TWO CHARACTERIZATION THEOREMS FOR IRROTATIONAL LIGHTLIKE GEOMETRY

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Abstract. We study irrotational half lightlike submanifolds $M$ of a semi-Riemannian space form with a semi-symmetric non-metric connection such that its structure vector field is tangent to $M$. We prove two characterization theorems for such an irrotational half lightlike submanifold.

1. Introduction

The theory of lightlike submanifolds is an important topic of research in differential geometry due to its application in mathematical physics, especially in the electromagnetic field theory. The study of such notion was initiated by Duggal and Bejancu [3] and later studied by many authors (see [4, 7]).

The notion of a semi-symmetric non-metric connection on a Riemannian manifold was introduced by Ageshe and Chaflle [1]. Recently several authors ([8]-[11], [13], [17]) studied lightlike submanifolds of semi-Riemannian manifolds admitting semi-symmetric non-metric connections.

Călin proved the following result in his thesis [2]: For any lightlike submanifolds $M$ of indefinite almost contact manifolds $\tilde{M}$, if the structure vector field $\zeta$ of $\tilde{M}$ is tangent to $M$, then it belongs to $S(TM)$. After Călin’s work, many earlier works [5, 6, 14] which have been written on either lightlike submanifolds of indefinite almost contact manifolds or lightlike submanifolds of semi-Riemannian manifolds admitting semi-symmetric non-metric connections obtained their results by using the Călin’s result described in above.

In this paper, first we prove that the afore cited Călin’s result is not true for any irrotational half lightlike submanifolds $M$ of a semi-Riemannian space form $\bar{M}(c)$ admitting a semi-symmetric non-metric connection (see Theorem 3.2 and Corollary 3.3). Next several authors [12, 16] have agreed the assertion that two screen conformalities, which are called screen conformal and screen quasi-conformal, of $M$ are dependent to each other. We prove that such two screen conformalities are independent (see Theorem 3.2 and Theorem 3.5).

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2. Semi-symmetric non-metric connection

Let \( (\overline{M}, \tilde{g}) \) be a semi-Riemannian manifold. A connection \( \tilde{\nabla} \) on \( \overline{M} \) is called a semi-symmetric non-metric connection [1] if \( \tilde{\nabla} \) and its torsion tensor \( \tilde{T} \) satisfy

\[
(\tilde{\nabla}_X \tilde{g})(Y, Z) = -\pi(Y)\tilde{g}(X, Z) - \pi(Z)\tilde{g}(X, Y) - \tilde{T}(X, Y) - \pi(X)Y,
\]

for any vector fields \( X, Y \) and \( Z \) on \( \overline{M} \), where \( \pi \) is a 1-form associated with a non-vanishing vector field \( \zeta \), which is called the structure vector field, by

\[
\pi(X) = \tilde{g}(X, \zeta).
\]

A submanifold \( (M, g) \) of codimension 2 is called half lightlike submanifold if the radical distribution \( \text{Rad}(TM) = TM \cap TM^\perp \) is a vector subbundle of the tangent bundle \( TM \) and the normal bundle \( TM^\perp \) of \( M \), with rank 1. In this case, there exists complementary non-degenerate distributions \( S(TM) \) and \( S(TM^\perp) \) of \( \text{Rad}(TM) \) in \( TM \) and \( TM^\perp \), respectively, which are called the screen and co-screen distributions on \( M \), respectively, such that

\[
TM = \text{Rad}(TM) \oplus \text{orth} S(TM), \quad TM^\perp = \text{Rad}(TM) \oplus \text{orth} S(TM^\perp),
\]

