UNIQUENESS OF ENTIRE FUNCTIONS CONCERNING DIFFERENTIAL POLYNOMIALS

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Abstract. In this paper, we study the uniqueness of entire functions concerning differential polynomials and deficient value. The results extend and improve Theorem 2 in Yi [13].

1. Introduction and main results

Let $f$ be a nonconstant meromorphic function in the whole complex plane $\mathbb{C}$, we will use the standard notations of Nevanlinna’s value distribution theory such as $T(r,f)$, $N(r,f)$, $N'(r,f)$, $m(r,f)$ and so on, as found in [11]. In particular, we denote by $S(r,f)$ any function satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$, possibly outside a set of $r$ of finite linear measure. For $a \in \mathbb{C} \cup \{\infty\}$, we set $E(a,f) = \{z \mid f(z) - a = 0, \text{counting multiplicities}\}$ and $\bar{E}(a,f) = \{z \mid f(z) - a = 0, \text{ignoring multiplicities}\}$ respectively.

Let $f$ and $g$ be two nonconstant meromorphic functions, we say that $f$ and $g$ share the value $a$ CM (IM) provided that $E(a,f) = E(a,g)$ ($\bar{E}(a,f) = \bar{E}(a,g)$).

The quantity $\lambda(f) = \lim_{r \to \infty} \frac{\log^+ T(r,f)}{\log r}$ is called the order of $f(z)$. Also

$$\delta(a,f) = \lim_{r \to \infty} \frac{m(r, \frac{1}{f-a})}{T(r,f)} = 1 - \lim_{r \to \infty} \frac{N(r, \frac{1}{f-a})}{T(r,f)}$$

is called the deficiency of $a$ with respect to $f(z)$. If $\delta(a,f) > 0$, then the complex number $a$ is named a deficient value of $f(z)$.

In 1976, Yang [8] posed the following question:

What can be said about the relationship between two nonconstant entire functions $f$ and $g$ if $f$ and $g$ share the value $0$ CM and $f'$ and $g'$ share the value $1$ CM?

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The above problem has been studied by K. Shibazaki [7], Yi [12, 13], Yang-Yi [10], Hua [2], Muse-Reinders [6] and I. Lahiri [3]. And Yi [13] has proved the following theorem.

**Theorem 1.1** ([13, Theorem 2]). Let \( f \) and \( g \) be two nonconstant entire functions and let \( k \) be a nonnegative integer. If \( f \) and \( g \) share the value 0 CM, \( f^{(k)} \) and \( g^{(k)} \) share the value 1 CM and \( \delta(0, f) > \frac{1}{2} \), then \( f \equiv g \) unless \( f^{(k)} \cdot g^{(k)} \equiv 1 \).

Let \( h \) be a nonconstant meromorphic function. We denote by \( P(h) = h^{(k)} + a_1h^{(k-1)} + a_2h^{(k-2)} + \cdots + a_{k-1}h' + a_kh \) the differential polynomial of \( h \), where \( a_1, a_2, \ldots, a_k \) are finite complex numbers and \( k \) is a positive integer.

**Remark 1.2.** The following example shows that in Theorem 1.1 the functions \( f^{(k)} \) and \( g^{(k)} \) cannot be replaced by \( P(f) \) and \( P(g) \). Let \( f = \frac{1}{2}e^{-2z} \) and \( g = e^{-2z} \). Then \( f \) and \( g \) share the value 0 CM, \( f'' + 2f' \) and \( g'' + 2g' \) share the value 1 CM and \( \delta(0, f) > \frac{1}{2} \), but \( f \neq g \) and \( (f'' + 2f')(g'' + 2g') \neq 1 \).

In this paper, we shall prove the following general results which extend and improve Theorem 1.1.

**Theorem 1.3.** Let \( f \) and \( g \) be two nonconstant entire functions. Suppose that \( f \) and \( g \) share the value 0 CM, \( P(f) \) and \( P(g) \) share the value 1 CM and \( \delta(0, f) > \frac{1}{2} \). If \( \lambda(f) \neq 1 \), then \( f \equiv g \) unless \( P(f) \cdot P(g) \equiv 1 \).

