_1\).

With the Littlewood-Paley characterisation of Sobolev spaces and Besov spaces given in (9) and (10) we have the following theorem:

**Theorem 3.** In the framework of the 2-adic group \(\mathbb{Z}_2\) we have, for a function \(f\) such that \(f \in W^{s_1,p}(\mathbb{Z}_2)\) and \(f \in B^{-\beta,\infty}_\infty(\mathbb{Z}_2)\), the inequality

\[
\|f\|_{W^{s,q}} \leq C \|f\|_{W^{s_1,p}} \|f\|_{B^{-\beta,\infty}_\infty}^{1-\theta},
\]

where \(1 < p < q < +\infty\), \(\theta = p/q\), \(s = \theta s_1 - (1 - \theta)\beta\) and \(-\beta < s < s_1\).
Proof. We start with an interpolation result: let \((a_j)_{j \in \mathbb{N}}\) be a sequence, let 
\(s = \theta s_1 - (1 - \theta) s_2\) with \(\theta = p/q\), then we have for \(r, r_1, r_2 \in [1, +\infty]\) the estimate
\[
\|2^{js}a_j\|_{r} \leq C\|2^{js}a_j\|_{r_1}^{\theta}\|2^{-js}a_j\|_{r_2}^{1-\theta}.
\]
See [2] for a proof. Apply this estimate to the dyadic blocks \(\Delta_j f\) to obtain
\[
\left(\sum_{j \in \mathbb{Z}} 2^{2js}|\Delta_j f(x)|^2\right)^{1/2} \leq C \left(\sum_{j \in \mathbb{Z}} 2^{2js}|\Delta_j f(x)|^2\right)^{\theta/2} \left(\sup_{x \in \mathbb{Z}} 2^{-j\beta}|\Delta_j f(x)|\right)^{1-\theta}.
\]
To finish, compute the \(L^q\) norm of the preceding quantities. \(\square\)

4. The \(BV(\mathbb{Z}_2)\) space and the proof of Theorem 1

We study in this section the space of functions of bounded variation \(BV\) and we will prove some surprising facts in the framework of 2-adic group \(\mathbb{Z}_2\). Let us start recalling the definition of this space:

**Definition 4.1.** If \(f\) is a real-valued function over \(\mathbb{Z}_2\), we will say that \(f \in BV(\mathbb{Z}_2)\) if there exists a constant \(C > 0\) such that
\[
\int_{\mathbb{Z}_2} |f(x + y) - f(x)|\,dx \leq C|y|_2, \quad (\forall y \in \mathbb{Z}_2).
\]

We prove now Theorem 1 which asserts that in \(\mathbb{Z}_2\), the \(BV\) space can be identified to the Besov space \(\dot{B}^{1,\infty}_1\). For this, we will use two steps given by Propositions 4.1 and 4.2 below.

**Proposition 4.1.** If \(f\) is a real-valued function over \(\mathbb{Z}_2\) belonging to the Besov space \(\dot{B}^{1,\infty}_1\), then \(f \in BV\) and we have the inclusion \(\dot{B}^{1,\infty}_1 \subseteq BV\).

**Proof.** Let \(f \in \dot{B}^{1,\infty}_1(\mathbb{Z}_2)\) and let us fix \(|y|_2 = 2^{-m}\). We have to prove the following estimation for all \(m > 0\)
\[
I = \int_{\mathbb{Z}_2} |f(x + y) - f(x)|\,dx \leq C 2^{-m}.
\]
Using the Littlewood-Paley decomposition given in (6), we will work on the formula below
\[
I = \left\| \left( S_0 f(x + y) + \sum_{j \geq 0} \Delta_j f(x + y) \right) - \left( S_0 f(x) + \sum_{j \geq 0} \Delta_j f(x) \right) \right\|_{L^1}.
\]
Then, by the dyadic block’s properties we have to study
\[
I \leq \|S_m f(x + y) - S_m f(x)\|_{L^1} + \sum_{j = m+1}^{+\infty} \|\Delta_j f(x + y) - \Delta_j f(x)\|_{L^1}.
\]
We estimate this inequality with the two following lemmas.

**Lemma 4.1.** The first term in (12) is identically zero.

**Proof.** Since we have fixed $|y|_2 = 2^{-m}$, then for $x \in Q_{m,k}$, we have $x + y \in Q_{m,k}$ with $k = \{0, \ldots, 2^m - 1\}$. Applying the operators $S_m$ to the functions $f(x+y)$ and $f(x)$ we get the desired result. □

The second term in (12) is treated by the next lemma.