where \( \oplus \text{orth} \) denotes the orthogonal direct sum. We denote such a half lightlike submanifold by \( M = (M, g, S(TM)) \). Denote by \( F(M) \) the algebra of smooth functions on \( M \), by \( \Gamma(E) \) the \( F(M) \) module of smooth sections of a vector bundle \( E \) over \( M \) and by (2.3), the \( i \)-th equation of (2.3). We use same notations for any others. Choose \( L \in \Gamma(S(TM^\perp)) \) as a spacelike unit vector field without loss of generality, i.e., \( \tilde{g}(L, L) = 1 \). Consider the orthogonal complementary distribution \( S(TM^\perp) \) to \( S(TM) \) in \( TM \). Certainly \( \text{Rad}(TM) \) and \( S(TM^\perp) \) are subbundles of \( S(TM^\perp) \). As \( S(TM^\perp) \) is non-degenerate, we have

\[
S(TM^\perp) = S(TM^\perp) \oplus \text{orth} S(TM^\perp),
\]

where \( S(TM^\perp) \) is the orthogonal complementary to \( S(TM^\perp) \) in \( S(TM^\perp) \).

It is well-known [3] that, for any null section \( \xi \) of \( \text{Rad}(TM) \) on a coordinate neighborhood \( U \subset M \), there exists a uniquely defined lightlike vector bundle \( \text{ltr}(TM) \) and a null vector field \( N \) of \( \text{ltr}(TM) \) on \( U \) satisfying

\[
\tilde{g}(\xi, N) = 1, \quad \tilde{g}(N, N) = \tilde{g}(N, X) = \tilde{g}(N, L) = 0, \quad \forall X \in \Gamma(S(TM)).
\]

We call \( N, \text{ltr}(TM) \) and \( \text{tr}(TM) = S(TM^\perp) \oplus \text{orth} \text{ltr}(TM) \) the lightlike transversal vector field, lightlike transversal vector bundle and transversal vector bundle of \( M \) with respect to \( S(TM) \), respectively [13]. Then \( TM \) is decomposed as

\[
TM = \{\text{Rad}(TM) \oplus \text{tr}(TM)\} \oplus \text{orth} S(TM)
\]

\[
= \{\text{Rad}(TM) \oplus \text{ltr}(TM)\} \oplus \text{orth} S(TM) \oplus \text{orth} S(TM^\perp).
\]

In the entire discussion of this article, we shall assume that \( \zeta \) to be a spacelike unit tangent vector field of \( M \). In the sequel, we take \( X, Y, Z, W \in \Gamma(TM) \) unless otherwise specified. Let \( P \) be the projection morphism of \( TM \) on \( S(TM) \)
with respect to the decomposition (2.3). Then the local Gauss and Weingarten formulas of \( M \) and \( S(TM) \) are given respectively by

\[
\begin{align*}
\tilde{\nabla}_XY &= \nabla_XY + B(X,Y)N + D(X,Y)L, \\
\tilde{\nabla}_XN &= -A_{\xi}X + \tau(X)N + \rho(X)L, \\
\tilde{\nabla}_XL &= -A_LX + \phi(X)N; \\
\nabla_XPY &= \nabla_XPY + C(X,PY)\xi; \\
\nabla_X\xi &= -A_{\xi}X - \tau(X)\xi,
\end{align*}
\]

where \( \nabla \) and \( \nabla^* \) are induced linear connections on \( TM \) and \( S(TM) \), respectively, \( B \) and \( D \) are called the \textit{local second fundamental forms} of \( M \), \( C \) is called the \textit{local second fundamental form} on \( S(TM) \). \( A_{\xi}, A^*_\xi \) and \( A_L \) are linear operators on \( TM \), which are called the \textit{shape operators}, and \( \tau, \rho \) and \( \phi \) are 1-forms on \( TM \). We say that \( h(X,Y) = B(X,Y)N + D(X,Y)L \) is the \textit{global second fundamental form tensor} of \( M \). Using (2.1), (2.2) and (2.5), we have

\[
\begin{align*}
(\nabla_Xg)(Y,Z) &= B(X,Y)\eta(Z) + B(X,Z)\eta(Y) - \pi(Y)g(X,Z) - \pi(Z)g(X,Y), \\
T(X,Y) &= \pi(Y)X - \pi(X)Y,
\end{align*}
\]

and \( B \) and \( D \) are symmetric on \( TM \), where \( T \) is the torsion tensor with respect to the induced connection \( \nabla \) and \( \eta \) is a 1-form on \( TM \) such that

\[
\eta(X) = \tilde{g}(X,N).
\]