**Theorem 1.4.** Let \( f \) and \( g \) be two nonconstant entire functions. Suppose \( f \) and \( g \) share the value 0 CM, \( P(f) \) and \( P(g) \) share the value 1 IM and \( \delta(0, f) > \frac{1}{2} \). If \( \lambda(f) \neq 1 \), then \( f \equiv g \) unless \( P(f) \cdot P(g) \equiv 1 \).

2. Some lemmas

**Lemma 2.1** ([5]). Let \( f \) be a nonconstant meromorphic function and let \( k \) be a nonnegative integer. Then

\[
T(r, P(f)) \leq T(r, f) + k\tilde{N}(r, f) + S(r, f).
\]

**Lemma 2.2.** Suppose that \( f(z) \) is a nonconstant meromorphic function in the complex plane and \( a(z) \) is a small function of \( f(z) \), that is, \( T(r, a) = S(r, f) \). If \( f(z) \) is not a polynomial, then

\[
N(r, \frac{1}{P(f) - P(a)}) \leq T(r, P(f)) - T(r, f) + N(r, \frac{1}{f - a}) + S(r, f)
\]

and

\[
N(r, \frac{1}{P(f) - P(a)}) \leq N(r, \frac{1}{f - a}) + k\tilde{N}(r, f) + S(r, f).
\]

**Proof.** By the Nevanlinna’s first fundamental theorem and the lemma of logarithmic derivatives, we have

\[
T(r, f) - N(r, \frac{1}{f - a}) = m(r, \frac{1}{f - a}) + S(r, f)
\]
We get (2) by transposition. And we obtain (3) combined with (1) and (2), which proves this lemma.

Next, we introduce some notations.

Let $F$ and $G$ be two nonconstant meromorphic functions such that $F$ and $G$ share the value $1$ IM. We denote by $\bar{N}_L(r, \frac{1}{F-1})$ the reduced counting function for zeros of both $F - 1$ and $G - 1$ about which $F - 1$ has larger multiplicity than $G - 1$, $N_{E}^{\downarrow}(r, \frac{1}{F-1})$ the counting function for common simple zeros of both $F - 1$ and $G - 1$, and $\bar{N}_E(r, \frac{1}{F-1})$ the reduced counting function for common multiple zeros of both $F - 1$ and $G - 1$. In the same way, we can define $\bar{N}_L(r, \frac{1}{G-1})$, $N_{E}^{\downarrow}(r, \frac{1}{G-1})$, and $\bar{N}_E(r, \frac{1}{G-1})$. Also we denote by $N_{1}(r, \frac{1}{F})$ the counting function for simple zeros of $F$, and $\bar{N}_L(r, \frac{1}{F})$ the reduced counting function for multiple zeros of $F$.

**Lemma 2.3.** Let $F$ and $G$ be two nonconstant meromorphic functions such that $F$ and $G$ share the value $1$ IM. Let

$$ H = \frac{F''}{F'} - \frac{2F'}{F-1} - \frac{G''}{G-1} + \frac{2G'}{G-1}. $$

If $H \neq 0$, then

$$ T(r, F) \leq N(r, \frac{1}{F}) + 2N(r, F) + N(r, \frac{1}{G}) + 2\bar{N}_E(r, \frac{1}{F-1}) $$
$$ + 2\bar{N}(r, G) + \bar{N}_L(r, \frac{1}{G-1}) + S(r, F) + S(r, G). $$

