A FIXED POINT APPROACH TO THE STABILITY OF THE GENERALIZED POLYNOMIAL FUNCTIONAL EQUATION OF DEGREE 2

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Abstract. In this paper, we investigate a stability of the functional equation
\[ \sum_{i=0}^{3} (-1)^{i+1} f(i x + y) = 0 \]
by using the fixed point theory in the sense of L. Cădariu and V. Radu.

1. Introduction

In 1940, S. M. Ulam [24] raised a question concerning the stability of homomorphisms: Given a group \( G_1 \), a metric group \( G_2 \) with the metric \( d(\cdot, \cdot) \), and a positive number \( \varepsilon \), does there exist a \( \delta > 0 \) such that if a mapping \( f : G_1 \to G_2 \) satisfies the inequality
\[ d(f(xy), f(x)f(y)) < \delta \]
for all \( x, y \in G_1 \), then there exists a homomorphism \( F : G_1 \to G_2 \) with
\[ d(f(x), F(x)) < \varepsilon \]
for all \( x \in G_1 \)? When this problem has a solution, we say that the homomorphisms from \( G_1 \) to \( G_2 \) are \textit{stable}. In the next year, D. H. Hyers [6] gave a partial solution of Ulam’s problem for the case of approximate additive mappings under the assumption that \( G_1 \) and \( G_2 \) are Banach spaces. Hyers’ result was generalized by T. Aoki [1] for additive mappings, and by Th. M. Rassias [22] for linear mappings, by considering the stability problem with unbounded Cauchy differences. The paper of Th. M. Rassias had much influence in the development of stability problems. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians (see [5], [10]-[19]).

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Almost all subsequent proofs, in this very active area, have used Hyers’ method. Namely, the solution \( F \) of a functional equation, starting from the given mapping \( f \), is explicitly constructed by the formulae

\[
F(x) = \lim_{n \to \infty} \frac{1}{2^n} f(2^n x) \quad \text{or} \quad F(x) = \lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right).
\]

We call it a direct method. In 2003, L. Cădariu and V. Radu [2] observed that the existence of the solution \( F \) of a functional equation and the estimation of the difference with the given mapping \( f \) can be obtained from the fixed point theory alternative. They applied this method to prove stability theorems of Jensen’s functional equation:

\[
2f\left(\frac{x + y}{2}\right) - f(x) - f(y) = 0.
\]

This method is called a fixed point method. In 2005, L. Cădariu [3] obtained a stability of the quadratic functional equation:

\[
f(x + y) + f(x - y) - 2f(x) - 2f(y) = 0
\]

by using the fixed point method. If we consider the functions \( f_1, f_2 : \mathbb{R} \to \mathbb{R} \) defined by \( f_1(x) = ax + b \) and \( f_2(x) = ax^2 \), where \( a \) and \( b \) are real constants, then \( f_1 \) satisfies the equation (1.1) and \( f_2 \) satisfies the equation (1.2), respectively. Now we consider the functional equation

\[
\sum_{i=0}^{3} 3C_i (-1)^{3-i} f(ix + y) = 0
\]

which is called the generalized polynomial functional equation of degree 2. The function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = ax^2 + bx + c \) satisfies this functional equation. We call a solution of (1.3) a general quadratic mapping. On the other hand, a solution of (1.1) with the condition \( f(0) = 0 \) is called an additive mapping and a solution of (1.2) a quadratic mapping, respectively. In [14], Lee obtained a stability of the functional equation (1.3) by handling the odd part and the even part of the given mapping \( f \), respectively. In the processing, Lee needed to take an additive mapping \( A \) which is close to the odd part \( \frac{f(x) - f(-x)}{2} \) of \( f \) and a quadratic mapping \( Q \) which is approximate to the even part \( \frac{f(x) + f(-x)}{2} - f(0) \) of it, and then combining \( A \) and \( Q \) to prove the existence of a general quadratic mapping \( F \) which is close to the given mapping \( f \).

In this paper, we will prove the stability of a generalized polynomial functional equation of degree 2 (1.3) by using the fixed point theory. In the previous results of stability problems of (1.3), as we mentioned above, we get a solution by using the direct method to the odd part and even part, respectively. Instead of splitting the given mapping \( f : X \to Y \) into two parts, in this paper, we can take the desired solution \( F \) at once. Precisely, we introduce a strictly contractive mapping with Lipschitz constant \( 0 < L < 1 \). Using the fixed point theory in the sense of L. Cădariu and V. Radu, together with suitable conditions, we
can show that the contractive mapping has a fixed point. Actually the fixed point \( F \) becomes the precise solution of the functional equation (1.3). Recently, using such an idea, the authors and S.-M. Jung have proved several kinds of stability results of functional equations (see [7]-[9], [20], [21]). In Section 2, we consider the fundamental result in the fixed point theory and construct some strictly contractive self-mappings. In Section 3, we prove several stability results of the functional equation (1.3) using the fixed point theory (see Theorem 3.2, Theorem 3.3, Theorem 3.4, and Theorem 3.5). In Section 4, we use the results in the previous sections to get a stability of Jensen’s functional equation (1.1) and that of the quadratic functional equation (1.2), respectively.

