HESITANT FUZZY BI-IDEALS IN SEMIGROUPS

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Abstract. Characterizations of hesitant fuzzy left (right) ideals are considered. The notion of hesitant fuzzy (generalized) bi-ideals is introduced, and related properties are investigated. Relations between hesitant fuzzy generalized bi-ideals and hesitant fuzzy semigroups are discussed, and characterizations of (hesitant fuzzy) generalized bi-ideals and hesitant fuzzy bi-ideals are considered. Given a hesitant fuzzy set $H$ on a semigroup $S$, hesitant fuzzy (generalized) bi-ideals generated by $H$ are established.

1. Introduction

The hesitant fuzzy set which is introduced by Torra [5] is a useful generalization of the fuzzy set that is designed for situations in which it is difficult to determine the membership of an element to a set owing to ambiguity between a few different values. The hesitant fuzzy set permits the membership degree of an element to a set to be represented by a set of possible values between 0 and 1 (see [5] and [6]). The hesitant fuzzy set therefore provides a more accurate representation of peoples hesitancy in stating their preferences over objects than the fuzzy set or its classical extensions. Hesitant fuzzy set theory has been applied to several practical problems, primarily in the area of decision making (see [4], [6], [7], [8], [9], [10], [11]). Jun and Song applied the notion of hesitant fuzzy sets to MTL-algebras and EQ-algebras (see [2] and [3]). Jun et al. [1] applied notion of hesitant fuzzy sets to semigroups. They introduced the notions of hesitant fuzzy semigroups and hesitant fuzzy left (resp. right) ideals, and investigated several properties. They considered characterizations of (hesitant fuzzy) subsemigroups and (hesitant fuzzy) left (resp. right) ideals. Also they proved that the hesitant intersection of hesitant fuzzy left (resp. right) ideals (resp. hesitant fuzzy semigroups) is also hesitant fuzzy left (resp. right) ideals (resp. hesitant fuzzy semigroups). They introduced the concept...
of hesitant fuzzy quasi-ideals, and discussed characterizations of a quasi-ideal and a regular semigroup.

In this paper, we consider characterizations of hesitant fuzzy left (right) ideals. We introduce the notion of hesitant fuzzy (generalized) bi-ideals, and investigated related properties. We discuss relations between hesitant fuzzy generalized bi-ideals and hesitant fuzzy semigroups, and studied characterizations of (hesitant fuzzy) generalized bi-ideals and hesitant fuzzy bi-ideals. Given a hesitant fuzzy set \( H \) on a semigroup \( S \), we establish hesitant fuzzy (generalized) bi-ideals generated by \( H \).

2. Preliminaries

Let \( S \) be a semigroup. Let \( A \) and \( B \) be subsets of \( S \). Then the multiplication of \( A \) and \( B \) is defined as follows:

\[
AB = \{ab \in S \mid a \in A \text{ and } b \in B\}.
\]

A semigroup \( S \) is said to be regular if for every \( x \in S \) there exists \( a \in S \) such that \( xax = x \).

A semigroup \( S \) is said to be left (resp., right) zero if \( xy = x \) (resp., \( xy = y \)) for all \( x, y \in S \).

A nonempty subset \( A \) of \( S \) is called

(i) a subsemigroup of \( S \) if \( AA \subseteq A \), that is, \( ab \in A \) for all \( a, b \in A \),
(ii) a left (resp., right) ideal of \( S \) if \( SA \subseteq A \) (resp., \( AS \subseteq A \)), that is, \( xa \in A \) (resp., \( ax \in A \)) for all \( x \in S \) and \( a \in A \).
(iii) a two-sided ideal of \( S \) if it is both a left and a right ideal of \( S \).
(iv) a generalized bi-ideal of \( S \) if \( ASA \subseteq A \).
(v) a bi-ideal of \( S \) if it is both a semigroup and a generalized bi-ideal of \( S \).

Torra [5] defined hesitant fuzzy sets in terms of a function that returns a set of membership values for each element in the domain. We display the basic notions on hesitant fuzzy sets. For more details, we refer to references.

Let \( S \) be a reference set. Then we define \textit{hesitant fuzzy set} on \( S \) in terms of a function \( H \) that when applied to \( X \) returns a subset of \([0, 1]\).

For a hesitant fuzzy set \( H \) on \( S \) and \( x, y, z \in S \), we use the notations \( H_x := H(x) \), \( H^y_x := H(x) \cap H(y) \), and \( H^y_x[z] := H(x) \cap H(y) \cap H(z) \). It is clear that \( H^y_x = H^x_y \), and

\[
H_x = H_y \iff H_x \subseteq H_y, \ H_y \subseteq H_x
\]

for all \( x, y \in S \).

