ATTRACTORS AND LYAPUNOV FUNCTION FOR CLOSED RELATIONS

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ABSTRACT. In this paper, we will study the characterizations of attractors and Lyapunov function which were studied in flows and homeomorphisms will also be satisfied in compact closed relation dynamics. In particular, we will find the necessary and sufficient conditions of the existence of the strict Lyapunov function.

1. Introduction and preliminaries

The main purpose of this paper is to prove following two theorems.

Theorem 2.10. *Let A be a Nf -positively invariant closed subset of X . Then there exists a Lyapunov function $h : X \rightarrow [0, 1]$ for f such that $A = h^{-1}(0)$.*

Theorem 2.11. *Let f be a compact continuous relation on a locally compact space X and A be a nonempty compact invariant subset of X . Assume that there exists a strict Lyapunov function $h : X \rightarrow [0, \infty)$ for f satisfying $h^{-1}(0) = A$. Then A is an attractor for f .*

To begin with, we introduce the necessary definitions and results without proof.

Let (X_1, d_1) , (X_2, d_2) be arbitrary metric spaces. A relation $f : X_1 \rightarrow X_2$ is considered as a map from X_1 to the power set of X_2 , that is, each $x \in X_1$ corresponds to a subset $f(x)$ of X_2 , or a subset of $X_1 \times X_2$ so that $y \in f(x)$ means $(x, y) \in f$. We define the domain of f by

$$\text{Dom}(f) = \{x \in X_1 \mid f(x) \neq \emptyset\}.$$

i.e., $x \in \text{Dom}(f)$ if and only if there exists $y \in X_2$ such that $(x, y) \in f$.

For relations $f : X_1 \rightarrow X_2$ and $g : X_2 \rightarrow X_3$ we define the inverse $f^{-1} : X_2 \rightarrow X_1$, and the composition $g \circ f : X_1 \rightarrow X_3$ by

$$x \in f^{-1}(y) \iff y \in f(x),$$

Received July 31, 2013.

2010 *Mathematics Subject Classification.* 37B40.

Key words and phrases. closed relation, attractors, Lyapunov function.

i.e., $f^{-1} = \{(y, x) \mid (x, y) \in f\}$, and

$$y \in (g \circ f)(x) \iff z \in f(x) \text{ and } y \in g(z) \text{ for some } z \in X_2.$$

The usual composition properties of associativity, identity, and inversion generalize to the relation, e.g., $1_{X_2} \circ f = f = f \circ 1_{X_1}$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. There are additional algebraic properties as well. For example, composition distributes over the union:

$$(\cup_m g_m) \circ (\cup_n f_n) = \cup_{m,n} (g_m \circ f_n).$$

For $f : X \rightarrow X$ we define f^n to be the n -fold composition of f ($n = 0, 1, 2, \dots$ with $f^0 = 1_X$ and $f^1 = f$ by definition). f^{-n} is defined to be $(f^{-1})^n$ (which equals $(f^n)^{-1}$). For a relation $f : X_1 \rightarrow X_2$ and a subset A of X_1 the *image* $f(A) \subset X_2$ is defined by

$$f(A) = \{y \mid (x, y) \in f \text{ for some } x \in A\} = \cup\{f(x) \mid x \in A\}.$$

$f : X_1 \rightarrow X_2$ is called a *closed relation* if it is a closed subset of $X_1 \times X_2$.

$f : X_1 \rightarrow X_2$ is called a *compact relation* if $f(x)$ is a compact subset of X_2 for any $x \in X_1$.

Definition 1.1 ([1]). A relation $f : X_1 \rightarrow X_2$ is said to be *upper semicontinuous* at $x \in X_1$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d_1(x, y) < \delta \text{ implies } f(y) \subset B(f(x), \epsilon).$$

A relation $f : X_1 \rightarrow X_2$ is said to be *lower semicontinuous* at $x \in X_1$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d_1(x, y) < \delta \text{ implies } f(x) \subset B(f(y), \epsilon).$$

A relation f is said to be *continuous* at x if f is both upper semicontinuous at x and lower semicontinuous at x .