2. Preliminaries

We recall the fundamental result in the fixed point theory.

**Theorem 2.1** ([4] or [23]). Suppose that a complete generalized metric space \((X,d)\), which means that the metric \(d\) may assume infinite values, and a strictly contractive mapping \(A : X \to X\) with the Lipschitz constant \(0 < L < 1\) are given. Then, for each given element \(x \in X\), either
\[
d(A^n x, A^{n+1} x) = +\infty, \quad \forall n \in \mathbb{N} \cup \{0\}
\]
or there exists a nonnegative integer \(k\) such that:
1. \(d(A^n x, A^{n+1} x) < +\infty\) for all \(n \geq k\);
2. the sequence \(\{A^n x\}\) is convergent to a fixed point \(y^*\) of \(A\);
3. \(y^*\) is the unique fixed point of \(A\) in \(Y := \{y \in X, d(A^k x, y) < +\infty\}\);
4. \(d(y, y^*) \leq (1/(1-L))d(y, Ay)\) for all \(y \in Y\).

Throughout this paper, let \(V\) be a (real or complex) linear space and \(Y\) a Banach space. For a mapping \(\varphi : (V \setminus \{0\})^2 \to [0, \infty)\), we will introduce a generalized metric \(d_{\varphi}\) on the set \(S := \{g : V \to Y \mid g(0) = 0\}\) by following
\[
d_{\varphi}(g, h) := \inf \{ K \in \mathbb{R}^+ \mid \|g(x) - h(x)\| \leq K(\varphi(x, -x) + \varphi(-x, x)) \}
\]
for all \(x \in V \setminus \{0\}\).

**Lemma 2.2.** Let \(0 < L < 1\) and let \(\varphi, \tilde{\varphi} : (V \setminus \{0\})^2 \to [0, \infty)\) satisfy
\[
(2.1) \quad \varphi(2x, 2y) \leq 2L\varphi(x, y),
\]
\[
(2.2) \quad L\tilde{\varphi}(2x, 2y) \geq 4\tilde{\varphi}(x, y)
\]
for all \(x, y \in V \setminus \{0\}\), respectively. Consider the mappings \(A, \tilde{A} : S \to S\) defined by
\[
A_g(x) := \frac{g(2x) - g(-2x)}{4} + \frac{g(2x) + g(-2x)}{8},
\]
\[
\tilde{A}_g(x) := g\left(\frac{x}{2}\right) - g\left(-\frac{x}{2}\right) + 2\left(g\left(\frac{x}{2}\right) + g\left(-\frac{x}{2}\right)\right)
\]
for all \( g \in S \) and \( x \in V \). Then \( A \) and \( \tilde{A} \) are strictly contractive self mappings of \( S \) with the Lipschitz constant \( L \) with respect to the generalized metric \( d_\varphi \) and \( d_{\tilde{\varphi}} \), respectively.

**Proof.** By the induction on \( n \in \mathbb{N} \), we get

\[
A^n g(x) = \frac{g(2^n x) - g(-2^n x)}{2^{n+1}} + \frac{g(-2^n x) + g(2^n x)}{2} \\
\tilde{A}^n g(x) = 2^{n-1} \left( \frac{g(2^n x)}{2^n} - g\left( -\frac{x}{2} \right) \right) + \frac{4^n}{2} \left( g\left( \frac{x}{2^n} \right) + g\left( -\frac{x}{2^n} \right) \right)
\]

for all \( x \in V \). For any \( g, h \in S \), let \( d_\varphi(g, h) < K \). Then

\[
\|Ag(x) - Ah(x)\| \leq \frac{3}{8} \|g(2x) - h(2x)\| + \frac{1}{8} \|g(-2x) - h(-2x)\| \\
\leq \frac{1}{2} (\varphi(2x, -2x) + \varphi(-2x, 2x)) \\
\leq \sqrt{L} (\varphi(x, -x) + \varphi(-x, x))
\]

for all \( x \in V \setminus \{0\} \). This implies that

\[
d_\varphi(Ag, Ah) \leq LK
\]

as well as

\[
d_\varphi(Ag, Ah) \leq Ld_\varphi(g, h)
\]

for all \( g, h \in S \). On the other hand, if \( d_{\tilde{\varphi}}(g, h) < K \), then

\[
\|\tilde{A}g(x) - \tilde{A}h(x)\| \leq 3 \left\| \frac{x}{2^n} \right\| - h\left( \frac{x}{2^n} \right) + \left\| g\left( -\frac{x}{2^n} \right) - h\left( -\frac{x}{2^n} \right) \right\| \\
\leq 4K \left( \varphi\left( \frac{x}{2^n}, \frac{x}{2^n} \right) + \varphi\left( -\frac{x}{2^n}, -\frac{x}{2^n} \right) \right) \\
\leq \sqrt{LK} (\varphi(x, -x) + \varphi(-x, x))
\]

for all \( x \in V \setminus \{0\} \). So we also have

\[
d_{\tilde{\varphi}}(\tilde{A}g, \tilde{A}h) \leq Ld_{\tilde{\varphi}}(g, h)
\]

for all \( g, h \in S \). \( \Box \)

### 3. The stability of Eq.(1.3)

In this section, we consider stability results of the functional equation (1.3). For a given mapping \( f : V \to Y \), we use the following abbreviation

\[
Df(x, y) := \sum_{i=0}^{3} 3C_i(-1)^{3-i} f(ix + y)
\]

for all \( x, y \in V \).