Let \( H \) and \( G \) be two hesitant fuzzy sets on \( S \). The hesitant union \( H \sqcup G \) and hesitant intersection \( H \sqcap G \) of \( H \) and \( G \) are defined to be hesitant fuzzy sets on \( S \) as follows:

\[
H \sqcup G : S \to \mathcal{P}([0, 1]), \ x \mapsto H_x \cup G_x
\]

(1)
and
\[ (2) \quad \mathcal{H} \cap \mathcal{G} : S \to \mathcal{P}([0, 1]), \; x \mapsto \mathcal{H}_x \cap \mathcal{G}_x, \]
respectively.

For any hesitant fuzzy sets \( \mathcal{H} \) and \( \mathcal{G} \) on \( S \), we define
\[ \mathcal{H} \sqsubseteq \mathcal{G} \text{ if } \mathcal{H}_x \subseteq \mathcal{G}_x \text{ for all } x \in S. \]
The hesitant fuzzy product of \( \mathcal{H} \) and \( \mathcal{G} \) is defined to be the hesitant fuzzy set \( \mathcal{H} \ast \mathcal{G} \) on \( S \) which is given by
\[ (\mathcal{H} \ast \mathcal{G})_x = \begin{cases} \bigcup \mathcal{H}_y \cap \mathcal{G}_z & \text{if } \exists y, z \in S \text{ such that } x = yz \\ \emptyset & \text{otherwise.} \end{cases} \]
For a hesitant fuzzy set \( \mathcal{H} \) on \( S \) and a subset \( \varepsilon \) of \([0, 1]\), the set
\[ S(\mathcal{H}; \varepsilon) := \{ x \in S \mid \varepsilon \subseteq \mathcal{H}_x \}, \]
is called the hesitant level set of \( \mathcal{H} \).

### 3. Hesitant fuzzy ideals

In what follows, we take a semigroup \( S \) as a reference set unless otherwise specified.

**Definition 3.1** ([1]). A hesitant fuzzy set \( \mathcal{H} \) on \( S \) is called a *hesitant fuzzy semigroup* on \( S \) if it satisfies:
\[ (3) \quad (\forall x, y \in S) (\mathcal{H}_y^x \subseteq \mathcal{H}_{xy}). \]

**Definition 3.2** ([1]). A hesitant fuzzy set \( \mathcal{H} \) on \( S \) is called a *hesitant fuzzy left* (resp., *right*) *ideal* on \( S \) if it satisfies:
\[ (4) \quad (\forall x, y \in S) (\mathcal{H}_{xy} \supseteq \mathcal{H}_y \text{ (resp., } \mathcal{H}_{xy} \supseteq \mathcal{H}_x)). \]

If a hesitant fuzzy set \( \mathcal{H} \) on \( S \) is both a hesitant fuzzy left ideal and a hesitant fuzzy right ideal on \( S \), we say that \( \mathcal{H} \) is a hesitant fuzzy two-sided ideal on \( S \).

Obviously, every hesitant fuzzy left (resp., right) ideal on \( S \) is a hesitant fuzzy semigroup on \( S \). But the converse is not true in general (see [1]).

**Proposition 3.1.** Let \( \mathcal{H} \) be a hesitant fuzzy left ideal on \( S \). If \( G \) is a left zero subsemigroup of \( S \), then the restriction of \( \mathcal{H} \) to \( G \) is constant, that is, \( \mathcal{H}_x = \mathcal{H}_y \) for all \( x, y \in G \).

**Proof.** Let \( x, y \in G \). Then \( xy = x \) and \( yx = y \). Thus
\[ \mathcal{H}_x = \mathcal{H}_{xy} \supseteq \mathcal{H}_y = \mathcal{H}_{yx} \supseteq \mathcal{H}_x, \]
and so \( \mathcal{H}_x = \mathcal{H}_y \) for all \( x, y \in G \). \( \square \)

Similarly, we have the following proposition.
Proposition 3.2. Let \( \mathcal{H} \) be a hesitant fuzzy right ideal on \( S \). If \( G \) is a right zero subsemigroup of \( S \), then the restriction of \( \mathcal{H} \) to \( G \) is constant, that is, \( \mathcal{H}_x = \mathcal{H}_y \) for all \( x, y \in G \).

Proposition 3.3. Let \( \mathcal{H} \) be a hesitant fuzzy left (resp., right) ideal on \( S \). If the set
\[
I(S) := \{ x \in S \mid x \text{ is an idempotent element of } S \}
\]
forms a left (resp., right) zero subsemigroup of \( S \), then the restriction of \( \mathcal{H} \) to \( I(S) \) is constant, that is, \( \mathcal{H}_x = \mathcal{H}_y \) for all \( x, y \in I(S) \).