From the facts \( B(X,Y) = \tilde{g}(\tilde{\nabla}_XY,\xi) \) and \( D(X,Y) = \tilde{g}(\tilde{\nabla}_XN,L) \), we know that \( B \) and \( D \) are independent of the choice of \( S(TM) \) and satisfy

\[
B(X,\xi) = 0, \quad D(X,\xi) = -\phi(X).
\]

The above three local second fundamental forms \( M \) and \( S(TM) \) are related to their shape operators by

\[
\begin{align*}
g(A^*_\xi X, Y) &= B(X,Y), & \tilde{g}(A^*_\xi X, N) &= 0, \\
g(A_L X, Y) &= D(X,Y) + \phi(X)\eta(Y), & \tilde{g}(A_L X, N) &= \rho(X), \\
g(A_{\tilde{\nabla}} X, PY) &= C(X,PY) - fg(X,PY) - \eta(X)\pi(PY), & \tilde{g}(A_{\tilde{\nabla}} X, N) &= -f\eta(X),
\end{align*}
\]

where \( f \) is the smooth function given by \( f = \pi(N) \). From (2.12) and (2.13), we show that \( A^*_\xi \) is \( S(TM) \)-valued self-adjoint and satisfies

\[
A^*_\xi = 0.
\]

Denote by \( \tilde{R}, R \) and \( R^* \) the curvature tensors of the connections \( \tilde{\nabla}, \nabla \) and \( \nabla^* \), respectively. Using the Gauss-Weingarten equations for \( M \) and \( S(TM) \), we obtain the Gauss-Codazzi equations for \( M \) and \( S(TM) \):

\[
\tilde{R}(X,Y)Z = R(X,Y)Z + B(X,Z)A_{\tilde{\nabla}}Y - B(Y,Z)A_{\tilde{\nabla}}X
\]
+ D(X, Z)A_\xi X - D(Y, Z)A_\xi X
+ \{(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z)
+ B(Y, Z)[\tau(X) - \pi(X)] - B(X, Z)[\tau(Y) - \pi(Y)]
+ D(Y, Z)\phi(X) - D(X, Z)\phi(Y)\}N
+ \{(\nabla_X D)(Y, Z) - (\nabla_Y D)(X, Z) + B(Y, Z)\rho(X)
- B(X, Z)\rho(Y) - D(Y, Z)\pi(X) + D(X, Z)\pi(Y)\}L,
\end{equation}

\begin{equation}
\tilde{R}(X, Y)N = -\nabla_X (A_\xi Y) + \nabla_Y (A_\xi X) + A_\xi [X, Y]
+ \tau(X)A_\xi Y - \tau(Y)A_\xi X + \rho(X)A_\xi Y - \rho(Y)A_\xi X
+ \{B(Y, A_\xi X) - B(Y, X, A_\xi Y) + 2d\tau(X, Y)
+ \phi(X)\rho(Y) - \phi(Y)\rho(X)\}N
+ \{D(Y, A_\xi X) - D(X, A_\xi Y) + 2d\phi(X, Y)
+ \rho(X)\tau(Y) - \rho(Y)\tau(X)\}L;
\end{equation}

\begin{equation}
\begin{split}
R(X, Y)PZ &= R^\ast(X, Y)PZ + C(X, PZ)A_\xi^\ast Y - C(Y, PZ)A_\xi X \\
&\quad + \{(\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ)
+ C(X, PZ)[\tau(Y) + \pi(Y)] - C(Y, PZ)[\tau(X) + \pi(X)]\}\xi,
\end{split}
\end{equation}

\begin{equation}
\begin{split}
R(X, Y)\xi &= -\nabla_X^\ast (A_\xi^\ast Y) + \nabla_Y^\ast (A_\xi^\ast X) + A_\xi^\ast [X, Y]
+ \tau(X)A_\xi^\ast Y - \tau(Y)A_\xi^\ast X
+ \{C(Y, A_\xi^\ast X) - C(X, A_\xi^\ast Y) - 2d\tau(X, Y)\}\xi.
\end{split}
\end{equation}