**Lemma 3.1.** If \( f : V \to Y \) is a mapping such that \( Df(x, y) = 0 \) for all \( x, y \in V \setminus \{0\} \), then \( f \) is a general quadratic mapping.
Proof. Notice that we get
\[ 3Df(x, 0) = -Df(x, x) + Df(2x, -2x) - 3Df(x, -x) - Df(x, -2x) = 0 \]
for all \( x \in V \setminus \{0\} \). Moreover, we have
\[ Df(0, y) = \sum_{i=0}^{3} \beta C_i (-1)^{3-i} f(y) = 0 \]
for all \( y \in V \). So we can say that \( Df(x, y) = 0 \) for all \( x, y \in V \). □

Now we can prove the stability of the functional equation \( Df \equiv 0 \) using the fixed point theory.

**Theorem 3.2.** Let \( \varphi : (V \setminus \{0\})^2 \to [0, \infty) \) satisfy the condition (2.1) for given \( 0 < L < 1 \) and let \( f : V \to Y \). If
\[
\|Df(x, y)\| \leq \varphi(x, y)
\]
for all \( x, y \in V \setminus \{0\} \), then there exists a unique general quadratic mapping \( F : V \to Y \) such that
\[
\|f(x) - F(x)\| \leq \frac{3}{8(1-L)} (\varphi(x, -x) + \varphi(-x, x))
\]
for all \( x \in V \setminus \{0\} \). In particular, the mapping \( F \) is represented by
\[
F(x) = \lim_{n \to \infty} \left( \frac{f(2^n x) + f(-2^n x)}{2 \cdot 4^n} + \frac{f(2^n x) - f(-2^n x)}{2^{n+1}} \right) + f(0)
\]
for all \( x \in V \). Moreover, if \( 0 < L < \frac{1}{2} \) and \( \varphi \) is continuous, then \( f \equiv F \), i.e., \( f \) is itself a general quadratic mapping.

**Proof.** Consider the mapping \( \tilde{f} : V \to Y \) such that \( \tilde{f}(x) = f(x) - f(0) \) for all \( x \in V \). Then \( \tilde{f}(0) = 0 \) and
\[
D\tilde{f}(x, y) = Df(x, y)
\]
for all \( x, y \in V \setminus \{0\} \). Notice that, if we consider the mapping \( A \) in Lemma 2.2, then we have
\[
\|\tilde{f}(x) - A\tilde{f}(x)\| = \frac{1}{8} \| -3D\tilde{f}(x, -x) - D\tilde{f}(-x, x) \|
\leq \frac{3}{8} (\varphi(x, -x) + \varphi(-x, x))
\]
for all \( x \in V \setminus \{0\} \), i.e., \( d_{\varphi} \left( \tilde{f}, A\tilde{f} \right) \leq \frac{3}{8} < \infty \). By Lemma 2.2, this implies that
\[
d_{\varphi} \left( A^n \tilde{f}, A^{n+1} \tilde{f} \right) < \infty
\]
for all \( n \geq 0 \). So we can apply (2) and (3) of Theorem 2.1 to get a unique fixed point \( \tilde{F} : V \to Y \) of the strictly contractive mapping \( A \), which is defined by

\[
(3.4) \quad \tilde{F}(x) := \lim_{n \to \infty} A^n \tilde{f}(x) = \lim_{n \to \infty} \left( \frac{\tilde{f}(2^n x) + \tilde{f}(-2^n x)}{2 \cdot 4^n} + \frac{\tilde{f}(2^n x) - \tilde{f}(-2^n x)}{2^{n+1}} \right)
\]

for all \( x \in V \). Since

\[
d_\varphi(\tilde{f}, \tilde{F}) \leq \frac{1}{1 - L} d_\varphi(\tilde{f}, A \tilde{f}) \leq \frac{3}{8(1 - L)}
\]

we have

\[
(3.5) \quad \| \tilde{F}(x) - \tilde{f}(x) \| \leq \frac{3}{8(1 - L)} (\varphi(x, -x) + \varphi(-x, x))
\]

for all \( x \in V \setminus \{ 0 \} \). Replacing \( x \) by \( 2^n x \) and \( y \) by \( 2^n y \) in (3.1), we obtain

\[
\| DA^n \tilde{f}(x,y) \| \leq \frac{1}{2n+1} \left( \| D\tilde{f}(2^n x, 2^n y) \| + \| D\tilde{f}(-2^n x, -2^n y) \| \right) + \frac{1}{2 \cdot 4^n} \left( \| D\tilde{f}(2^n x, 2^n y) \| + \| D\tilde{f}(-2^n x, -2^n y) \| \right)
\]

\[
\leq \left( \frac{1}{2n+1} + \frac{1}{2 \cdot 4^n} \right) (\varphi(2^n x, 2^n y) + \varphi(-2^n x, -2^n y))
\]

\[
\leq \left( \frac{1}{2n+1} + \frac{1}{2 \cdot 4^n} \right) 2^n L^n (\varphi(x, y) + \varphi(-x, -y)).
\]