Proof. Assume that the set \( I(S) \) is a left zero subsemigroup of \( S \). Let \( u, v \in I(S) \). Then \( uv = u \) and \( vu = v \). Hence
\[
\mathcal{H}_u = \mathcal{H}_{uv} \supseteq \mathcal{H}_v = \mathcal{H}_{vu} \supseteq \mathcal{H}_u
\]
and thus \( \mathcal{H}_u = \mathcal{H}_v \). \( \square \)

For a nonempty subset \( A \) of \( S \) and \( \varepsilon, \delta \in \mathcal{P}([0,1]) \) with \( \varepsilon \supseteq \delta \), define a map \( [\chi_A^{(\varepsilon,\delta)}] \) as follows:
\[
[\chi_A^{(\varepsilon,\delta)}] : S \to \mathcal{P}([0,1]), \quad x \mapsto \left\{ \begin{array}{ll} \varepsilon & \text{if } x \in A, \\ \delta & \text{otherwise.} \end{array} \right.
\]
Then \( [\chi_A^{(\varepsilon,\delta)}] \) is a hesitant fuzzy set on \( S \), which is called the \((\varepsilon, \delta)\)-characteristic hesitant fuzzy set. The hesitant fuzzy set \( [\chi_S^{(\varepsilon,\delta)}] \) is called the \((\varepsilon, \delta)\)-identity hesitant fuzzy set on \( S \). The \((\varepsilon, \delta)\)-characteristic hesitant fuzzy set with \( \varepsilon = [0, 1] \) and \( \delta = \emptyset \) is called the characteristic hesitant fuzzy set, and is denoted by \( [\chi_A] \). The \((\varepsilon, \delta)\)-identity hesitant fuzzy set with \( \varepsilon = [0, 1] \) and \( \delta = \emptyset \) is called the identity hesitant fuzzy set, and is denoted by \( [\chi_S] \).

Lemma 3.4. Let \( [\chi_A^{(\varepsilon,\delta)}] \) and \( [\chi_B^{(\varepsilon,\delta)}] \) be \((\varepsilon, \delta)\)-characteristic hesitant fuzzy sets on \( S \) where \( A \) and \( B \) are nonempty subsets of \( S \). Then the following properties hold.

1. \( [\chi_A^{(\varepsilon,\delta)}] \cap [\chi_B^{(\varepsilon,\delta)}] = [\chi_{A\cap B}^{(\varepsilon,\delta)}] \).
2. \( [\chi_A^{(\varepsilon,\delta)}] \cap [\chi_B^{(\varepsilon,\delta)}] ] = [\chi_{A\cap B}^{(\varepsilon,\delta)}] \).

Proof. (1) Let \( x \in S \). If \( x \in A \cap B \), then \( x \in A \) and \( x \in B \). Thus we have
\[
([\chi_A^{(\varepsilon,\delta)}] \cap [\chi_B^{(\varepsilon,\delta)}]) \cap = [\chi_A^{(\varepsilon,\delta)}] \cap [\chi_B^{(\varepsilon,\delta)}] \cap = [\chi_{A\cap B}^{(\varepsilon,\delta)}] \cap .
\]
If \( x \notin A \cap B \), then \( x \notin A \) or \( x \notin B \). Hence we have
\[
([\chi_A^{(\varepsilon,\delta)}] \cap [\chi_B^{(\varepsilon,\delta)}]) \cap = [\chi_A^{(\varepsilon,\delta)}] \cap [\chi_B^{(\varepsilon,\delta)}] \cap = [\chi_{A\cap B}^{(\varepsilon,\delta)}] \cap .
\]
Therefore \( [\chi_A^{(\varepsilon,\delta)}] \cap [\chi_B^{(\varepsilon,\delta)}] = [\chi_{A\cap B}^{(\varepsilon,\delta)}] \).
(2) For any \( x \in S \), if \( x \in AB \), then \( x = ab \) for some \( a \in A \) and \( b \in B \). Thus we have
\[
\left( [\chi_A^{(\varepsilon, \delta)}] \tilde{\circ} [\chi_B^{(\varepsilon, \delta)}] \right)_x = \bigcup_{x \in yz} \left\{ [\chi_A^{(\varepsilon, \delta)}]_y \cap [\chi_B^{(\varepsilon, \delta)}]_z \right\}
\]
\[
\supseteq [\chi_A^{(\varepsilon, \delta)}]_a \cap [\chi_B^{(\varepsilon, \delta)}]_b = \varepsilon,
\]
and so \( \left( [\chi_A^{(\varepsilon, \delta)}] \tilde{\circ} [\chi_B^{(\varepsilon, \delta)}] \right)_x = \varepsilon \). Since \( x \in AB \), we get \( [\chi_{AB}^{(\varepsilon, \delta)}]_x = \varepsilon \). Suppose \( x \notin AB \). Then \( x \neq ab \) for all \( a \in A \) and \( b \in B \). If \( x = yz \) for some \( y, z \in S \), then \( y \notin A \) or \( z \notin B \). Hence
\[
\left( [\chi_A^{(\varepsilon, \delta)}] \tilde{\circ} [\chi_B^{(\varepsilon, \delta)}] \right)_x = \bigcup_{x \in yz} \left\{ [\chi_A^{(\varepsilon, \delta)}]_y \cap [\chi_B^{(\varepsilon, \delta)}]_z \right\} = \delta = [\chi_{AB}^{(\varepsilon, \delta)}]_x.
\]
If \( x \neq yz \) for all \( x, y \in S \), then
\[
\left( [\chi_A^{(\varepsilon, \delta)}] \tilde{\circ} [\chi_B^{(\varepsilon, \delta)}] \right)_x = \delta = [\chi_{AB}^{(\varepsilon, \delta)}]_x.
\]
In any case, we have \( [\chi_A^{(\varepsilon, \delta)}] \tilde{\circ} [\chi_B^{(\varepsilon, \delta)}] = [\chi_{AB}^{(\varepsilon, \delta)}] \). \qed