Proposition 1.1. Let f be a relation on X and $x \in X$. Then the following are true: (1) f is upper semicontinuous at x if and only if for sequences $(x_n), (y_n)$ in X with $x_n \rightarrow x$, $y_n \rightarrow y$, and $y_n \in f(x_n)$, $y \in f(x)$. (2) f is lower semicontinuous at x if and only if for every sequence $(x_n), (y_n)$ in X with $x_n \rightarrow x$, $y_n \rightarrow y$, and $y_n \in f(x_n)$, $d(y_n, f(x_n))$ converges to 0 as $n \rightarrow \infty$.

Lemma 1.2. Let $f : X_1 \rightarrow X_2$ be a compact relation. (1) f is upper semicontinuous at x if and only if for every neighborhood U of $f(x)$, there exists a neighborhood V of x such that $y \in V$ implies $f(y) \subset U$. (2) f is lower semicontinuous at x if and only if for every open set U with $U \cap f(x) \neq \emptyset$, there exists a neighborhood V of x such that $y \in V$ implies $f(y) \cap U \neq \emptyset$.

Theorem 1.3. Let $f : X_1 \rightarrow X_2$ be compact lower semicontinuous. Then $f(\overline{A}) \subset \overline{f(A)}$ for any subset A of X_1 .

Proposition 1.4. Let $f : X_1 \rightarrow X_2$ be compact upper semicontinuous and let X_2 be a locally compact space. Then for every compact subset K of X_1 , $f(K)$ is a compact subset of X_2 .

Proposition 1.5. *Let $f : X_1 \rightarrow X_2$ be compact continuous and let X_2 be locally compact. For any compact subset C of X_1 and neighborhood U of $f(C)$ there exists a neighborhood V of C such that $f(V) \subset U$.*

Corollary 1.6. *Let (X, d) be a locally compact space and $f : X \rightarrow X$ be compact continuous. Then n -fold composition of f , $f^n : X \rightarrow X$, is compact continuous.*

2. Attractors and Lyapunov function

Many mathematicians have been researched on the attractor and Lyapunov function for flows and homeomorphisms ([2], [3]). This section is devoted to the study of notions of attractor and the characterization of some properties of them in terms of the so called Lyapunov function.

Let f be a closed continuous relation on X .

For every subset A of X , the ω -limit set $\omega(A)$ of A is denoted to be the set:

$$\omega(A) = \bigcap_{n \geq 0} \overline{\bigcup_{k \geq n} f^k(A)}.$$

Now, let us explore the properties of ω -limit set.

Proposition 2.1. *Let A be a subset of X . Then $x \in \omega(A)$ if and only if there exist sequences (a_i) in A , (n_i) in $\mathbb{N}$, and (x_i) in X such that $n_i \rightarrow \infty$, $x_i \in f^{n_i}(a_i)$, and $x_i \rightarrow x$.*

Proof. Let $x \in \omega(A)$. For every $i \in \mathbb{N}$, since $x \in \overline{\bigcup_{n \geq i} f^n(A)}$,

$$B(x, \frac{1}{i}) \cap \bigcup_{n \geq i} f^n(A) \neq \emptyset.$$

Let $x_i \in B(x, \frac{1}{i}) \cap \bigcup_{n \geq i} f^n(A)$. Then there exist $n_i \geq i$, $a_i \in A$ such that $x_i \in f^{n_i}(a_i)$. It is clear that $n_i \rightarrow \infty$ and $x_i \rightarrow x$.

Suppose that there exist sequences (a_i) in A , (n_i) in $\mathbb{N}$, and (x_i) in X such that $n_i \rightarrow \infty$, $x_i \in f^{n_i}(a_i)$, and $x_i \rightarrow x$. Then for every $n \geq 0$ and neighborhood U of x , there exist $x_i \in U$ and $n_i \geq n$. Since $x_i \in \overline{f^{n_i}(a_i)} \subset \overline{f^{n_i}(A)} \subset \bigcup_{k \geq n} f^k(A)$, $U \cap \bigcup_{k \geq n} f^k(A) \neq \emptyset$. Thus $x \in \omega(A) = \bigcap_{n \geq 0} \overline{\bigcup_{k \geq n} f^k(A)}$. $\square$

Definition 2.1. A subset A of X is said to be *positively invariant* if $f(A) \subset A$. A subset A of X is said to be *invariant* if $f(A) = A$ or equivalently if A is positively invariant and $f^{-1}(x) \cap A \neq \emptyset$ for all $x \in A$.