The right hand side tends to 0 as \( n \to \infty \), since \( 0 < L < 1 \). This implies that

\[D\tilde{F}(x, y) = 0 \quad \text{for all } x, y \in V \setminus \{ 0 \}.\]

From Lemma 3.1, we also have

\[D\tilde{F}(x, y) = 0 \quad \text{for all } x, y \in V.\]

Put \( F = \tilde{F} + \tilde{f}(0) \). Then (3.2) and (3.3) follow from (3.5) and (3.4), respectively. Now let \( 0 < L < \frac{1}{2} \) and \( \varphi \) be continuous. Then we get

\[
\lim_{n \to \infty} \varphi((a \cdot 2^n + b)x, (c \cdot 2^n + d)y) \leq \lim_{n \to \infty} \left( (2L)^n \varphi \left( \frac{a \cdot 2^n + b}{2^n}, \frac{c \cdot 2^n + d}{2^n} \right) \right) = 0 \cdot \varphi(ax, cy) = 0
\]

for all \( x, y \in V \setminus \{ 0 \} \) and for any fixed integers \( a, b, c, d \) with \( a, c \neq 0 \). Therefore, we obtain

\[
3\|f(x) - F(x)\| \leq \lim_{n \to \infty} \left( \| Df ((2^n + 1)x, -2^n x) - DF ((2^n + 1)x, -2^n x) \| + \| (F - f) ((2^n + 1)x) \| + 3\| (f - F) ((2^n + 2)x) \| \right)
\]

\[
\leq \lim_{n \to \infty} \varphi((2^n + 1)x, -2^n x) + \varphi(-2^n x, 2^n x) + 3\varphi((2^n + 2)x, -(2^n + 2)x) + 3\varphi(-(2^n + 2)x, (2^n + 2)x)
\]
\[ + \varphi \left( \left( 2^{n+1} + 3 \right) x, - \left( 2^{n+1} + 3 \right) x \right) \\
+ \varphi \left( - \left( 2^{n+1} + 3 \right) x, \left( 2^{n+1} + 3 \right) x \right) \]
\[ = 0 \]

for all \( x \in V \setminus \{ 0 \} \). Since \( f(0) = F(0) \), we have shown that \( f \equiv F \). \( \square \)

**Theorem 3.3.** Suppose that \( f : V \to Y \) satisfies the inequality (3.1) for all \( x, y \in V \setminus \{ 0 \} \), where \( \varphi \) has the property (2.2) with \( 0 < L < 1 \). Then there exists a unique general quadratic mapping \( F : V \to Y \) such that

\[
\| f(x) - F(x) \| \leq \frac{L}{4(1 - L)} (\varphi(x, -x) + \varphi(-x, x))
\]

for all \( x \in V \setminus \{ 0 \} \). In particular, \( F \) is represented by

\[
F(x) = \lim_{n \to \infty} \left( 2^{n-1} \left( f \left( \frac{x}{2^n} \right) - f \left( -\frac{x}{2^n} \right) \right) \\
+ \frac{4^n}{2} \left( f \left( \frac{x}{2^n} \right) + f \left( -\frac{x}{2^n} \right) - 2f(0) \right) \right) + f(0)
\]

for all \( x \in V \).

**Proof.** Let \( \tilde{f} = f - f(0) \). Then \( \tilde{f} \) satisfies (3.1), \( \tilde{f}(0) = 0 \), and \( D\tilde{f} = Df \). If we consider the mapping \( \tilde{A} \) as in Lemma 2.2, then we see that

\[
\| \tilde{f}(x) - \tilde{A}f(x) \| = \| D\tilde{f} \left( \frac{x}{2^n}, -\frac{x}{2^n} \right) \|
\]

\[
\leq \varphi \left( \frac{x}{2^n}, -\frac{x}{2^n} \right) \leq \frac{L}{4} \left( \varphi(x, -x) + \varphi(-x, x) \right)
\]

for all \( x \in V \setminus \{ 0 \} \), which implies that \( d_{\varphi}(\tilde{f}, \tilde{A}f) \leq \frac{L}{4} < \infty \). By Lemma 2.2, we get

\[
d_{\varphi} \left( \tilde{A}^{n+1} \tilde{f}, \tilde{A}^{n+1} \tilde{f} \right) < \infty
\]

for all \( n \geq 0 \). We can apply (2) and (3) of Theorem 2.1 to get a unique fixed point \( \tilde{F} : V \to Y \) of the strictly contractive mapping \( \tilde{A} \), which is defined by

\[
\tilde{F}(x) := \lim_{n \to \infty} \tilde{A}^n \tilde{f}(x)
\]

\[
= \lim_{n \to \infty} \left( 2^{n-1} \left( \tilde{f} \left( \frac{x}{2^n} \right) - \tilde{f} \left( -\frac{x}{2^n} \right) \right) + \frac{4^n}{2} \left( \tilde{f} \left( \frac{x}{2^n} \right) + \tilde{f} \left( -\frac{x}{2^n} \right) \right) \right)
\]