**Theorem 3.5.** For the \((\varepsilon, \delta)\)-identity hesitant fuzzy set \([\chi_S^{(\varepsilon, \delta)}]\), let \( \mathcal{H} \) be a hesitant fuzzy left ideal on \( S \) such that \( \mathcal{H}_x \subseteq \varepsilon \) for all \( x \in S \). Then the following assertions are equivalent:

1. \( \mathcal{H} \) is a hesitant fuzzy left ideal on \( S \).
2. \([\chi_S^{(\varepsilon, \delta)}] \tilde{\circ} \mathcal{H} \subseteq \mathcal{H} \).

**Proof.** Suppose that \( \mathcal{H} \) is a hesitant fuzzy left ideal on \( S \). Let \( x \in S \). If \( x = yz \) for some \( y, z \in S \), then
\[
\left( [\chi_S^{(\varepsilon, \delta)}] \tilde{\circ} \mathcal{H} \right)_x = \bigcup_{x \in yz} \left\{ [\chi_S^{(\varepsilon, \delta)}]_y \cap \mathcal{H}_z \right\}
\]
\[
\subseteq \bigcup_{x \in yz} \{ \varepsilon \cap \mathcal{H}_y \} = \mathcal{H}_x.
\]
Otherwise, we have \( \left( [\chi_S^{(\varepsilon, \delta)}] \tilde{\circ} \mathcal{H} \right)_x = \emptyset \subseteq \mathcal{H}_x \). Therefore \( [\chi_S^{(\varepsilon, \delta)}] \tilde{\circ} \mathcal{H} \subseteq \mathcal{H} \).

Conversely, assume that \([\chi_S^{(\varepsilon, \delta)}] \tilde{\circ} \mathcal{H} \subseteq \mathcal{H} \). For any \( x, y \in S \), we have
\[
\mathcal{H}_{xy} \supseteq \left( [\chi_S^{(\varepsilon, \delta)}] \tilde{\circ} \mathcal{H} \right)_{xy} \supseteq [\chi_S^{(\varepsilon, \delta)}]_x \cap \mathcal{H}_y = \varepsilon \cap \mathcal{H}_y = \mathcal{H}_y.
\]
Hence \( \mathcal{H} \) is a hesitant fuzzy left ideal on \( S \). \qed

Similarly, we have the following theorem.

**Theorem 3.6.** For the \((\varepsilon, \delta)\)-identity hesitant fuzzy set \([\chi_S^{(\varepsilon, \delta)}]\), let \( \mathcal{H} \) be a hesitant fuzzy set on \( S \) such that \( \mathcal{H}_x \subseteq \varepsilon \) for all \( x \in S \). Then the following assertions are equivalent:

1. \( \mathcal{H} \) is a hesitant fuzzy right ideal on \( S \).
Corollary 3.7. For the \((\varepsilon, \delta)\)-identity hesitant fuzzy set \([\chi_S^{(\varepsilon, \delta)}]\), let \(\mathcal{H}\) be a hesitant fuzzy set on \(S\) such that \(\mathcal{H}_x \subseteq \varepsilon\) for all \(x \in S\). Then the following assertions are equivalent:

1. \(\mathcal{H}\) is a hesitant fuzzy two-sided ideal on \(S\).
2. \([\chi_S^{(\varepsilon, \delta)}] \circ \mathcal{H} \subseteq \mathcal{H}\) and \(\mathcal{H} \circ [\chi_S^{(\varepsilon, \delta)}] \subseteq \mathcal{H}\).

