A SHARP SCHWARZ AND CARATHÉODORY INEQUALITY ON THE BOUNDARY

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Abstract. In this paper, a boundary version of the Schwarz and Carathéodory inequality are investigated. New inequalities of the Carathéodory’s inequality and Schwarz lemma at boundary are obtained by taking into account zeros of \( f(z) \) function which are different from zero. The sharpness of these inequalities is also proved.

1. Introduction

Let \( f \) be a function which is holomorphic on the \( D = \{ z : |z| < 1 \} \) and vanish at \( z = 0 \), and suppose that \( |f| < 1 \) for all \( z \in D \). Then the inequality

\[
|f(z)| \leq |z|
\]

holds for all \( z \in D \), and moreover

\[
|f'(0)| \leq 1. 
\]

Equality is achieved in (1.1) (for some nonzero \( z \in D \)) or in (1.2) if and only if \( f \) is an entire linear function of the form \( f(z) = e^{i\alpha}z \), where \( \alpha \) is a real number(\([2] \), p. 381).

Let the zeros of \( f \) be \( z_1, z_2, \ldots, z_n \). If we apply inequality (1.1) to the function \( f(z) \prod_{k=1}^{n} \left( \frac{1-zk}{1-zkz} \right) \), we can conclude in the following Schwarz’s inequality:

\[
|f(z)| \leq |z| \prod_{k=1}^{n} \left| \frac{z - zk}{1 - zkz} \right|
\]

and

\[
|f'(0)| \leq \prod_{k=1}^{n} |zk|.
\]
If \( f(z) = c_p z^p + c_{p+1} z^{p+1} + \cdots, c_p \neq 0, \) is a holomorphic function in \( D \) and \(|f| \leq 1 \) for \( z \in D \), then at each \( z \in D \) we have the inequality

\[
|f(z)| \leq |z|^p \prod_{k=1}^{n} \left| \frac{z - z_k}{1 - z_k \overline{z}} \right|
\]

and

\[
|c_p| \leq \prod_{k=1}^{n} |z_k|.
\]

If, in addition, the function \( f \) can be extended by continuity to a point \( z_0 \in \partial D, |f(z_0)| = 1 \), and the derivative \( f'(z_0) \) exists, then (1.1) implies the inequality \(|f'(z_0)| \geq 1\), which is known as the Schwarz lemma on the boundary. Previously, R. Osserman, examined sharp Schwarz inequality at the boundary (see [3]).

If the function \( f \) has an angular limit \( f(z_0) \) at \( z_0 \in \partial D, |f(z_0)| = 1 \), then by Julia-Wolff-Lemma the angular derivative \( f'(z_0) \) exists and \( 1 \leq |f'(z_0)| \leq \infty \) ([4]).

We will obtain more general results at the boundary. In the following Theorems 1.1-1.2, new inequalities of Schwarz inequality at the boundary are obtained and the sharpness of these inequalities is proved.

Introducing the notation

\[
\Phi = \sum_{i_1=1}^{n-k+1} \sum_{i_2=i_1+1}^{n-k+2} \cdots \sum_{i_k=i_{k-1}+1}^{n} (|z_{i_1}| |z_{i_2}| \cdots |z_{i_k}|).
\]

**Theorem 1.1.** Let \( f \) be a holomorphic function in the disc \( D, |f| < 1 \) for \(|z| < 1, f(0) = 0 \) and \( z_1, z_2, \ldots, z_n \) are zeros of the function \( f \) in the unit disc that are different from \( z = 0 \). Further assume that, for some \( z_0 \in \partial D, f \) has an angular limit \( f(z_0) \) at \( z_0, |f(z_0)| = 1 \). Then

\[
|f'(z_0)| \geq \frac{n+1+\sum_{k=1}^{n} (n-2k+1) \Phi}{\prod_{k=1}^{n} (1+|z_k|)}.
\]

The inequality (1.5) is sharp, with equality for the function \( f(z) = z \prod_{k=1}^{n} \frac{z - z_k}{1 - z_k \overline{z}} \), where \( z_1, z_2, \ldots, z_n \) are negative real numbers.