for all \( x \in V \). Moreover, we can say that

\[
d_{\varphi}(\tilde{f}, \tilde{F}) \leq \frac{1}{1 - L} d_{\varphi}(f, A\tilde{f}) \leq \frac{L}{4(1 - L)}
\]

that is

\[
\| \tilde{F}(x) - \tilde{f}(x) \| \leq \frac{L}{4(1 - L)} (\varphi(x, -x) + \varphi(-x, x))
\]
for all $x \in V \setminus \{0\}$. Replacing $x$ by $\frac{x}{2^n}$ and $y$ by $\frac{y}{2^n}$ in (3.1), we obtain
\[
\|D\tilde{A}^n\tilde{f}(x,y)\| = \left\|2^{n-1} \left( D\tilde{f}\left(\frac{x}{2^n}, \frac{y}{2^n}\right) - D\tilde{f}\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right)\right) + \frac{4^n}{2} \left( D\tilde{f}\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + D\tilde{f}\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right)\right)\right\|
\leq \left(2^{n-1} + \frac{4^n}{2}\right) (\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + \varphi\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right))
\leq \frac{L^n}{4^n} \left(2^{n-1} + \frac{4^n}{2}\right) (\varphi(x,y) + \varphi(-x,-y))
\]
for all $x, y \in V \setminus \{0\}$. In a similar way of the proof of Theorem 3.2, this implies that
\[
D\tilde{F}(x,y) = 0
\]
for all $x, y \in V$. Put $F = \tilde{F} + f(0)$. Then (3.6) and (3.7) follow from (3.9) and (3.8), respectively, too. Since the uniqueness of $F$ is clear in the fixed point theory, we have proved this theorem. \hfill \Box

**Theorem 3.4.** Let $\varphi : (V \setminus \{0\})^2 \to [0, \infty)$ satisfy the condition (2.1) for given $0 < L < 1$ with $\varphi(x,y) = \varphi(-x,-y)$ for all $x, y \in V \setminus \{0\}$. If $f$ satisfies the inequality (3.1) for all $x, y \in V \setminus \{0\}$, then there exists a unique general quadratic mapping $F : V \to Y$ such that
\[
\|f(x) - F(x)\| \leq \frac{1}{2(1-L)} \varphi(x,-x)
\]
for all $x \in V \setminus \{0\}$. Moreover, if $0 < L < \frac{1}{2}$ and $\varphi$ is continuous, then $f \equiv F$, i.e., $f$ is itself a general quadratic mapping.

**Proof.** If we consider the mappings $A$, $\tilde{f}$ and $\tilde{F}$ in Theorem 3.2, then we have
\[
\|\tilde{f}(x) - A\tilde{f}(x)\| = \left\|\frac{3}{8}D\tilde{f}(x,-x) + \frac{1}{8}D\tilde{f}(-x,x)\right\|
\leq \frac{1}{4} (\varphi(x,-x) + \varphi(-x,x))
\]
and so
\[
\|f(x) - F(x)\| = \|\tilde{f}(x) - \tilde{F}(x)\| = \frac{1}{1-L}\|\tilde{f}(x) - A\tilde{f}(x)\| \leq \frac{\varphi(x,-x)}{2(1-L)}
\]
for all $x \in V \setminus \{0\}$. \hfill \Box

**Theorem 3.5.** Suppose that $f : V \to Y$ satisfies the inequality (3.1) for all $x, y \in V \setminus \{0\}$, where $\varphi$ has the property (2.2) with $0 < L < 1$. If $\varphi(x,y) = \varphi(-x,-y)$ for all $x, y \in V \setminus \{0\}$, then there exists a unique general quadratic mapping $F : V \to Y$ such that
\[
\|f(x) - F(x)\| \leq \frac{L}{4(1-L)} \varphi(x,-x)
\]
for all $x \in V \setminus \{0\}$. 

Proof. If we consider the mappings $\hat{A}$, $\hat{f}$, and $\tilde{F}$ in Theorem 3.3, then we have

$$\left\| \tilde{f}(x) - \hat{f}(x) \right\| = \left\| D\tilde{f} \left( \frac{x}{2}, -\frac{x}{2} \right) \right\| \leq \frac{L}{8}(\varphi(x, -x) + \varphi(-x, x))$$

and so

$$\left\| f(x) - F(x) \right\| = \left\| \tilde{f}(x) - \hat{F}(x) \right\| = \frac{1}{1-L}\left\| \tilde{f}(x) - \hat{A}f(x) \right\| \leq \frac{L\varphi(x, -x)}{4(1-L)}$$

for all $x \in V \setminus \{0\}$. □

Now we obtain the Hyers-Ulam stability results in the framework of normed spaces using Theorem 3.4 and Theorem 3.5.