Proof. Let $z_0$ be a common simple zero of $F - 1$ and $G - 1$. By (4), we have $H(z_0) = 0$ and $m(r, H) = S(r, F) + S(r, G)$, then

$$ N_{E}^{\downarrow}(r, \frac{1}{F-1}) \leq N(r, \frac{1}{H}) \leq T(r, H) + O(1) $$

and

$$ N_{E}^{\downarrow}(r, \frac{1}{G-1}) \leq N(r, H) + S(r, F) + S(r, G). $$

By the Nevanlinna’s second fundamental theorem, we have

$$ T(r, F) + T(r, G) \leq \bar{N}(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{F-1}) + \bar{N}(r, F) - N_{0}(r, \frac{1}{F'}) $$
$$ + S(r, F) + \bar{N}(r, \frac{1}{G}) + \bar{N}(r, \frac{1}{G-1}) $$
$$ + \bar{N}(r, G) - N_{0}(r, \frac{1}{G}) + S(r, G). $$

We get (2) by transposition. And we obtain (3) combined with (1) and (2), which proves this lemma.
where \( N_0(r, 1/F) \) denotes the counting function corresponding to the zeros of \( F' \) that are not zeros of \( F \) and \( F - 1 \) and \( N_0(r, 1/G') \) denotes the counting function corresponding to the zeros of \( G' \) that are not zeros of \( G \) and \( G - 1 \). Since \( F \) and \( G \) share the value 1 IM, we get

\[
\bar{N}(r, \frac{1}{F - 1}) = N_{E}^{1}(r, \frac{1}{F - 1}) + \bar{N}_{L}(r, \frac{1}{F - 1}) + \bar{N}_{L}(r, \frac{1}{G - 1})
\]

\[
+ \bar{N}_{E}^{2}(r, \frac{1}{G - 1}) + S(r, F) + S(r, G)
\]

\[
= \bar{N}(r, \frac{1}{F - 1}) + S(r, F) + S(r, G).
\]

Then

\[
N(r, \frac{1}{F - 1}) + \bar{N}(r, \frac{1}{G - 1}) = N_{E}^{1}(r, \frac{1}{F - 1}) + \bar{N}_{L}(r, \frac{1}{F - 1})
\]

\[
+ \bar{N}_{L}(r, \frac{1}{G - 1}) + \bar{N}_{E}^{2}(r, \frac{1}{G - 1})
\]

\[
+ \bar{N}(r, \frac{1}{G - 1}) + S(r, F) + S(r, G)
\]

\[
\leq N_{E}^{1}(r, \frac{1}{F - 1}) + \bar{N}_{L}(r, \frac{1}{F - 1})
\]

\[
+ N(r, \frac{1}{G - 1}) + S(r, F) + S(r, G)
\]

\[
\leq N_{E}^{1}(r, \frac{1}{F - 1}) + \bar{N}_{L}(r, \frac{1}{F - 1})
\]

\[
+ \bar{N}(r, \frac{1}{G - 1}) - N_0(r, \frac{1}{F}) - N_0(r, \frac{1}{G}) + S(r, F) + S(r, G).
\]

From (7) and (8), we obtain

\[
T(r, F) \leq \bar{N}(r, \frac{1}{F}) + \bar{N}(r, F) + \bar{N}(r, \frac{1}{G}) + N_{E}^{1}(r, \frac{1}{F - 1})
\]

\[
+ \bar{N}_{L}(r, \frac{1}{F - 1}) - N_0(r, \frac{1}{F}) - N_0(r, \frac{1}{G}) + S(r, F) + S(r, G).
\]

By (4), we get

\[
N(r, H) \leq \bar{N}(r, \frac{1}{F}) + \bar{N}(r, F) + \bar{N}(r, \frac{1}{G}) + \bar{N}(r, \frac{1}{G - 1})
\]

\[
+ \bar{N}_{L}(r, \frac{1}{F - 1}) + \bar{N}_{L}(r, \frac{1}{G - 1}) + N_0(r, \frac{1}{F - 1}) + N_0(r, \frac{1}{G - 1})
\]

\[
+ S(r, F) + S(r, G).
\]