Note that the hesitant intersection of hesitant fuzzy left (right, two-sided) ideals on \(S\) is a hesitant fuzzy left (right, two-sided) ideal on \(S\). In fact, the hesitant intersection of hesitant fuzzy left (right, two-sided) ideals containing a hesitant fuzzy set \(\mathcal{H}\) on \(S\) is the smallest hesitant fuzzy left (right, two-sided) ideal on \(S\).

For any hesitant fuzzy set \(\mathcal{H}\) on \(S\), the smallest hesitant fuzzy left (resp., right, two-sided) ideal on \(S\) containing \(\mathcal{H}\) is called the hesitant fuzzy left (resp., right, two-sided) ideal on \(S\) generated by \(\mathcal{H}\), and is denoted by \((\mathcal{H}_l)\) (resp., \((\mathcal{H}_r), (\mathcal{H}_2))\).

Theorem 3.8. Let \(S\) be a monoid with identity \(e\). Given a hesitant fuzzy set \(\mathcal{H}\) on \(S\), the smallest hesitant fuzzy left ideal \((\mathcal{H}_l)\) on \(S\) containing \(\mathcal{H}\) is given as follows:

\[
(\mathcal{H}_l) : S \to \mathcal{P}([0, 1]), \ a \mapsto \bigcup \{\mathcal{H}_y \mid a = xy, \ x, y \in S\}.
\]

Proof. Let \(\mathcal{G}\) be a hesitant fuzzy set on \(S\) given as follows:

\[
\mathcal{G}_a := \bigcup \{\mathcal{H}_y \mid a = xy, \ x, y \in S\}
\]

for any \(a \in S\). Since \(a = ea\), we have \(\mathcal{G}_a \supseteq \mathcal{H}_a\), and so \(\mathcal{H} \subseteq \mathcal{G}\). For all \(x, y \in S\), we have

\[
\mathcal{G}_{xy} = \bigcup \{\mathcal{H}_{x_2} \mid xy = x_1x_2, \ x_1, x_2 \in S\} \\
\supseteq \bigcup \{\mathcal{H}_{z_2} \mid xy = (xz_1)z_2, \ y = z_1z_2, \ z_1, z_2 \in S\} \\
\supseteq \bigcup \{\mathcal{H}_{z_2} \mid y = z_1z_2, \ z_1, z_2 \in S\} = \mathcal{G}_y.
\]

Thus \(\mathcal{G}\) is a hesitant fuzzy left ideal on \(S\). Now let \(\mathcal{A}\) be a hesitant fuzzy left ideal on \(S\) such that \(\mathcal{H} \subseteq \mathcal{A}\). Then \(\mathcal{H}_a \subseteq \mathcal{A}_a\) for all \(a \in S\) and

\[
\mathcal{G}_a = \bigcup \{\mathcal{H}_{x_2} \mid a = x_1x_2, \ x_1, x_2 \in S\} \\
\supseteq \bigcup \{\mathcal{A}_{x_2} \mid a = x_1x_2, \ x_1, x_2 \in S\} \\
\supseteq \bigcup \{\mathcal{A}_{x_2} \mid a = x_1x_2, \ x_1, x_2 \in S\} = \mathcal{A}_a,
\]

which implies that \(\mathcal{G} \subseteq \mathcal{A}\). Therefore \((\mathcal{H}_l) = \mathcal{G}\). \(\Box\)
Similarly, we have the following theorem.

**Theorem 3.9.** Let $S$ be a monoid with identity $e$. Given a hesitant fuzzy set $H$ on $S$, the smallest hesitant fuzzy right ideal $(H_r)$ on $S$ containing $H$ is given as follows:

$$(8) \quad (H_r) : S \to \mathcal{P}([0,1]), \quad a \mapsto \bigcup \{H_x \mid a = xy, \ x, y \in S\}.$$ 

4. Hesitant fuzzy bi-ideals

**Definition 4.1.** A hesitant fuzzy set $H$ on $S$ is called a hesitant fuzzy generalized bi-ideal on $S$ if it satisfies:

$$(9) \quad (\forall x, y, z \in S) (H_{xyz} \supseteq H_z^x).$$

We know that any hesitant fuzzy generalized bi-ideal may not be a hesitant fuzzy semigroup by the following example.