Proposition 2.2. *For every subset A of compact metric space X , $\omega(A)$ is closed and invariant.*

Proof. By definition of ω limit set, $\omega(A)$ is closed.

$$\begin{aligned} f(\omega(A)) &= f\left(\bigcap_{n \geq 0} \overline{\bigcup_{k \geq n} f^k(A)}\right) \subset \bigcap_{n \geq 0} \overline{f\left(\bigcup_{k \geq n} f^k(A)\right)} \\ &\subset \bigcap_{n \geq 0} \overline{f\left(\bigcup_{k \geq n} f^k(A)\right)} \\ &= \bigcap_{n \geq 0} \overline{\bigcup_{k \geq n} f^{k+1}(A)} \end{aligned}$$

$$\begin{aligned} &\subset \bigcap_{n \geq 0} \overline{\bigcup_{k \geq n} f^k(A)} \\ &= \omega(A). \end{aligned}$$

Let $x \in \omega(A)$. By Proposition 2.1, there exist sequences (a_i) in A , (n_i) in $\mathbb{N}$, and (x_i) in X such that $n_i \rightarrow \infty$, $x_i \in f^{n_i-1}(a_i)$, and $x_i \rightarrow x$. Since $x_i \in f^{n_i}(a_i) = f(f^{n_i-1}(a_i))$, we can find $y_i \in f^{n_i-1}(a_i)$ with $x_i \in f(y_i)$. By compactness of X , the sequence (y_i) has a convergent subsequence. Let $y_i \rightarrow y$. Since $n_i - 1 \rightarrow \infty$, $y \in \omega(A)$. By Proposition 2.1, $x \in f(y) \subset f(\omega(A))$. Thus $f(\omega(A)) = \omega(A)$. $\square$

Proposition 2.3. *If $A \subset X$ is positively invariant, then $\overline{A}$ is positively invariant.*

Proof. Since $f(\overline{A}) \subset \overline{f(A)} \subset \overline{A}$, $\overline{A}$ is positively invariant. $\square$

Proposition 2.4. *If $A \subset X$ is f -positively invariant, then A is $\overline{f}$ -positively invariant.*

Proof. Let $y \in \overline{f(A)}$, then there exists $x \in A$ such that $(x, y) \in \overline{f}$. There exists a sequence $((x_n, y_n))$ in f such that $(x_n, y_n) \rightarrow (x, y)$. Since $x_n \rightarrow x$, $y_n \rightarrow y$, and $y_n \in f(x_n)$ for all n , we have $y \in f(x) \subset f(A) \subset A$. Thus $\overline{f(A)} \subset A$. $\square$

We start with the definitions of attractor and attractor block.

Definition 2.2 ([3]). A subset A of X is said to be an *attractor* for f if there exists a neighborhood U of A such that $\omega(U) = A$.

By Proposition 2.2, attractor is a closed invariant set.

Definition 2.3 ([3]). A subset B of X is said to be an *attractor block* for f if $f(\overline{B}) \subset \text{Int}(B)$.

Theorem 2.5. *Attractor block B is a neighborhood $\omega(B)$. Thus $\omega(B)$ is an attractor.*

Proof. Since $\overline{B}$ is compact, $f(\overline{B})$ is compact. Thus $\overline{f(\overline{B})} = f(\overline{B})$.

$$\omega(B) = \bigcap_{n \geq 0} \overline{f^n(B)} \subset \overline{f(B)} \subset \overline{f(\overline{B})} = f(\overline{B}) \subset \text{Int}(B).$$

Thus B is a neighborhood $\omega(B)$. $\square$

Theorem 2.6. *If A is an attractor for f and V is a neighborhood of A , then there exists a closed attractor block B for f such that $B \subset V$ and $\omega(B) = A$.*

Proof. Since A is an attractor, there exists a neighborhood U of A such that $\omega(U) = A$. For a neighborhood $U \cap A$ of A , there exists a closed neighborhood B of A such that $B \subset U \cap V$. Since $A \subset B \subset U$, we have

$$A = \omega(A) \subset \omega(B) \subset \omega(U) = A.$$

Thus $\omega(B) = A$. $\square$

Suppose that A is an attractor, that B is an attractor block, and that $A = \omega(B)$. Then A will be called the *attractor associated with B* , while B will be called an *attractor block associated with A* . Theorem 2.5 states that an attractor block always has an associated attractor. Theorem 2.6 states that an attractor always has an associated attractor block in any neighborhood.