**Lemma 4.2.** Under the hypothesis of Proposition 4.1 and for $|y|_2 = 2^{-m}$ we have

$$\sum_{j=m+1}^{+\infty} \|\Delta_j f(x+y) - \Delta_j f(x)\|_{L^1} \leq C 2^{-m}.$$

**Proof.** Indeed,

$$\sum_{j=m+1}^{+\infty} \|\Delta_j f(x+y) - \Delta_j f(x)\|_{L^1} \leq 2 \sum_{j=m+1}^{+\infty} \|\Delta_j f\|_{L^1}.$$  

We use now the fact $\|\Delta_j f\|_{L^1} \leq C 2^{-j}$ for all $j$, since $f \in \dot{B}^{1,\infty}_1$, to get

$$\sum_{j=m+1}^{+\infty} \|\Delta_j f(x+y) - \Delta_j f(x)\|_{L^1} \leq C 2^{-m}.$$  

With these two lemmas, and getting back to (12), we deduce the following inequality for all $y \in \mathbb{Z}_2$:

$$\int_{\mathbb{Z}_2} |f(x+y) - f(x)| dx \leq C |y|_2$$

and this concludes the proof of Proposition 4.1. □

Our second step in order to prove Theorem 1 is the next result.

**Proposition 4.2.** In $\mathbb{Z}_2$ we have the inclusion $BV(\mathbb{Z}_2) \subseteq \dot{B}^{1,\infty}_1(\mathbb{Z}_2)$.

**Proof.** Observe that we can characterize the Besov space $\dot{B}^{1,\infty}_1(\mathbb{Z}_2)$ by the condition

$$\|f(\cdot + y) + f(\cdot - y) - 2 f(\cdot)\|_{L^1} \leq C |y|_2, \quad \forall y \neq 0.$$  

Let $f$ be a function in $BV(\mathbb{Z}_2)$, then we have

$$\|f(\cdot + y) - f(\cdot)\|_{L^1} \leq C |y|_2.$$  

Summing $\|f(\cdot - y) - f(\cdot)\|_{L^1}$ in both sides of the previous inequality we obtain

$$\|f(\cdot + y) - f(\cdot)\|_{L^1} + \|f(\cdot - y) - f(\cdot)\|_{L^1} \leq C |y|_2 + \|f(\cdot - y) - f(\cdot)\|_{L^1}$$

and by the triangular inequality we have

$$\|f(\cdot + y) + f(\cdot - y) - 2 f(\cdot)\|_{L^1} \leq C |y|_2 + \|f(\cdot - y) - f(\cdot)\|_{L^1}.$$
We thus obtain
\[ \|f(\cdot + y) + f(\cdot - y) - 2f(\cdot)\|_{L^1} \leq 2C |y|_2. \]

We have proved, in the setting of the 2-adic group \( \mathbb{Z}_2 \), the inequalities
\[ C_1 \|f\|_{\dot{B}_1^{1,\infty}} \leq \|f\|_{BV} \leq C_2 \|f\|_{\dot{B}^{-1}_{1,\infty}}, \]
so Theorem 1 follows.

5. Improved Sobolev inequalities, \( BV \) space and proof of Theorem 2

We do not give here a global treatment of the family of inequalities of type (2); instead we focus on the next inequality
\[ \|f\|_{L^2} \leq C \|f\|_{BV} \|f\|_{\dot{B}^{-1}_{1,\infty}} \]
and we want to know if this estimation is true in a 2-adic framework. Since in the \( \mathbb{Z}_2 \) setting we have the identification \( \|f\|_{BV} \simeq \|f\|_{\dot{B}_1^{1,\infty}} \), the estimation (13) becomes
\[ \|f\|_{L^2} \leq C \|f\|_{\dot{B}_1^{1,\infty}} \|f\|_{\dot{B}^{-1}_{1,\infty}}. \]
This remark lead us to Theorem 2 which states that the previous inequalities are false.