Corollary 3.6. Let $X$ be a normed space and $Y$ a Banach space. Suppose that, for $\theta \geq 0$ and $p \in \mathbb{R} \setminus [1, 2]$, the mapping $f : X \to Y$ satisfies an inequality of the form

$$\|Df(x,y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X \setminus \{0\}$. Then there exists a unique general quadratic mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \leq \begin{cases} \frac{2\theta}{1-2p}\|x\|^p & \text{if } 0 \leq p < 1 \\ \frac{2\theta}{2-p}\|x\|^p & \text{if } p > 2 \end{cases}$$

for all $x \in X \setminus \{0\}$ and $f$ is itself a general quadratic mapping if $p < 0$.

Proof. Let $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X \setminus \{0\}$. If $p < 1$, then $\varphi$ satisfies (2.1) with $L = 2^{p-1} < 1$. In particular, if $p < 0$, then $0 < L < \frac{1}{2}$ and it is clear that $\varphi$ is continuous on $(X \setminus \{0\})^2$. On the other hand, if $p > 2$, then $\varphi$ satisfies (2.2) with $L = 2^{2-p} < 1$. So we can prove this corollary by using Theorem 3.4 and Theorem 3.5, respectively. □

Corollary 3.7. Let $X$ be a normed space and $Y$ a Banach space. Suppose that, for $\theta \geq 0$ and $p + q \in \mathbb{R} \setminus [1, 2]$, the mapping $f : X \to Y$ satisfies an inequality of the form

$$\|Df(x,y)\| \leq \theta(\|x\|^p|y|^q)$$

for all $x, y \in X \setminus \{0\}$. Then there exists a unique general quadratic mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \leq \begin{cases} \frac{\theta}{2^{p+q-1}}\|x\|^{p+q} & \text{if } 0 \leq p + q < 1 \\ \frac{\theta}{2^{p+q-2}}\|x\|^{p+q} & \text{if } p + q > 2 \end{cases}$$

for all $x \in X \setminus \{0\}$ and $f$ is itself a general quadratic mapping if $p + q < 0$.

Proof. Let $\varphi(x,y) := \theta(\|x\|^p|y|^q)$ for all $x, y \in X \setminus \{0\}$. If $p + q < 1$, then $\varphi$ satisfies (2.1) with $L = 2^{p+q-1} < 1$. In particular, if $p + q < 0$, then $0 < L < \frac{1}{2}$ and it is clear that $\varphi$ is continuous on $(X \setminus \{0\})^2$. Therefore we can prove the corollary in this case by Theorem 3.4. On the other hand, if $p + q > 2$, then $\varphi$ satisfies (2.2) with $L = 2^{2-p+q} < 1$. By Theorem 3.5, the proof of this corollary completes. □
4. Applications to Jensen’s functional equation and the quadratic functional equation

For a given mapping \( f : V \to Y \), we use the following abbreviations

\[
Jf(x, y) := 2f\left(\frac{x + y}{2}\right) - f(x) - f(y),
\]

\[
Qf(x, y) := f(x + y) + f(x - y) - 2f(x) - 2f(y)
\]

for all \( x, y \in V \). Using the previous results, we can prove stability results about Jensen’s functional equation \( Jf \equiv 0 \) and the quadratic functional equation \( Qf \equiv 0 \) by followings.

**Corollary 4.1.** Let \( \psi_i : V^2 \to [0, \infty), i = 1, 2 \), be given functions. Suppose that for each \( i = 1, 2 \), \( f_i : V \to Y \) satisfies

\[
\|Jf_i(x, y)\| \leq \psi_i(x, y)
\]

for all \( x, y \in V \), respectively. If there exists \( 0 < L < 1 \) such that \( \psi_1 \) has the property (2.1) and \( \psi_2 \) satisfies (2.2) for all \( x, y \in V \), then there exist unique Jensen mappings \( F_i : V \to Y \), \( i = 1, 2 \), such that

\[
\|f_i(x) - F_i(x)\| \leq \frac{3\Psi_1(x)}{8(1 - L)}
\]

(4.2)

\[
\|f_2(x) - F_2(x)\| \leq \frac{L}{4(1 - L)}\Psi_2(x)
\]

(4.3)

for all \( x \in V \), where \( \Psi_i(x) := \psi_i(x, -x) + \psi_i(2x, 0) + \psi_i(-x, x) + \psi_i(-2x, 0) \). In particular, the desired mappings \( F_1, F_2 \) are represented by

\[
F_1(x) = \lim_{n \to \infty} \frac{f_1(2^n x)}{2^n} + f_1(0),
\]

(4.4)

\[
F_2(x) = \lim_{n \to \infty} 2^n \left( f_2\left(\frac{x}{2^n}\right) - f_2(0)\right) + f_2(0)
\]

(4.5)

for all \( x \in V \). Moreover, if \( 0 < L < \frac{1}{2} \) and \( \psi_1 \) is continuous, then \( f_1 \) is itself a Jensen mapping.