For a relation f on X , we define the relations Of , Nf on X by

$$Of = \cup_{n \in \mathbb{N}} f^n, \quad Nf = \overline{Of}.$$

Proposition 2.7. *If $A \subset X$ is an f -positively invariant closed set, then A is Nf -positively invariant.*

Proof. Since $f(A) \subset A$, we have

$$A \supset f(A) \supset f^2(A) \supset \dots$$

Thus $Of(A) = \cup_{n \in \mathbb{N}} f^n(A) = f(A) \subset A$. By Proposition 2.4, we obtain $Nf(A) = \overline{\cup_{n \in \mathbb{N}} f^n(A)} \subset A$. $\square$

Proposition 2.8. *For a closed subset A of X , the following conditions are equivalent: (1) $Nf(A) \subset A$ (2) For every neighborhood U of A there exists a neighborhood $V \subset U$ of A such that $Of(x) \subset U$ for all $x \in V$. (3) For every neighborhood U of A there exists a neighborhood $V \subset U$ of A such that $f(U) \subset V$.*

Proof. (1) $\Rightarrow$ (3). Since Nf is a closed relation on X , there exists $\delta > 0$ such that $B[A, \delta] \subset U$ and $Nf(B[A, \delta]) \subset U$. Let $V = B(A, \delta) \cup Of(B(A, \delta))$. Since $B(A, \delta) \subset V$, V is a neighborhood of A . Clearly, $f(V) \subset V$. We have

$$V \subset B[A, \delta] \cup Nf(B[A, \delta]) \subset U.$$

(3) $\Rightarrow$ (2). This is obvious.

(2) $\Rightarrow$ (1). Let $x \in A$ and $y \in Nf(x)$. Given any $\epsilon > 0$, since $B(A, \epsilon)$ is a neighborhood of A , there exists a neighborhood $U \subset B(A, \epsilon)$ of A such that $Of(x) \subset B(A, \epsilon)$ for all $x \in U$. There exists δ with $0 < \delta < \epsilon$ such that $B(A, \delta) \subset U$. Since

$$(x, y) \in Nf = \overline{Of},$$

we have

$$(B(x, \delta) \times B(y, \delta)) \cap Of \neq \emptyset$$

and so $(B(x, \delta) \times B(y, \delta)) \cap f^n \neq \emptyset$ for some $n \geq 1$. Let $(p, q) \in (B(x, \delta) \times B(y, \delta)) \cap f^n$. Then $d(x, p) < \delta$, $d(y, q) < \delta$ and $q \in f^n(p)$. Since $d(p, A) \leq d(p, x) < \delta$, $p \in B(A, \delta) \subset U$. Thus $q \in f^n(p) \subset B(A, \epsilon)$. Then $d(y, A) \leq d(y, q) + d(q, A) < \delta + \epsilon < 2\epsilon$. Hence $d(y, A) = 0$ and so $y \in A$. Therefore $Nf(A) \subset A$. $\square$

Proposition 2.9. *Let f be a transitive relation on X and A be a closed positively invariant subset of X . For every neighborhood U of A , there exist positively invariant open set V_1 and positively invariant closed set V_2 such that*

$$A \subset V_1 \subset V_2 \subset U.$$

Proof. Since f is transitive, we have $Nf = Of = f$. Since $Nf(A) = f(A) \subset A$, by the proof of (1) $\Rightarrow$ (3) in Proposition 2.8, $V_2 = B[A, \delta] \cup f(B[A, \delta]) \subset U$ is a positively invariant closed set. Let $V_1 = \{x \in \text{Int}(V_2) \mid f(x) \subset \text{Int}(V_2)\}$. Since f is transitive, V_1 is a positively invariant open set. Since A is positively invariant and $B(A, \delta) \subset \text{Int}(V_2)$, we have $A \subset V_1$. $\square$

We now introduce the fundamental definitions of Lyapunov function and strict Lyapunov function.

Definition 2.4. Let f be a compact continuous relation on X . We say that h is a *Lyapunov function* for f if $h : X \rightarrow [0, \infty)$ is continuous, $A = h^{-1}(0)$ is nonempty invariant and $h(y) \leq h(x)$ for all $x \in X, y \in f(x)$.