Combine (6), (9) and (10), we have

\[
T(r, F) \leq \bar{N}(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{F}) + 2 \bar{N}(r, F) + N_{E}^{1}(r, \frac{1}{G})
\]

\[
+ \bar{N}_{E}^{2}(r, \frac{1}{G}) + 2 \bar{N}(r, G) + 2 \bar{N}_{L}(r, \frac{1}{F - 1})
\]

\[
+ \bar{N}(r, \frac{1}{G - 1}) - N_0(r, \frac{1}{F}) - N_0(r, \frac{1}{G}) + S(r, F) + S(r, G).
\]
It is obvious that

\begin{align}
\hat{N}(r, \frac{1}{F}) + \hat{N}(r, \frac{1}{G}) & \leq N(r, \frac{1}{F}), \\
\hat{N}(r, \frac{1}{G}) + \hat{N}(r, \frac{1}{F}) & \leq N(r, \frac{1}{G}).
\end{align}

From (11), (12) and (13), we get (5), which completes the proof. \(\square\)

**Lemma 2.4 ([9]).** Suppose \(f_j (j = 1, 2, \ldots, m + 1)\) and \(g_j (j = 1, 2, \ldots, m)\) are entire functions satisfying the following conditions:

- \[\sum_{j=1}^{m} f_j(z) e^{g_j(z)} \equiv f_{m+1}(z);\]
- The order of \(f_j(z)\) is less than the order of \(e^{g_k(z)}\) for \(1 \leq j \leq m + 1, 1 \leq k \leq m\); And furthermore, the order of \(f_j(z)\) is less than the order of \(e^{g_l(z)} - g_k(z)\) for \(m \geq 2\) and \(1 \leq j \leq m + 1, 1 \leq l, k \leq m, l \neq k.\)

Then \(f_j \equiv 0 (j = 1, 2, \ldots, m + 1).\)

### 3. Proof of Theorem 1.4

We just prove Theorem 1.4, and the proof of Theorem 1.3 is similar. Next we consider two cases.

**Case 1.** Assume that \(P(f), P(g) \neq c\), where \(c\) is a finite complex constant.

Since \(f\) and \(g\) share the value 0 CM and \(P(f)\) and \(P(g)\) share the value 1 IM, by Milloux’s basic result we have

\[
T(r, f) \leq \hat{N}(r, f) + N(r, \frac{1}{f}) + \hat{N}(r, \frac{1}{P(f) - 1}) + S(r, f)
\]

\[
= N(r, \frac{1}{g}) + \hat{N}(r, \frac{1}{P(g) - 1}) + S(r, f)
\]

\[
\leq T(r, g) + T(r, P(g)) + S(r, f).
\]

By Lemma 2.1, we get

\[
(14) \quad T(r, f) \leq (k + 2)T(r, g) + S(r, f) + S(r, g).
\]

Similarly we can get

\[
(15) \quad T(r, g) \leq (k + 2)T(r, f) + S(r, f) + S(r, g).
\]

Then

\[
(16) \quad S(r, f) = S(r, g).
\]

Let \(F = P(f), G = P(g)\) and let \(H\) be defined by (4), then \(F\) and \(G\) share the value 1 IM. If \(H \neq 0\), then by Lemma 2.3 we have

\[
(17) \quad T(r, F) \leq N(r, \frac{1}{F}) + N(r, \frac{1}{G}) + 2\hat{N}(r, \frac{1}{F - 1})
\]
\[ + \tilde{N}_L(r, \frac{1}{G-1}) + S(r, F) + S(r, G). \]

From (3), we obtain
\[ (18) \quad \tilde{N}_L(r, \frac{1}{F-1}) \leq N(r, \frac{1}{F}) \leq N(r, \frac{1}{F}) + \tilde{N}(r, F) + S(r, F), \]
\[ \tilde{N}_L(r, \frac{1}{G-1}) \leq N(r, \frac{1}{G}) \leq N(r, \frac{1}{G}) + \tilde{N}(r, G) + S(r, G). \]

Substituting (18) into (17), we deduce that
\[ (19) \quad T(r, F) \leq 3N(r, \frac{1}{F}) + 2N(r, \frac{1}{G}) + S(r, F) + S(r, G). \]