Proof. We will construct a counterexample by means of the Littlewood-Paley decomposition, so it is worth to recall very briefly the dyadic bloc characterization of the norms involved in inequality (14). For the \( L^2 \) norm we have \( \|f\|_{L^2}^2 = \sum_{j \in \mathbb{N}} \|\Delta_j f\|_{L^2}^2 \), while for the Besov spaces \( \dot{B}_1^{1,\infty} \) and \( \dot{B}^{-1}_{1,\infty} \) we have
\[ \|f\|_{\dot{B}_1^{1,\infty}} = \sup_{j \in \mathbb{N}} 2^j \|\Delta_j f\|_{L^1} \quad \text{and} \quad \|f\|_{\dot{B}^{-1}_{1,\infty}} = \sup_{j \in \mathbb{N}} 2^{-j} \|\Delta_j f\|_{L^\infty}. \]
We construct a function \( f : \mathbb{Z}_2 \rightarrow \mathbb{R} \) by considering his values over the dyadic blocs and we will use for this the sets \( Q_{j,k} \) defined in (7). First fix \( \alpha \) and \( \beta \) two non negative real numbers and \( j_0, j_1 \) two integers such that \( 0 \leq j_0 \leq j_1 \) with the condition
\[ 2^{2j_0} \leq \frac{\beta}{\alpha}. \]
Now define \( N_j \) as a function of \( \alpha \) and \( \beta \):
\[ N_j = 2^j \quad \text{if} \ 0 \leq j \leq j_0 \quad \text{and} \quad N_j = \frac{\beta}{\alpha} 2^{-j} \leq 2^j \quad \text{if} \ j_0 < j \leq j_1. \]
and write

\[
\Delta_j f(x) = \begin{cases} 
\alpha 2^j & \text{over } Q_{j+1,0}, \\
-\alpha 2^j & \text{over } Q_{j+1,1}, \\
\alpha 2^j & \text{over } Q_{j+1,2}, \\
-\alpha 2^j & \text{over } Q_{j+1,3}, \\
\vdots \\
\alpha 2^j & \text{over } Q_{j+1,2N_j-2}, \\
-\alpha 2^j & \text{over } Q_{j+1,2N_j-1}, \\
0 & \text{elsewhere.}
\end{cases}
\]

Once this function is fixed, we compute the following norms

- \( \|\Delta_j f\|_{L^1} = \sum_{k=0}^{N_j} \alpha 2^j 2^{-j} = \alpha N_j \),
- \( \|\Delta_j f\|_{L^\infty} = \alpha 2^j \),
- \( \|\Delta_j f\|_{L^2}^2 = \sum_{k=0}^{N_j} \alpha^2 2^{2j} 2^{-j} = \alpha^2 2^j N_j \),

and we build from these quantities the Besov and Lebesgue norms in the following manner:

1) For the Besov space \( \dot{B}^{-1,\infty}_{\infty} \):
   \[
   \|f\|_{\dot{B}^{-1,\infty}_{\infty}} = \sup_{0 \leq j \leq j_1} 2^{-j} \alpha 2^j = \alpha,
   \]
   2) For the Besov space \( \dot{B}_{1,\infty}^1 \):
   By the definition (15) of \( N_j \) we have \( 2^j \|\Delta_j f\|_{L^1} = 2^j \alpha N_j = 2^j \alpha \) if \( 0 \leq j \leq j_0 \) and \( 2^j \|\Delta_j f\|_{L^1} = \beta \) if \( j_0 < j \leq j_1 \). Since \( 2^{2j_0} \leq \frac{\beta}{\alpha} \) we have:
   \[
   \|f\|_{\dot{B}_{1,\infty}^1} = \beta.
   \]
   3) For the Lebesgue space \( L^2 \):
   \[
   \|f\|_{L^2}^2 = \sum_{j=0}^{j_1} \alpha^2 2^j N_j = \sum_{j=0}^{j_0} \alpha^2 2^{2j} + \sum_{j>j_0}^{j_1} \alpha^2 2^j \frac{\beta}{\alpha} 2^{-j} = \sum_{j=0}^{j_0} \alpha^2 2^{2j} + (j_1 - j_0) \alpha \beta
   \]
   \[
   = \alpha \beta \left( \frac{\alpha}{\beta} \sum_{j=0}^{j_0} 2^{2j} + (j_1 - j_0) \right).
   \]
   With the condition \( 2^{2j_0} \leq \frac{\beta}{\alpha} \), we obtain from the previous formula that
   \[
   \|f\|_{L^2}^2 \simeq \alpha \beta (j_1 - j_0) = \|f\|_{\dot{B}_{1,\infty}^1} \|f\|_{\dot{B}^{-1,\infty}_{\infty}} (j_1 - j_0).
   \]
   Thus, getting back to (14) and therefore to (13), we have for an universal constant \( C \) the inequality
   \[
   \|f\|_{\dot{B}_{1,\infty}^1} \|f\|_{\dot{B}^{-1,\infty}_{\infty}} (j_1 - j_0) \leq C \|f\|_{\dot{B}_{1,\infty}^1} \|f\|_{\dot{B}^{-1,\infty}_{\infty}} \]
   \[
   \iff (j_1 - j_0) \leq C,
   \]
   which is false since we can freely choose the values of \( j_1 \) and \( j_0 \). Theorem 2 is proved. \(\square\)
References


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