Remark 2.1. In general, the Lyapunov function h need not have the property $m < n$ implies $\max h(f^n(x)) < \min h(f^m(x))$ for all $x \notin A$.

For example, let $X = [0, 1]$, $h : X \rightarrow [0, \infty)$ be the identity function and let $f = \{(x, y) \in X \times X \mid \frac{1}{3}x \leq y \leq \frac{2}{3}x\}$. Then h is a Lyapunov function, but h does not satisfy the above condition. So we introduce the following.

Definition 2.5. A Lyapunov function h is called a *strict Lyapunov function* if there is a neighborhood W of A such that $h^{-1}([0, 1]) \subset W$ and $m < n$ implies $\max h(f^n(x)) < \min h(f^m(x))$ for all $x \in W - A$. In particular, if h is a strict Lyapunov function, then $h(y) < h(x)$ for all $x \notin A, y \in f(x)$.

We now give an important characterization of Lyapunov function.

Theorem 2.10. Let A be a Nf -positively invariant closed subset of X . Then there exists a Lyapunov function $h : X \rightarrow [0, 1]$ for f such that $A = h^{-1}(0)$.

Proof. Let R be the set of all rationals r of form $\frac{k}{2^n}$, $0 < \frac{k}{2^n} \leq 1$. We first show that with $r \in R$, we can associate an Nf -positively invariant open subset $U(r)$ of X such that

- (1) $A \subset U(r)$,
- (2) $r_1 < r_2$ implies $\overline{U(r_1)} \subset U(r_2)$.

We proceed by induction on the exponent of the dyadic fractions, letting

$$D_m = \{U(\frac{k}{2^m}) \mid k = 1, 2, \dots, 2^m\}.$$

D_1 consists of $U(1) = X$ and $U(\frac{1}{2}) =$ some Nf -positively invariant open set satisfying

$$A \subset U(\frac{1}{2}) \subset \overline{U(\frac{1}{2})} \subset U(1),$$

which exists by Proposition 2.9. Assume D_{m-1} constructed, and note that only $U(\frac{k}{2^m})$ for odd k requires definition; for each odd k , we have from D_{m-1} that

$$\overline{U(\frac{k-1}{2^m})} \subset U(\frac{k+1}{2^m}),$$

so we define $U(\frac{k}{2^m})$ to be an Nf -positively invariant open set U satisfying

$$\overline{U(\frac{k-1}{2^m})} \subset U \subset \overline{U} \subset U(\frac{k+1}{2^m}).$$

This completes the inductive step.

Let $D = \cup_{m \in \mathbb{N}} D_m$ and let

$$h(x) = \inf\{r \mid x \in U(r)\}.$$

Clearly, $0 \leq h(x) \leq 1$. Furthermore $h(A) = 0$, since $A \subset U(r)$ for all r . Because $U(r)$ is positively invariant, $x \in U(r)$ implies $f(x) \subset U(r)$ and so for $y \in f(x)$ we have

$$\{r \mid x \in U(r)\} \subset \{r \mid y \in U(r)\}.$$

Thus $h(y)$, the sup of the larger set, is at least $h(x)$.

It remains to prove continuity. Let $h(x_0) = r_0$, and choose any $W = (r_0 - \epsilon, r_0 + \epsilon)$. If $r_0 \neq 0, 1$, choose also r_1, r_2 such that

$$r_0 - \epsilon < r_1 < r_0 < r_2 < r_0 + \epsilon,$$

then $U = U(r_2) - \overline{U(r_1)}$ is a neighborhood of x_0 and $h(U) \subset W$, since $x \in U(r_2)$ implies $h(x) \leq r_2$, and $x \notin \overline{U(r_1)}$ implies $h(x) \geq r_1$. If $r_0 = 0$ (or 1), then the neighborhood $U(r_2)$ (or $X - \overline{U(r_1)}$) of x_0 alone suffices. This completes the proof. $\square$

We are now in a position to prove the following important result.

Theorem 2.11. *Let f be a compact continuous relation on a locally compact space X and A be a nonempty compact invariant subset of X . Assume that there exists a strict Lyapunov function $h : X \rightarrow [0, \infty)$ for f satisfying $h^{-1}(0) = A$. Then A is an attractor for f .*

Proof. Since X is locally compact, there is a compact neighborhood K of A such that $K \subset W$. For any $\epsilon > 0$, let $K_\epsilon = \{x \in K \mid h(x) \leq \epsilon\}$. Then K_ϵ is a compact subset of K . We claim that for every neighborhood U of A there exists a constant $\delta > 0$ such that $0 < \epsilon \leq \delta$ implies $K_\epsilon \subset U$.