**Proof.** Using the upper bound (1.3) and if \( z_0, c \in \partial D \) with \( f(z_0) = c \), then we obtain

\[
\frac{|f(z) - c|}{|z| - |z_0|} \geq \frac{1-|f(z)|}{1-|z|} \geq \frac{1-|z|}{1-|z|} \prod_{k=1}^{n} \frac{|z - z_k|}{1 - z_k \overline{z}} \geq \frac{1-|z|}{1-|z|} \prod_{k=1}^{n} \frac{|z| + |z_k|}{1 - |z_k|}
\]

\[
= \prod_{k=1}^{n} \frac{(1+|z_k|)|z| - |z|}{(1-|z|) \prod_{k=1}^{n} (1+|z_k|)|z|}
\]
Proof. Using the upper bound \((1.4)\) and if 

$$f \left( \frac{|z|}{z} \right) = \frac{1}{|z|} \prod_{k=1}^{n} |z_k|||z_k^2|\ldots|z_k^n|,$$

Passing to the angular limit in the last inequality yields

$$n+1 \sum_{k=1}^{n} (n-2k+1) \sum_{i=1}^{n-k+1} \sum_{j=1}^{n} \sum_{k=1}^{n} \left( |z_i^k| |z_j^k| \ldots |z_n^k| \right) (1+|z_k||z|)$$

The equality in \((1.5)\) is obtained for the function \(f(z) = z \prod_{k=1}^{n} \frac{z-z_k}{z_k^2}, \) as shown by simple calculations.

**Theorem 1.2.** Let \(f(z) = c_p z^p + c_{p+1} z^{p+1} + \ldots, \) \(c_p \neq 0, \ p \geq 1, \) be a holomorphic function in the disc \(D, \) \(|f| < 1\) for \(|z| < 1\) and \(z_1, z_2, \ldots, z_n\) are zeros of the function \(f\) in the unit disc that are different from \(z = 0.\) Further assume that, for some \(z_0 \in \partial D, \) \(f\) has an angular limit \(f(z_0)\) at \(z_0,\) \(|f(z_0)| = 1.\) Then

\[
|f'(z_0)| \geq \frac{n+p+ \sum_{k=1}^{n} (n-2k+p) z_k^p}{\prod_{k=1}^{n} (1+|z_k||z|)}.
\]

The inequality \((1.6)\) is sharp, with equality for the function

\[
f(z) = z^p \prod_{k=1}^{n} \frac{z-z_k}{z_k^2},
\]

where \(z_1, z_2, \ldots, z_n\) are negative real numbers.

**Proof.** Using the upper bound \((1.4)\) and if \(z_0, c \in \partial D\) with \(f(z_0) = c,\) then we obtain

\[
\left| \frac{f(z) - c}{z - z_0} \right| \geq \frac{1-|f(z)|}{1-|z|} \geq \frac{1-|z|^p \prod_{k=1}^{n} |z_k^k||\ldots|z_k^n|}{1-|z|} \geq \frac{1-|z|^p \prod_{k=1}^{n} |z_k^k||\ldots|z_k^n|}{1-|z|}.
\]

Passing to the angular limit in the last inequality yields

\[
\frac{n+p+ \sum_{k=1}^{n} (n-2k+p) \sum_{i=1}^{n-k+1} \sum_{j=1}^{n} \sum_{k=1}^{n} \left( |z_i^k| |z_j^k| \ldots |z_n^k| \right)}{\prod_{k=1}^{n} (1+|z_k||z|)}.
\]
The equality in (1.6) is obtained for the function \( f(z) = z^p \prod_{k=1}^{n} \frac{z - z_k}{1 - z_k} \), as shown by simple calculations.

In the following theorems, we formulated boundary “Carathéodory inequality” (see [1]) as long the Schwarz lemma at the boundary (see [3]). Besides, we have following results, which can be offered as the boundary refinement of the Carathéodory inequality.