A semi-Riemannian manifold $\tilde{M}$ of constant curvature $c$ is called a semi-Riemannian space form and denote it by $\tilde{M}(c)$. In this case, the curvature tensor $\tilde{R}$ of $\tilde{M}(c)$ is given by

\begin{equation}
\tilde{R}(X, Y)Z = c\{\tilde{g}(Y, Z)X - \tilde{g}(X, Z)Y\}, \quad \forall X, Y, Z \in \Gamma(T\tilde{M}).
\end{equation}

Taking the scalar product with $\xi$ and $L$ to (2.22), we get

\begin{equation}
\tilde{g}(\tilde{R}(X, Y)Z, \xi) = \tilde{g}(\tilde{R}(X, Y)Z, L) = 0, \quad \forall X, Y, Z \in \Gamma(TM).
\end{equation}
From this results and (2.17), for all \( X, Y, Z \in \Gamma(TM) \), we obtain
\[
\tilde{R}(X,Y)Z = R(X,Y)Z + B(X,Z)A_\nu Y - B(Y,Z)A_\mu X \\
+ D(X,Z)A_\nu Y - D(Y,Z)A_\mu X.
\] (2.24)

3. Characterization theorems

**Definition.** A half lightlike submanifold \( M \) of a semi-Riemannian manifold \( \tilde{M} \) is said to be irrotational [15] if \( \tilde{\nabla}_X \xi \in \Gamma(TM) \) for any \( X \in \Gamma(TM) \).

From (2.5) and (2.12), we show that the above definition is equivalent to the condition: \( D(X, \xi) = 0 = \phi(X) \) for all \( X \in \Gamma(TM) \).

**Lemma 3.1.** Let \( M \) be an irrotational half lightlike submanifold of a semi-Riemannian manifold \( \tilde{M} \) admitting a semi-symmetric non-metric connection such that its structure vector field \( \xi \) is tangent to \( M \). Then \( \xi \) is conjugate to any vector field \( X \) on \( M \), i.e., \( \xi \) satisfies \( h(X, \xi) = 0 \).

**Proof.** Taking the scalar product with \( \xi \) to (2.18) and \( N \) to (2.17) such that \( Z = \xi \) by turns and using (2.12), (2.21) and the fact \( \phi = 0 \), we obtain
\[
\tilde{g}(\tilde{R}(X,Y)\xi, N) = B(X, A_\nu Y) - B(Y, A_\mu X) - 2d\tau(X,Y) \\
= C(Y, A_\nu^* X) - C(X, A_\mu^* Y) - 2d\tau(X,Y).
\]

From these two representations, we obtain
\[
B(X, A_\nu Y) - B(Y, A_\mu X) = C(Y, A_\nu^* X) - C(X, A_\mu^* Y).
\]

Using (2.13)\(_1\), (2.15)\(_2\) and the fact \( A_\nu^* \) is self-adjoint, we have
\[
\pi(A_\nu^* X)\eta(Y) = \pi(A_\mu^* Y)\eta(X).
\]

Replacing \( Y \) by \( \xi \) to this equation and using (2.16), we have
\[
B(X, \xi) = \pi(A_\nu^* X) = 0.
\]

As \( D \) is symmetric and \( \phi = 0 \), we show that \( A_\nu \) is self-adjoint. Taking the scalar product with \( L \) to (2.18) and \( N \) to (2.19) with \( \phi = 0 \), we obtain
\[
\tilde{g}(\tilde{R}(X,Y)N, L) = \tilde{g}(\nabla_X (A_\nu Y) - \nabla_Y (A_\mu X) - A_\mu [X,Y], N) \\
= D(Y, A_\nu X) - D(X, A_\nu Y) + 2d\rho(X,Y) + \rho(X)\tau(Y) - \rho(Y)\tau(X).
\]