**Example 4.2.** Let $S = \{a, b, c, d\}$ be a semigroup with the Cayley table which is appeared in Table 1.

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Let $H$ be a hesitant fuzzy set on $S$ defined as follows:

$$H : S \to \mathcal{P}([0,1]), \quad x \mapsto \begin{cases} 
(0.2, 0.8) & \text{if } x = a, \\
(0.4) \cup (0.5, 0.7) & \text{if } x \in \{b, d\}, \\
(0.3, 0.7) & \text{if } x = c.
\end{cases}$$

Then $H$ is a hesitant fuzzy generalized bi-ideal on $S$. But it is not a hesitant fuzzy semigroup on $S$ since $H_{cc} = [0.3, 0.7] \not\subseteq (0.4) \cup (0.5, 0.7) = H_b = H_{bc}.

If a hesitant fuzzy set $H$ on $S$ is both a hesitant fuzzy semigroup and a hesitant fuzzy generalized bi-ideal on $S$, then we say that $H$ is a hesitant fuzzy bi-ideal on $S$.

We provide a condition for a hesitant fuzzy generalized bi-ideal to be a hesitant fuzzy semigroup.

**Theorem 4.1.** In a regular semigroup $S$, every hesitant fuzzy generalized bi-ideal is a hesitant fuzzy semigroup, and so a hesitant fuzzy bi-ideal.
Proof. Let $\mathcal{H}$ be a hesitant fuzzy generalized bi-ideal on $S$ and let $x$ and $y$ be any elements of $S$. Then there exists $a \in S$ such that $y = yax$, and so

$$H_{xy} = H_{x(yax)} = H_{x(ya)y} \supseteq H_y^x.$$ 

Therefore $\mathcal{H}$ is a hesitant fuzzy semigroup on $S$.

We consider characterizations of a hesitant fuzzy (generalized) bi-ideal.

Lemma 4.2. Let $A$ be a nonempty subset of $S$. Then $A$ is a generalized bi-ideal of $S$ if and only if the $(\varepsilon, \delta)$-characteristic hesitant fuzzy set $[\chi_A^{(\varepsilon, \delta)}]$ on $S$ is a hesitant fuzzy generalized bi-ideal on $S$ for any $\varepsilon, \delta \in \mathcal{P}([0, 1])$ with $\varepsilon \supseteq \delta$.

Proof. Assume that $A$ is a generalized bi-ideal of $S$. Let $\varepsilon, \delta \in \mathcal{P}([0, 1])$ with $\varepsilon \supseteq \delta$ and $x, y, z \in S$. If $x, z \in A$, then $[\chi_A^{(\varepsilon, \delta)}]_{x+y} = \varepsilon = [\chi_A^{(\varepsilon, \delta)}]_z$ and $xyz \in ASA \subseteq A$. Hence

$$[\chi_A^{(\varepsilon, \delta)}]_{x+y} = \varepsilon = [\chi_A^{(\varepsilon, \delta)}]_z.$$

If $x \notin A$ or $z \notin A$, then $[\chi_A^{(\varepsilon, \delta)}]_{x+y} = \delta$ or $[\chi_A^{(\varepsilon, \delta)}]_z = \delta$. Hence

$$[\chi_A^{(\varepsilon, \delta)}]_{x+y} \supseteq \delta = [\chi_A^{(\varepsilon, \delta)}]_z.$$

Therefore $[\chi_A^{(\varepsilon, \delta)}]$ is a hesitant fuzzy generalized bi-ideal on $S$ for any $\varepsilon, \delta \in \mathcal{P}([0, 1])$ with $\varepsilon \supseteq \delta$.

Conversely, suppose that the $(\varepsilon, \delta)$-characteristic hesitant fuzzy set $[\chi_A^{(\varepsilon, \delta)}]$ on $S$ is a hesitant fuzzy generalized bi-ideal on $S$ for any $\varepsilon, \delta \in \mathcal{P}([0, 1])$ with $\varepsilon \supseteq \delta$. Let $a$ be any element of $ASA$. Then $a = xyz$ for some $x, z \in A$ and $y \in S$. Then

$$[\chi_A^{(\varepsilon, \delta)}]_a = [\chi_A^{(\varepsilon, \delta)}]_{x+y} \supseteq [\chi_A^{(\varepsilon, \delta)}]_z = \varepsilon,$$

and so $[\chi_A^{(\varepsilon, \delta)}]_a = \varepsilon$. Thus $a \in A$, which shows that $ASA \subseteq A$. Therefore $A$ is a generalized bi-ideal of $S$.

Theorem 4.3. A hesitant fuzzy set $\mathcal{H}$ on $S$ is a hesitant fuzzy generalized bi-ideal on $S$ if and only if the nonempty hesitant level set $S(\mathcal{H}; \varepsilon)$ of $\mathcal{H}$ is a generalized bi-ideal of $S$ for all $\varepsilon \in \mathcal{P}([0, 1])$.