**Theorem 1.3.** Let \( f \) be a holomorphic function in the disc \( D \), \( f(0) = 0 \) and \( z_1, z_2, \ldots, z_n \) are zeros of the function \( f \) in the unit disc that are different from \( z = 0 \). Let \( \Re f \leq A \) for \( |z| < 1 \). Further assume that, for some \( z_0 \in \partial D \), \( f \) has an angular limit \( f(z_0) \) at \( z_0 \), \( \Re f(z_0) = A \). Then

\[
|f'(z_0)| \geq \frac{A^{n+1} + n \sum_{k=1}^{n} (n-2k+1)\Phi}{\prod_{k=1}^{n} (1+|z_k|)}.
\]

The inequality (1.7) is sharp, with equality for the function

\[
f(z) = \frac{2A z \prod_{k=1}^{n} \frac{z - z_k}{1 - z_k}}{1 + z \prod_{k=1}^{n} \frac{z - z_k}{1 - z_k}},
\]

where \( z_1, z_2, \ldots, z_n \) are negative real numbers.

**Proof.** Consider the function

\[
w(z) = \frac{f(z)}{f(z) - 2A}, \quad |z| < 1.
\]

Since \( \Re f(z_0) = A \), \( |w(z_0)| = 1 \) for \( z_0 \in \partial D \) and the function \( w(z) \) is defined in (1.8) satisfies the assumptions of Theorem 1.1, we obtain

\[
|w'(z_0)| \geq \frac{A^{n+1} + n \sum_{k=1}^{n} (n-2k+1)\Phi}{\prod_{k=1}^{n} (1+|z_k|)}
\]

and

\[
\frac{2A |f'(z_0)|}{|f(z_0) - 2A|^2} \geq \frac{A^{n+1} + n \sum_{k=1}^{n} (n-2k+1)\Phi}{\prod_{k=1}^{n} (1+|z_k|)}.
\]

Since \( |f(z_0) - 2A|^2 \geq |\Re (f(z_0) - 2A)|^2 = A^2 \), we obtain inequality (1.7) with an obvious equality case.

**Theorem 1.4.** Let \( f(z) = c_p z^p + c_{p+1} z^{p+1} + \cdots, \) \( c_p \neq 0, \) \( p \geq 1 \), be a holomorphic function in the disc \( D \) and \( z_1, z_2, \ldots, z_n \) are zeros of the function \( f \) in the
The inequality
\[|f'(z_0)| \geq \frac{A^{n+p} \prod_{k=1}^{n} (1+|z_k|)}{2} \]

is sharp, with equality for the function
\[f(z) = \frac{2Az^p \prod_{k=1}^{n} 1+\frac{z}{z_k}}{1+z^p \prod_{k=1}^{n} \frac{1+\frac{z}{z_k}}{1+\frac{z}{z_k}}},\]

where \(z_1, z_2, \ldots, z_n\) are negative real numbers.

**Proof.** The function \(w(z)\) is defined in (1.8) satisfies the assumptions of Theorem 1.2, we obtain
\[|w'(z_0)| \geq \frac{A^{n+p} \prod_{k=1}^{n} (1+|z_k|)}{2} \]

and
\[\frac{2A|f'(z_0)|}{|f(z_0)-2A|} \geq \frac{A^{n+p} \prod_{k=1}^{n} (1+|z_k|)}{2} \]

Since \(|f(z_0)-2A|^2 \geq |\Re (f(z_0)-2A)|^2 = A^2\), we obtain inequality (1.9) with an obvious equality case. \(\square\)

**Theorem 1.5.** Let \(f\) be a holomorphic function in the disc \(D\) and \(z_1, z_2, \ldots, z_n\) are zeros of the function \(f(z) - f(0)\) in the unit disc that are different from \(z = 0\). Let \(\Re f \leq A\) for \(|z| < 1\). Further assume that, for some \(z_0 \in \partial D\), \(f\) has an angular limit \(f(z_0)\) at \(z_0\), \(\Re f(z_0) = A\). Then
\[|f'(z_0)| \geq \left(\frac{A - \Re f(0)}{2}\right) \frac{A^{n+1+p} \prod_{k=1}^{n} (1+|z_k|)}{2} \]

The inequality (1.10) is sharp, with equality for the function
\[f(z) = f(0) + \frac{2(A - \Re f(0))}{1+z^p} \prod_{k=1}^{n} \frac{1+\frac{z}{z_k}}{1+\frac{z}{z_k}},\]

where \(z_1, z_2, \ldots, z_n\) are negative real numbers.