**Proof.** Notice that for \( f_i : V \to Y \), \( i = 1, 2 \), we have

\[
\|Df_i(x, y)\| = \|Jf_i(2x + y, u) - Jf_i(3x + y, x + y)\|
\]

\[
\leq \psi_i(2x + y, u) + \psi_i(3x + y, x + y)
\]

for all \( x, y \in V \). Put \( \varphi_i(x, y) := \psi_i(2x + y, y) + \psi_i(3x + y, x + y), i = 1, 2 \), for all \( x, y \in V \), then \( \varphi_1 \) satisfies (2.1) and \( \varphi_2 \) satisfies (2.2). Observe that \( \|Df_i(x, y)\| \leq \varphi_i(x, y), i = 1, 2 \), for all \( x, y \in V \), respectively. According to Theorem 3.2, we can take the unique general quadratic mapping \( F_1 \) by

\[
F_1(x) := \lim_{n \to \infty} \left( \frac{f_1(2^n x) + f_1(-2^n x)}{2 \cdot 4^n} + \frac{f_1(2^n x) - f_1(-2^n x)}{2^{n+1}} \right) + f_1(0)
\]

(4.6)
which satisfies (4.2) clearly. Observe that
\[
\frac{1}{2^{n+1}} \| Jf_1(2^n x) + f_1(0) \| \\
\leq \frac{1}{2^{n+1}} \| Jf_1(2^n x, -2^n x) \|
\leq \frac{L_n}{2} \psi_1(x, -x)
\]
for all \( x \in V \). Letting \( n \to \infty \), we get
\[
\lim_{n \to \infty} \frac{1}{2^{n+1}} \| Jf_1(2^n x) + f_1(0) \| = 0
\]
for all \( x, y \in V \). Together with (4.6), this implies (4.4). Notice that
\[
\frac{1}{2n+1} \| Jf_1(2^n x, 2^n y) \| \leq \psi_1(2^n x, 2^n y) \leq L_n \psi_1(x, y)
\]
for all \( x, y \in V \). Taking the limit as \( n \to \infty \), we obtain
\[
JF_1(x, y) = 0
\]
for all \( x, y \in V \). In particular, consider the case \( 0 < L < \frac{1}{2} \) and \( \psi_1 \) is continuous. Then \( \varphi_1 \) is continuous on \( (V \setminus \{0\})^2 \) and we can say that \( f_1 \equiv F_1 \) by Theorem 3.2. On the other hand, according Theorem 3.3, we can get
\[
F_2(x) := \lim_{n \to \infty} \left( 2^{n-1} \left( f_2 \left( \frac{x}{2^n} \right) - f_2 \left( \frac{-x}{2^n} \right) \right) \\
+ \frac{4^n}{2} \left( f_2 \left( \frac{x}{2^n} \right) + f_2 \left( \frac{-x}{2^n} \right) - 2f_2(0) \right) \right) + f_2(0)
\]
which is the unique general quadratic mapping satisfying (4.3). From (4.1) and (2.2), we have
\[
\lim_{n \to \infty} 2^{2n-1} \left( f_2 \left( \frac{x}{2^n} \right) + f_2 \left( \frac{-x}{2^n} \right) - 2f_2(0) \right) = \lim_{n \to \infty} 2^{2n-1} \| Jf_2 \left( \frac{x}{2^n}, -\frac{x}{2^n} \right) \|
\leq \lim_{n \to \infty} 2^{2n-1} \psi_2 \left( \frac{x}{2^n}, \frac{-x}{2^n} \right)
\leq \lim_{n \to \infty} \frac{L_n}{2} \psi_2(x, -x)
= 0
\]
as well as
\[
\lim_{n \to \infty} 2^{n-1} \left( f_2 \left( \frac{x}{2^n} \right) + f_2 \left( \frac{-x}{2^n} \right) - 2f_2(0) \right) = 0
\]
for all \( x \in V \). So we get (4.5) following (4.7). Observe that
\[
\left\| 2^n Jf_2 \left( \frac{x}{2^n}, \frac{y}{2^n} \right) \right\| \leq 2^n \psi_2 \left( \frac{x}{2^n}, \frac{y}{2^n} \right) \leq \frac{L_n}{2^n} \psi_2(x, y)
\]
for all \( x, y \in V \). Taking the limit as \( n \to \infty \), then we get
\[
JF_2(x, y) = 0
\]
for all $x, y \in V$. \hfill \square

**Corollary 4.2.** Let $\psi_i : V^2 \to [0, \infty)$, $i = 1, 2$, be given functions. Suppose that each $f_i : V \to Y$, $i = 1, 2$, satisfies

$$
(4.8) \quad \|Qf_i(x, y)\| \leq \psi_i(x, y)
$$

for all $x, y \in V$, respectively. If there exists $0 < L < 1$ such that the mappings $\psi_1$ and $\psi_2$ have the property (2.1) and (2.2) for all $x, y \in V$, respectively, then there exist unique quadratic mappings $F_1, F_2 : V \to Y$ such that

$$
(4.9) \quad \|f_1(x) - f_1(0) - F_1(x)\| \leq \frac{3\Psi_1(x)}{8(1 - L)},
$$

$$
(4.10) \quad \|f_2(x) - F_2(x)\| \leq \frac{L}{4(1 - L)} \Psi_2(x)
$$

for all $x \in V \setminus \{0\}$, where

$$
\Psi_i(x) := \psi_i(x, x) + \psi_i(x, 0) + 2\psi_i(0, 0) + \psi_i(0, -x) + \psi_i(-x, x) + \psi_i(-x, 0) + \psi_i(0, x),
$$

respectively. In particular, the desired mappings $F_1$ and $F_2$ are represented by

$$
(4.11) \quad F_1(x) = \lim_{n \to \infty} \frac{f_1(2^n x)}{4^n},
$$

$$
(4.12) \quad F_2(x) = \lim_{n \to \infty} 4^n f_2 \left( \frac{x}{2^n} \right)
$$

for all $x \in V$. Moreover, if $0 < L < \frac{1}{2}$ and $\psi_1$ is continuous, then $f_1 - f_1(0)$ is itself a quadratic mapping.