Using these two representations and (2.14)\(_2\), we show that
\[
D(Y, A_\nu X) - D(X, A_\nu Y) + 2d\rho(X,Y) + \rho(X)\tau(Y) - \rho(Y)\tau(X) \\
= \tilde{g}(\nabla_X (A_\mu Y), N) - \tilde{g}(\nabla_Y (A_\mu X), N) - \rho([X,Y]).
\]

Applying \( \tilde{\nabla} \) to \( \tilde{g}(A_\mu Y, N) = \rho(Y) \) and using (2.1), (2.5) and (2.6), we have
\[
\tilde{g}(\nabla_X (A_\mu Y), N) = X(\rho(Y)) + \pi(A_\mu Y)\eta(X) + fg(X, A_\mu Y) \\
+ g(A_\mu Y, A_\nu X) - \tau(X)\rho(Y).
\]
Substituting this equation into the last equation and using (2.14), we have
\[ \pi(A_{\xi}X)\eta(Y) = \pi(A_{\xi}Y)\eta(X). \]
Replacing \( Y \) by \( \xi \) to this equation, we have
\[ \pi(A_{\xi}X) = \pi(A_{\xi}X)\eta(X). \]
Taking \( X = \xi \) and \( Y = \zeta \) to (2.14), we get \( \pi(A_{\xi}\xi) = 0 \). Therefore we have
\[ D(X, \zeta) = \pi(A_{\xi}X) = 0. \]
From (3.1) and (3.2), we show that 
\[ (3.3) \]
where
\[ (3.4) \]
and
\[ (3.5) \]
\[ (3.6) \]
respectively. Applying \( \nabla_Y \) to (3.4), we have
\[ \nabla_X(A_{\xi}Y) = X[\varphi]A_{\xi}Y + \varphi \nabla_X(A_{\xi}Y) - X[f]Y - f \nabla_XY. \]
Substituting this into (3.5) and using (2.11) and (3.6), we get
\[ \begin{align*}
\tilde{g}(\tilde{R}(X,Y)N, PZ) &= \varphi\tilde{g}(\tilde{R}(X,Y)\xi, PZ) \\
&= \{Y[\varphi]B(\varphi)Y - 2\varphi\pi(Y)\}B(X, PZ) - \rho(Y)D(X, PZ) \\
&\quad - \{X[\varphi] - 2\varphi\pi(X)\}B(Y, PZ) + \rho(X)D(Y, PZ)
\end{align*} \]
Substituting (2.22) into the last equation and using (2.12), we get

\[ \{X[\varphi] - 2\varphi\tau(X)\}B(Y, Z) - \{Y[\varphi] - 2\varphi\tau(Y)\}B(X, Z) = \rho(X)D(Y, Z) - \rho(Y)D(X, Z) + \{X[f] - f\tau(X) - f\pi(X) + cn(X)\}g(Y, Z) - \{Y[f] - f\tau(Y) - f\pi(Y) + cn(Y)\}g(X, Z). \]

Taking \( X = Z = \zeta \) and \( Y = \xi \) to this and using (3.1) and (3.2), we have

\[ \xi[f] - f\tau(\xi) + c = 0. \]

Taking the scalar product with \( \xi \) to (2.17) and using (2.22), we have

\[ (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = B(Y, Z)\{\pi(X) - \tau(X)\} - B(X, Z)\{\pi(Y) - \tau(Y)\}. \]

Applying \( \tilde{\nabla}_X \) to \( \eta(Y) = \tilde{g}(Y, N) \) and using (2.1), (2.5) and (2.6), we have

\[ X(\eta(Y)) = -\pi(Y)\eta(X) - fg(X, Y) + \tilde{g}(\nabla_X Y, N) - g(A_\eta X, Y) + \tau(X)\eta(Y). \]