**Proof.** Introducing the notation \(\alpha = A - \Re f(z), \beta = A - \Re f(0)\).

If \(f\) is not identically constant, then \(\alpha > 0, \beta > 0, \Re (f(z) - f(0)) = \beta - \alpha < \beta\) and \(4\beta \Re (f(z) - f(0)) \leq 4\beta^2\). Therefore
\[|f(z) - f(0) - 2\beta|^2 = |f(z) - f(0)|^2 - 4\beta \Re (f(z) - f(0)) + 4\beta^2 > |f(z) - f(0)|^2.\]
The function
\[ \varphi(z) = \frac{f(z) - f(0)}{f(z) - f(0) - 2\beta} \]
is a holomorphic function in the disc \( D \), \( |\varphi(z)| < 1 \), \( \varphi(0) = 0 \) and \( |\varphi(z_0)| = 1 \) for \( z_0 \in \partial D \). The function \( \varphi(z) \) satisfies the assumptions of Theorem 1.1, we obtain
\[ |\varphi'(z_0)| \geq \frac{n+1+\sum_{k=1}^{n}(n-2k+1)\sum_{\epsilon_1=1}^{n-k+2}\ldots\sum_{\epsilon_k=1}^{n-k-1}(|z_{\epsilon_1}|^2|z_{\epsilon_2}|\ldots|z_{\epsilon_k}|)}{\prod_{k=1}^{1+|z_{\epsilon_k}|}}. \]

Therefore, we take
\[ \frac{2\beta |f'(z_0)|}{|f(z) - f(0) - 2\beta|^2} \geq \frac{n+1+\sum_{k=1}^{n}(n-2k+1)\Phi}{\prod_{k=1}^{1+|z_{\epsilon_k}|}}. \]

Since \( |f(z_0) - f(0)|^2 \geq |\Re(f(z_0) - f(0))|^2 = \beta^2 \), we obtain inequality (1.10) with an obvious equality case. \( \square \)

**Theorem 1.6.** Let \( f(z) = f(0) + c_1z^p + c_{p+1}z^{p+1} + \cdots \), \( c_p \neq 0 \), \( p \geq 1 \), be a holomorphic function in the disc \( D \) and \( z_1, z_2, \ldots, z_n \) are zeros of the function \( f(z) - f(0) \) in the unit disc that are different from \( z = 0 \). Let \( \Re f \leq A \) for \( |z| < 1 \). Further assume that, for some \( z_0 \in \partial D \), \( f \) has an angular limit \( f(z_0) \) at \( z_0 \), \( \Re f(z_0) = A \). Then

\[ |f'(z_0)| \geq \left( A - \Re f(0) \right) \frac{n+p+\sum_{k=1}^{n}(n-2k+p)\Phi}{2 \prod_{k=1}^{1+|z_{\epsilon_k}|}}. \]  

The inequality (1.11) is sharp, with equality for the function
\[ f(z) = f(0) + \frac{2(A - \Re f(0))z^p \prod_{k=1}^{\frac{1-z}{z-z_k}}}{1+z^p \prod_{k=1}^{\frac{1-z}{z-z_k}}} \]
where \( z_1, z_2, \ldots, z_n \) are negative real numbers.

**Proof.** The function \( \varphi(z) \) is defined in Theorem 1.5 satisfies the assumptions of Theorem 1.4, we obtain
\[ |\varphi'(z_0)| \geq \frac{n+p+\sum_{k=1}^{n}(n-2k+p)\Phi}{\prod_{k=1}^{1+|z_{\epsilon_k}|}}. \]

Therefore, we take
\[ \frac{2\beta |f'(z_0)|}{|f(z) - f(0) - 2\beta|^2} \geq \frac{n+p+\sum_{k=1}^{n}(n-2k+p)\Phi}{\prod_{k=1}^{1+|z_{\epsilon_k}|}}. \]

Since \( |f(z_0) - f(0)|^2 \geq |\Re(f(z_0) - f(0))|^2 = \beta^2 \), we obtain inequality (1.11) with an obvious equality case. \( \square \)
References


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