Suppose that the above claim is not true. Then there is a neighborhood U of A with the property that, for every $\delta > 0$, there is a positive constant ϵ with $0 < \epsilon \leq \delta$ such that $K_\epsilon \not\subset U$. For every $n \in \mathbb{N}$, there is a positive constant ϵ_n with $0 < \epsilon_n \leq \frac{1}{n}$ such that $K_{\epsilon_n} \not\subset U$. Then there is $x_n \in K_{\epsilon_n} - U \subset K - U$. Since $K - U$ is compact, (x_n) has a convergent subsequence. Let $x_n \rightarrow p \in K - U$. Since $p \notin A$, $h(p) > 0$. Since $h(x_n) \rightarrow h(p)$ and $0 \leq h(x_n) \leq \epsilon_n \leq \frac{1}{n}$, $h(p) = 0$. This is a contradiction.

For every $x \in A$,

$$f(x) \subset f(A) = A \subset \text{Int}(K).$$

Since $f(x)$ is compact, there is $\epsilon_x > 0$ such that $B(f(x), \epsilon_x) \subset \text{Int}(K)$. Since f is continuous at x , there exists $\delta_x > 0$ such that $B(x, \delta_x) \subset \text{Int}(K)$ and $d(x, y) < \delta_x$ implies $f(y) \subset B(f(x), \epsilon_x)$. $\{B(x, \delta_x) \mid x \in A\}$ is an open

cover of A . Since A is compact, there are finitely many elements $x_1, x_2, \dots, x_n$ in A such that

$$A \subset \bigcup_{i=1}^n B(x_i, \delta_{x_i}).$$

Let $U = \bigcup_{i=1}^n B(x_i, \delta_{x_i})$. Then U is a neighborhood of A and $U \subset \text{Int}(K)$. For every $y \in U$, there is i such that $y \in B(x_i, \delta_{x_i})$. Since $d(x_i, y) < \delta_{x_i}$,

$$f(y) \subset B(f(x_i), \epsilon_{x_i}) \subset \text{Int}(K).$$

Thus $f(U) \subset \text{Int}(K)$. There is $\delta > 0$ such that $0 < \epsilon \leq \delta$ implies $K_\epsilon \subset U$. We claim that $f(K_\epsilon) \subset \text{Int}(K_\epsilon)$. Let $x \in K_\epsilon$. There are two cases depending upon whether $x \in A$ or not.

For the first case $x \in A$, we have $f(x) \subset f(A) = A$. Since $h^{-1}(0) = A$, we have $f(x) \subset \text{Int}(K_\epsilon)$.

For the remaining case $x \notin A$, since $h(y) < f(x) \leq \epsilon$ for all $y \in f(x)$, we have $f(x) \subset \text{Int}(K_\epsilon)$. Thus $f(K_\epsilon) \subset \text{Int}(K_\epsilon)$ and so K_ϵ is an attractor block.

Since $A = f^n(A) \subset f^n(K_\epsilon)$ for all $n \geq 0$, we obtain

$$A \subset \bigcap_{n=0}^{\infty} f^n(K_\epsilon) \subset \bigcap_{n=0}^{\infty} \overline{f^n(K_\epsilon)}.$$

Now we will prove that $\bigcap_{n=0}^{\infty} \overline{f^n(K_\epsilon)} \subset A$. Since K_ϵ is compact, $f^n(K_\epsilon)$ is compact by Proposition 2.4. So $\overline{f^n(K_\epsilon)} = f^n(K_\epsilon)$. Thus it suffices to show that $\bigcap_{n=0}^{\infty} f^n(K_\epsilon) \subset A$.