Substituting this equation into the right term of the following relation

\[ 2d\eta(X, Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]) \]

and using (2.11), (3.4) and the fact \( A_\xi^* \) is self-adjoint, we get

\[ 2d\eta(X, Y) = \tau(X)\eta(Y) - \tau(Y)\eta(X). \]

Taking the scalar product with \( N \) to (2.17) and (2.20) by turns and using (2.14)\(_2\), (2.15)\(_2\) and (2.22), we get

\[ \tilde{g}(R(X, Y)PZ, N) = c\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\} + f\{B(X, PZ)\eta(Y) - B(Y, PZ)\eta(X)\} + \rho(X)D(Y, PZ) - \rho(Y)D(X, PZ), \]

\[ \tilde{g}(R(X, Y)PZ, N) = (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) + C(X, PZ)\{\pi(Y) + \tau(Y)\} - C(Y, PZ)\{\pi(X) + \tau(X)\}. \]

From the last two equations, we have the following equation:

\[ \{cg(Y, PZ) - fB(Y, PZ)\}\eta(X) - \{cg(X, PZ) - fB(X, PZ)\}\eta(Y) + \rho(X)D(Y, PZ) - \rho(Y)D(X, PZ) = (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) + C(X, PZ)\{\pi(Y) + \tau(Y)\} - C(Y, PZ)\{\pi(X) + \tau(X)\}. \]

Applying \( \nabla_X \) to \( C(Y, PZ) = \varphi B(Y, PZ) + \eta(Y)\pi(PZ) \), we have

\[ (\nabla_X C)(Y, PZ) = X[\varphi]B(Y, PZ) + \varphi(\nabla_X B)(Y, PZ) \]
+ \{X(\eta(Y)) - \eta(\nabla_X Y)\} \pi(PZ) \\
+ \eta(Y)\{X(\pi(PZ)) - \pi(\nabla_X PZ)\}.

Substituting this into (3.11) and using (3.3), (3.7), (3.9) and (3.10), we obtain
\begin{equation}
(3.12) \quad f\{\eta(Y)B(X, PZ) - \eta(X)B(Y, PZ)\} \\
= \{X[f] - f\pi(X) - f\tau(X)\}g(Y, PZ) \\
- \{Y[f] - f\pi(Y) - f\tau(Y)\}g(X, PZ) \\
+ \eta(Y)\{X(\pi(PZ)) - \pi(\nabla_X PZ)\} \\
- \eta(X)\{Y(\pi(PZ)) - \pi(\nabla_Y PZ)\}.
\end{equation}

Applying $\nabla_X$ to $\pi(PZ) = g(\zeta, PZ)$ and using (2.10) and (3.1), we have
\begin{equation}
(3.13) \quad X(\pi(PZ)) - \pi(\nabla_X PZ) \\
= -g(X, PZ) - \pi(X)\pi(PZ) + fB(X, PZ) + g(\nabla_X \zeta, PZ).
\end{equation}

Substituting this equation into (3.12), we obtain
\begin{equation}
(3.14) \quad 0 = \{X[f] - f\pi(X) - f\tau(X)\}g(Y, PZ) \\
- \{Y[f] - f\pi(Y) - f\tau(Y)\}g(X, PZ) \\
+ \eta(Y)\{g(Y, PZ) + \pi(Y)\pi(PZ) - g(\nabla_Y \zeta, PZ)\} \\
- \eta(X)\{g(X, PZ) + \pi(X)\pi(PZ) - g(\nabla_X \zeta, PZ)\}.
\end{equation}

Applying $\nabla_X$ to $g(\zeta, \zeta) = 1$ and using (2.10) and (3.1), we have
\begin{equation}
(3.15) \quad g(\nabla_X \zeta, \zeta) = \pi(\xi).
\end{equation}

Taking $X = \xi$ and $Y = Z = \zeta$ to (3.13) and using (2.3) and (3.14), we get
\begin{equation}
(3.16) \quad \xi[f] - f\tau(\xi) + 1 = 0.
\end{equation}

From this result and (3.8), we show that $c = 1$. \hfill \Box

**Corollary 3.3.** There exist no irrotational screen quasi-conformal half lightlike submanifolds $M$ of a semi-Riemannian space form $\tilde{M}(c)$ admitting a semi-symmetric non-metric connection such that $\zeta$ belongs to $S(TM)$.