By Lemma 2.2 and (19), we have
\[ (20) \quad T(r, P(f)) \leq T(r, P(f)) - T(r, f) + N(r, \frac{1}{F}) + 2N(r, \frac{1}{G}) + 2N(r, \frac{1}{g}) + S(r, f) + S(r, g). \]

Noting that \( f \) and \( g \) share the value 0 CM, by (16) and (20) we get
\[ T(r, f) \leq 5N(r, \frac{1}{F}) + S(r, f), \]
which contradicts the condition \( \delta(0, f) > \frac{4}{5} \). Thus \( H \equiv 0 \).

Solving this equation, we get
\[ (21) \quad F = \frac{AG + B}{CG + D} \quad (AD - BC \neq 0), \]
where \( A, B, C \) and \( D \) are finite complex constants. Next we consider three subcases.

**Subcase 1.1.** Assume that \( AC \neq 0 \). From (21), we know that \( \frac{A}{C} \) is a Picard exceptional value of \( F \). By the Nevanlinna’s second fundamental theorem, we have
\[ (22) \quad T(r, F) \leq N(r, \frac{1}{F}) + N(r, \frac{1}{F - \frac{A}{C}}) + N(r, F) + S(r, F) \]
\[ = N(r, \frac{1}{F}) + S(r, F). \]

From (3) and (22), we get
\[ T(r, P(f)) \leq T(r, P(f)) - T(r, f) + N(r, \frac{1}{F}) + S(r, f), \]
that is, \( T(r, f) \leq N(r, \frac{1}{F}) + S(r, f) \), which contradicts the condition \( \delta(0, f) > \frac{4}{5} \).

**Subcase 1.2.** Assume that \( A \neq 0 \) and \( C = 0 \). Then \( F = \frac{AG}{D} + \frac{B}{D} \). If \( B \neq 0 \), then \( N(r, \frac{1}{F}) = N(r, \frac{1}{f}) \). By the Nevanlinna’s second fundamental theorem, we have
\[ (23) \quad T(r, F) \leq N(r, \frac{1}{F}) + N(r, \frac{1}{F - \frac{A}{C}}) + N(r, F) + S(r, F) \]
\[ = N(r, \frac{1}{F}) + N(r, \frac{1}{G}) + S(r, F). \]
From Lemma 2.3 and (23), we obtain

\[
T(r, P(f)) \leq T(r, P(f)) - T(r, f) + N(r, \frac{1}{f}) + N(r, \frac{1}{g}) + S(r, f) + S(r, g).
\]

By (16) and (24), we have

\[
T(r, f) \leq N(r, \frac{1}{f}) + N(r, \frac{1}{g}) + S(r, f) = 2N(r, \frac{1}{f}) + S(r, f),
\]
a contradiction to the condition \(\delta(0, f) > \frac{4}{5}\). Thus \(B = 0\), that is, \(F = \frac{1}{g}\). If \(1\) is a Picard exceptional value of \(F\), then \(\frac{1}{4} = 1\). Otherwise, \(\frac{1}{4}\) is a Picard exceptional value of \(F\) that is different from 1, which contradicts the Deficiency Theorem [11]. Thus \(F \equiv G\). If \(1\) is not a Picard exceptional value of \(F\), then there is a complex number \(z_0\) such that \(F(z_0) = G(z_0) = 1\). Therefore, \(\frac{1}{4} = 1\), that is, \(F \equiv G\).

**Subcase 1.3.** Assume that \(A = 0\) and \(C \neq 0\). Proceeding as in the proof of subcase 1.2 we can get \(F \cdot G \equiv 1\).