Let $x \in K_\epsilon - A$ and $y_n \in f^n(x)$ for every n . We claim that

$$h(y_n) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Suppose that above statement is not true. Then there exists $\epsilon > 0$ such that for every $n \in \mathbb{N}$, there is a natural number $N > n$ such that $h(y_N) \geq \epsilon$. Since

$$K_\epsilon \supset f(K_\epsilon) \supset f^2(K_\epsilon) \supset \dots,$$

we have

$$y_{n_i} \in f^{n_i}(x) \subset f^{n_i}(K_\epsilon) \subset K_\epsilon.$$

Since K_ϵ is compact, (y_{n_i}) has a convergent subsequence. Let $y_{n_i} \rightarrow p \in K_\epsilon$. Then $h(y_{n_i}) \rightarrow h(p)$. Since $h(y_{n_i}) \geq \epsilon$ for all i , we have $h(p) \geq \epsilon$. Thus $p \notin A$. Suppose that $f^N(x) \subset A$ for some N . Then

$$f^{N+1}(x) \subset f(A) = A, f^{N+2}(x) \subset f(A) = A, \dots$$

Since $n_i \rightarrow \infty$, we can assume that $n_i \geq N$. Then $y_{n_i} \in f^{n_i}(x) \subset A$. Thus $p \in \overline{A} = A$, which is a contradiction. Hence $f^n(x) \not\subset A$ for all n and $p \in \omega(x)$.

Assume that $\omega(x) \cap A = \emptyset$. Since $\omega(x)$ is compact, there is $y \in \omega(x)$ such that $h(y) = \min h(\omega(x))$. Since $\omega(x)$ is invariant, $f(y) \subset \omega(x)$. Let $p \in f(y) \subset \omega(x)$. Since $y \notin A$, we have $h(y) \leq h(p) < h(y)$. This is a contradiction. Thus $\omega(x) \cap A \neq \emptyset$. Let $z \in \omega(x) \cap A$. There are natural numbers $m_1 < m_2 < \dots$ and $z_{m_i} \in f^{m_i}(x)$ for all i such that $z_{m_i} \rightarrow z$. Thus $h(z_{m_i}) \rightarrow h(z) = 0$. We can assume that $m_i < n_i$ for all i without loss of generality. Since $x \notin A$, $y_{n_i} \in f^{n_i}(x)$ and $z_{m_i} \in f^{m_i}(x)$, we have

$$\epsilon \leq h(y_{n_i}) < h(z_{m_i}) \rightarrow 0.$$

This is a contradiction. Thus $h(y_n) \rightarrow 0$.

Since $f^n(x)$ is compact, there is $z_x^n \in f^n(x)$ such that

$$h(z_x^n) = \max h(f^n(x)).$$

Since $h(z_x^n) \rightarrow 0$ as $n \rightarrow \infty$, for every $\delta > 0$, there is a natural number $N(x, \delta)$ such that

$$n \geq N(x, \delta) \text{ implies } h(z_x^n) < \delta.$$

There is a neighborhood V_x of x such that

$$y \in V_x \text{ implies } \max h(f^{N(x, \delta)}(y)) < \delta.$$

Here $\max h(f^{N(x, \delta)}(y)) = h(z_y^{N(x, \delta)})$.

If $n \geq N(x, \delta)$, then $h(z_y^n) \leq h(z_y^{N(x, \delta)}) < \delta$. $\{V_x \mid x \in K_\epsilon\}$ is an open cover of K_ϵ . Since K_ϵ is compact, there are finitely many $x_1, x_2, \dots, x_k \in K_\epsilon$ such that $K_\epsilon \subset \cup_{i=1}^k V_{x_i}$. Let $N = \max N(x_i, \delta)$. For any $y \in K_\epsilon$, there is i such that $y \in V_{x_i}$. If $n \geq N \geq N(x_i, \delta)$, then $h(z_y^n) = \max h(f^n(y)) < \delta$. Since

$$f^n(y) \subset f^n(K_\epsilon) \subset K_\epsilon \subset K,$$

we have $f^n(y) \subset K_\delta$ for all $n \geq N$. Thus $f^n(K_\epsilon) \subset K_\delta$ for all $n \geq N$.

Since $K_\epsilon \supset f(K_\epsilon) \supset f^2(K_\epsilon) \supset \dots$,

$$\cap_{n=0}^{\infty} f^n(K_\epsilon) \subset f^N(K_\epsilon) \subset K_\delta \text{ for all } \delta > 0.$$

Since $K_\delta \rightarrow A$ as $\delta \rightarrow 0$, we have

$$\cap_{n=0}^{\infty} f^n(K_\epsilon) \subset A.$$

Hence $\cap_{n=0}^{\infty} \overline{f^n(K_\epsilon)} = A$.

The proof of the theorem is complete. □

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