**Proof.** If $\zeta$ belongs to $S(TM)$, then $f = \tilde{g}(\zeta, N) = 0$. It follows from (3.15) that $1 = 0$. It implies that there exist no irrotational screen quasi-conformal half lightlike submanifolds $M$ of a semi-Riemannian space form $\tilde{M}(c)$ admitting a semi-symmetric non-metric connection such that $\zeta$ belongs to $S(TM)$. \hfill \Box

**Remark 3.4.** Călin [2] proved the following result: For any lightlike submanifolds $M$ of indefinite almost contact manifolds $\tilde{M}$ such that the structure vector field $\zeta$ of $\tilde{M}$ is tangent to $M$, if $\zeta$ belongs to $\text{Rad}(TM)$, then $\zeta$ is decompose as $\zeta = a\xi$ and $a \neq 0$. It follow that $1 \neq \tilde{g}(\zeta, \zeta) = a^2\tilde{g}(\xi, \xi) = 0$. It is a contradiction. Thus $\zeta$ does not belong to $\text{Rad}(TM)$. This enables one to choose a screen distribution $S(TM)$ which contains $\zeta$. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $S(TM)^2 = TM/\text{Rad}(TM)$.
Thus all screen distributions are mutually isomorphic. This implies that if $\zeta$ is tangent to $M$, then it belongs to $S(TM)$. Duggal and Sahin also proved this result (see pp. 318–319 of [7]). After Călin’s work, many earlier works [5, 6, 14] which have written on lightlike submanifolds of indefinite almost contact manifolds or lightlike submanifolds of semi-Riemannian manifolds admitting semi-symmetric non-metric connections obtained their results by using the Călin’s result described in above. However, we regret to indication that Călin’s result is not true for any irrotational half lightlike submanifolds $M$ of a semi-Riemannian space form $\tilde{M}(c)$ admitting a semi-symmetric non-metric connection by Theorem 3.2 and Corollary 3.3.

**Definition.** A half lightlike submanifold $M$ of a semi-Riemannian manifold $\tilde{M}$ admitting a semi-symmetric non-metric connection is **screen conformal** [4, 7, 9] if the second fundamental forms $B$ and $C$ satisfy

$$C(X, PY) = \varphi B(X, Y),$$

where $\varphi$ is a non-vanishing function on a coordinate neighborhood $U$ in $M$.

**Theorem 3.5.** Let $M$ be an irrotational half lightlike submanifold of a semi-Riemannian space form $\tilde{M}(c)$ admitting a semi-symmetric non-metric connection such that its the structure vector field $\zeta$ is tangent to $M$. If $M$ is screen conformal, then we have $c = 0$.

**Proof.** We show that the equation (3.9) and (3.11) follow by straightforward calculations from the Gauss-Codazzi equations and (2.22). Applying $\nabla_X$ to $C(Y, PZ) = \varphi B(Y, PZ)$, we have


Substituting this equation into (3.11) and using (3.9) and (3.16), we have

$$c\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\}$$

$$= \{X[\varphi] - 2\varphi\tau(X) + f\eta(X)\}B(Y, PZ)$$

$$- \{Y[\varphi] - 2\varphi\tau(Y) + f\eta(Y)\}B(X, PZ)$$

$$+ \rho(Y)D(X, PZ) - \rho(X)D(Y, PZ).$$

Taking $X = \xi$ and $Y = Z = \zeta$ to this and using (3.1), we have $c = 0$. □

**Remark 3.6.** From Theorem 3.2 and Theorem 3.5, we show that the two screen conformalities, which are called **screen conformal** and **screen quasi-conformal**, of $M$ are mutually independent to each other.

**References**


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