In conclusion, we know that \(F \equiv G\) unless \(F \cdot G \equiv 1\). If \(F \cdot G \equiv 1\), that is, \(P(f) \cdot P(g) \equiv 1\), then the result of theorem 1.4 is true. If the former is established, that is, \(P(f - g) \equiv 0\), solving this equation (see [1, 4]) we get

\[
f - g = \sum_{j=1}^{m} p_j(z)e^{\alpha_j z},
\]

where \(m \leq k\) is a positive integer, \(\alpha_j\) \((j = 1, \ldots, m)\) are distinct complex constants and \(p_j(z)\) \((j = 1, \ldots, m)\) are polynomials. Next we prove that if \(\lambda(f) \neq 1\), then \(f \equiv g\). We distinguish two cases below.

**Case I.** Assume that \(\lambda(f) < 1\). By (14) and (15), we know that \(\lambda(f) = \lambda(g)\). Since \(f\) and \(g\) share the value 0 CM, we can get \(\frac{f}{g} = e^{h(z)}\), where \(h(z)\) is an entire function. Then

\[
\lambda(e^{h(z)}) = \lambda\left(\frac{f}{g}\right) \leq \max\{\lambda(f), \lambda(\frac{1}{g})\} < 1.
\]

Thus \(e^{h(z)} \equiv c_0\), where \(c_0\) is a finite complex constant. We obtain \(f \equiv c_0g\), then \(P(f) \equiv c_0P(g)\). By \(P(f) \equiv P(g)\), we can get \(c_0 = 1\), that is, \(f \equiv g\).

**Case II.** Assume that \(\lambda(f) > 1\). By the Weierstrass’s factorization theorem, we have

\[
f(z) = \pi(z)e^{l_1(z)}, \quad g(z) = \pi(z)e^{l_2(z)},
\]

where \(\pi(z)\) is canonical product formed with common zeros of \(f\) and \(g\) and \(l_1(z)\) and \(l_2(z)\) are entire functions.

If \(l_1 \equiv l_2\), then \(f \equiv g\). If \(l_1 \neq l_2\), since \(\lambda(\pi)\) is equal to \(\tau(f)\) which is the exponent of convergence of zeros of \(f(z)\) and \(\tau(f) \leq \tau(f - g) \leq \lambda(f - g)\), by
(25) we have
\[ \lambda(\pi) \leq \lambda(f - g) = \lambda(\sum_{j=1}^{m} p_j(z)e^{\alpha_j z}) \leq 1. \]

Since \( \lambda(f) = \lambda(g) > 1 \) and \( f - g = (e^{l_1(z)} - 1)g \), we can get that \( \lambda(e^{l_1(z)} - l_2(z)) > 1 \) and \( \lambda(e^{l_1(z)} - l_2(z)) > 1 \). By \( \pi(z)e^{l_1(z)} - \pi(z)e^{l_2(z)} = \sum_{j=1}^{m} p_j(z)e^{\alpha_j z} \) and Lemma 2.4 we know that \( \sum_{j=1}^{m} p_j(z)e^{\alpha_j z} \equiv 0 \) and \( \pi(z) \equiv 0 \). Then \( f(z) \equiv 0 \), a contradiction.

**Case 2.** Assume that \( P(f) \equiv c \), where \( c \) is a finite complex constant.

We can know that \( f \equiv c_1 + \sum_{j=1}^{m} q_j(z)e^{\beta_j z} \), where \( c_1 \) is finite complex constant, \( q_j \) \((j = 1, 2, \ldots, m)\) are polynomials and \( \beta_j \) \((j = 1, 2, \ldots, m)\) are distinct finite complex constants. Since \( \lambda(f) \neq 1 \), we get \( \lambda(f) < 1 \). Then \( f \equiv c_1 + \sum_{j=1}^{m} q_j(z) \), that is, \( f \) is a polynomial. Suppose the degree of \( f \) is \( n \). Then

\[ N(r, f) = n \log r \quad \text{and} \quad T(r, f) = n \log r + O(1). \]

Therefore, \( \delta(0, f) = 1 - \lim_{r \to \infty} \frac{N(r, f)}{T(r, f)} = 0 < \frac{4}{5} \), which is a contradiction.

This completes the proof of Theorem 1.4.

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