Proof. Assume that $\mathcal{H}$ is a hesitant fuzzy generalized bi-ideal on $S$. Let $\varepsilon \in \mathcal{P}([0, 1])$ be such that $S(\mathcal{H}; \varepsilon) \neq \emptyset$. Let $a \in S$ and $x, y \in S(\mathcal{H}; \varepsilon)$. Then $\mathcal{H}_x \supseteq \varepsilon$ and $\mathcal{H}_y \supseteq \varepsilon$. It follows from (9) that

$$\mathcal{H}_{x+y} \supseteq \mathcal{H}_x^y \supseteq \varepsilon,$$

and that $xy \in S(\mathcal{H}; \varepsilon)$. Thus $S(\mathcal{H}; \varepsilon)$ is a generalized bi-ideal of $S$.

Conversely, suppose that the nonempty hesitant level set $S(\mathcal{H}; \varepsilon)$ of $\mathcal{H}$ is a generalized bi-ideal of $S$ for all $\varepsilon \in \mathcal{P}([0, 1])$. Let $x, y, z \in S$ be such that $\mathcal{H}_x = \varepsilon_x$ and $\mathcal{H}_z = \varepsilon_z$. Taking $\varepsilon = \varepsilon_x \cap \varepsilon_z$ implies that $x, z \in S(\mathcal{H}; \varepsilon)$. Hence $xyz \in S(\mathcal{H}; \varepsilon)$, and so

$$\mathcal{H}_{xyz} \supseteq \varepsilon = \varepsilon_x \cap \varepsilon_z = \mathcal{H}_x^z.$$
Therefore \( H \) is a hesitant fuzzy generalized bi-ideal on \( S \). \( \Box \)

**Lemma 4.4** ([1]). A hesitant fuzzy set \( H \) on \( S \) is a hesitant fuzzy semigroup on \( S \) if and only if the nonempty hesitant level set \( S(H; \varepsilon) \) of \( H \) is a semigroup of \( S \) for all \( \varepsilon \in \mathcal{P}([0, 1]) \).

Combining Theorem 4.3 and Lemma 4.4, we have the following characterization of a hesitant fuzzy bi-ideal.

**Theorem 4.5.** A hesitant fuzzy set \( H \) on \( S \) is a hesitant fuzzy bi-ideal on \( S \) if and only if the nonempty hesitant level set \( S(H; \varepsilon) \) of \( H \) is a bi-ideal of \( S \) for all \( \varepsilon \in \mathcal{P}([0, 1]) \).

**Theorem 4.6.** For the identity hesitant fuzzy set \([\chi_S]\) and a hesitant fuzzy set \( H \) on \( S \), the following are equivalent:

1. \( H \) is a hesitant fuzzy generalized bi-ideal on \( S \).
2. \( \mathcal{H}[\chi_S] \circ H \subseteq H \).

**Proof.** Assume that \( H \) is a hesitant fuzzy generalized bi-ideal on \( S \). Let \( a \) be any element of \( S \). If \( (\mathcal{H}[\chi_S] \circ H)_a = \emptyset \), then it is clear that \( \mathcal{H}[\chi_S] \circ H \subseteq H \).

Otherwise, there exist \( x, y, u, v \in S \) such that \( a = xy \) and \( x = uv \). Since \( H \) is a hesitant fuzzy generalized bi-ideal on \( S \), it follows from (9) that

\[
(\mathcal{H}[\chi_S] \circ H)_a = \bigcup_{a = xy} ((\mathcal{H}[\chi_S] \circ H)_x \cap H_y)
\]

and that

\[
(\mathcal{H}[\chi_S] \circ H)_a = \bigcup_{a = xy} \left( \bigcup_{x = uv} (H_u \cap [\chi_S]_v) \cap H_y \right)
\]

\[
= \bigcup_{a = xy} \left( \bigcup_{x = uv} (H_u \cap [0, 1]) \cap H_y \right)
\]

\[
= \bigcup_{a = uv} H_u \subseteq \bigcup_{a = uv} H_{uvy} = H_a.
\]

Therefore \( \mathcal{H}[\chi_S] \circ H \subseteq H \).

Conversely, suppose that \( \mathcal{H}[\chi_S] \circ H \subseteq H \). For any \( x, y, z \in S \), let \( a = xyz \).

Then

\[
H_{xyz} = H_a \supseteq (\mathcal{H}[\chi_S] \circ H)_a = \bigcup_{a = bc} ((\mathcal{H}[\chi_S])_b \cap H_c)
\]

\[
\supseteq (\mathcal{H}[\chi_S])_{xy} \cap H_z = \left( \bigcup_{xy = uv} (H_u \cap [\chi_S]_v) \right) \cap H_z
\]

\[
\supseteq (H_x \cap [\chi_S]_u) \cap H_z = (H_x \cap [0, 1]) \cap H_z = H_z^x.
\]

Therefore \( H \) is a hesitant fuzzy generalized bi-ideal on \( S \). \( \Box \)
**Theorem 4.7.** Let \( H \) be a hesitant fuzzy semigroup on \( S \). Then \( H \) is a hesitant fuzzy bi-ideal on \( S \) if and only if \( H \circ [\chi_S] \subseteq H \) where \([\chi_S]\) is the identity hesitant fuzzy set on \( S \).