**Proof.** Notice that

$$
\|Df_i(x, y)\| = \|Qf_i(x, 2x + y) - Qf_i(x, x + y) + Qf_i(0, y) - Qf_i(x, x + y)\|
$$

$$
\leq \psi_i(x, 2x + y) + \psi_i(x, x + y) + \psi_i(0, y) + \psi_i(x, x + y)
$$

for all $x, y \in V$, $i = 1, 2$. Put

$$
\varphi_i(x, y) := \psi_i(x, 2x + y) + \psi_i(x, x + y) + \psi_i(0, y) + \psi_i(x, x + y)
$$

for all $x, y \in V$, then $\|Df_i(x, y)\| \leq \varphi_i(x, y)$ for all $x, y \in V$, respectively. Moreover, $\varphi_1$ satisfies (2.1) and $\varphi_2$ holds (2.2). Therefore, according to Theorem 3.2, there exists a unique mapping $F_1 : V \to Y$ satisfying (4.9), which is represented by

$$
F_1(x) := \lim_{n \to \infty} \left( \frac{f_1(2^n x) + f_1(-2^n x) - f_1(-2^n x)}{2 \cdot 4^n} + \frac{f_1(2^n x) - f_1(-2^n x)}{2^n} \right).
$$

Observe that

$$
\lim_{n \to \infty} \left\| \frac{f_1(2^n x) - f_1(-2^n x)}{2^n} \right\| = \lim_{n \to \infty} \frac{1}{2^{n+1}} \|Qf_1(0, 2^n x)\|
\[ \leq \lim_{n \to \infty} \frac{1}{2^{n+1}} \psi_1(0, 2^n x) \]
\[ \leq \lim_{n \to \infty} \frac{L^n}{2} \psi_1(0, x) \]

and
\[ \lim_{n \to \infty} \left\| \frac{f_1(2^n x) - f_1(-2^n x)}{2 \cdot 4^n} \right\| \leq \lim_{n \to \infty} \frac{L^n}{2^{n+1}} \psi_1(0, x) = 0 \]

for all \( x \in V \). From this, we get (4.11). Notice that
\[ \left\| Q f_1(2^n x, 2^n y) \right\| \leq \psi_1(2^n x, 2^n y) \leq \frac{L^n}{2^n} \psi_1(x, y) \]

for all \( x, y \in V \). Taking the limit as \( n \to \infty \) in the above inequality, we get
\[ Q F_1(x, y) = 0 \]

for all \( x, y \in V \). If \( 0 < L < \frac{1}{2} \) and \( \psi_1 \) is continuous, then \( \varphi_1 \) is also continuous on \( (V \setminus \{0\})^2 \) and we can say that \( f_1 - f_1(0) \equiv F_1 \) by Theorem 3.2. On the other hand, since \( L \psi_2(0, 0) \geq 4 \psi_2(0, 0) \) and \( \|2 f_2(0)\| = \|Q f_2(0, 0)\| \leq \psi_2(0, 0) \), we can show that \( \psi_2(0, 0) = 0 \) and \( f_2(0) = 0 \). According to Theorem 3.3, there exists a unique mapping \( F_2 : V \to Y \) satisfying (4.10), which is represented by (4.7). We have
\[ \lim_{n \to \infty} \frac{4^n}{2} \left\| f_2 \left( \frac{x}{2^n} \right) + f_2 \left( -\frac{x}{2^n} \right) \right\| = \lim_{n \to \infty} \frac{4^n}{2} \left\| Q f_2(0, \frac{x}{2^n}) \right\| \]
\[ \leq \lim_{n \to \infty} \frac{4^n}{2} \psi_2 \left( 0, \frac{x}{2^n} \right) \]
\[ \leq \lim_{n \to \infty} \frac{L^n}{2} \psi_2(0, x) = 0 \]

as well as
\[ \lim_{n \to \infty} \frac{2^{n-1}}{2^{n}} \left\| f_2 \left( \frac{x}{2^n} \right) - f_2 \left( -\frac{x}{2^n} \right) \right\| = 0 \]

for all \( x \in V \). From these and (4.7), we get (4.12). Notice that
\[ \left\| 4^n Q f_2 \left( \frac{x}{2^n}, \frac{y}{2^n} \right) \right\| \leq 4^n \psi_2 \left( \frac{x}{2^n}, \frac{y}{2^n} \right) \leq L^n \psi_2(x, y) \]

for all \( x, y \in V \). Taking the limit as \( n \to \infty \), we have shown that
\[ Q F_2(x, y) = 0 \]

for all \( x, y \in V \).

\[ \square \]

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282 SUN-SOOK JIN AND YANG-HI LEE


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