**Proof.** It is the same as the proof of Theorem 4.6. \( \square \)

**Theorem 4.8.** If \( H \) and \( G \) are hesitant fuzzy generalized bi-ideals on \( S \), then so is the hesitant intersection \( H \cap G \).

**Proof.** For any \( a, b, x \in S \), we have
\[
(H \cap G)_{axb} = H_{axb} \cap G_{axb} \supseteq H_a^b \cap G_a^b = (H \cap G)_a^b.
\]
Thus \( H \cap G \) is a hesitant fuzzy generalized bi-ideal on \( S \). \( \square \)

**Theorem 4.9.** If \( H \) and \( G \) are hesitant fuzzy bi-ideals on \( S \), then so is the hesitant intersection \( H \cap G \).

**Proof.** It is the same as the proof of Theorem 4.8. \( \square \)

**Theorem 4.10.** If \( S \) is a group, then every hesitant fuzzy generalized bi-ideal is a constant function.

**Proof.** Let \( S \) be a group with identity \( e \) and let \( H \) be a hesitant fuzzy generalized bi-ideal on \( S \). For any \( x, y, z \in S \), we get
\[
H_x = H_e x e \supseteq H^e = H = H_e
\]
and so \( H_x = H_e \). Therefore \( H \) is a constant function. \( \square \)

For any hesitant fuzzy set \( H \) on \( S \), the smallest hesitant fuzzy (generalized) bi-ideal on \( S \) containing \( H \) is called a hesitant fuzzy (generalized) bi-ideal on \( S \) generated by \( H \), and is denoted by \( (H)_b \).

**Theorem 4.11.** Let \( S \) be a monoid with identity \( e \). Given a hesitant fuzzy set \( H \) on \( S \) with \( H_x \subseteq H_e \) for all \( x \in S \), the smallest hesitant fuzzy (generalized) bi-ideal \( (H_b) \) on \( S \) containing \( H \) is given as follows:
\[
(H_b) : S \rightarrow \mathcal{P}([0, 1]), \ a \mapsto \bigcup \{H^a_3 \mid a = x_1 x_2 x_3, x_1, x_2, x_3 \in S\}.
\]

**Proof.** Let \( G \) be a hesitant fuzzy set on \( S \) defined by
\[
G_a = \bigcup \{H^a_3 \mid a = x_1 x_2 x_3, x_1, x_2, x_3 \in S\}
\]
for all \( a \in S \). Since \( a = eea \), it follows from hypothesis that \( G_a \supseteq H^a_e = H_a \) and that \( H \subseteq G \). For any \( x, y, z \in S \), we get
\[
G_x = \bigcup \{H^a_3 \mid x = x_1 x_2 x_3, x_1, x_2, x_3 \in S\},
\]
\[
G_z = \bigcup \{H^a_3 \mid z = z_1 z_2 z_3, z_1, z_2, z_3 \in S\}.
\]
and
\[ G_{xyz} = \bigcup \left\{ H_{u_1}^{u_3} \mid xyz = u_1u_2u_3, u_1, u_2, u_3 \in S \right\} \]
\[ \supseteq \bigcup \left\{ H_{x_1}^{z_3} \mid xyz = x_1(x_2x_3yz_1z_2)z_3, x = x_1x_2x_3, z = z_1z_2z_3 \right\}. \]

Since
\[ G_{x} = \bigcup \left\{ H_{x_1}^{z_3} \cap H_{x_1}^{z_3} \mid x = x_1x_2x_3, z = z_1z_2z_3 \in S \right\}, \]
we have \( G_y \subseteq G_{xyz} \). Let \( y = e \). Then \( G_y \subseteq G_z \), for all \( x, z \in S \). Hence \( G \) is a hesitant fuzzy (generalized) bi-ideal on \( S \). Let \( A \) be a hesitant fuzzy (generalized) bi-ideal on \( S \) such that \( H \subseteq A \). For any \( a \in S \), we have
\[ G_a = \bigcup \left\{ H_{x_1}^{z_3} \mid a = x_1x_2x_3, x_1, x_2, x_3 \in S \right\} \]
\[ \subseteq \bigcup \left\{ A_{x_1}^{x_3} \mid a = x_1x_2x_3, x_1, x_2, x_3 \in S \right\} \]
\[ \subseteq \bigcup \left\{ A_{x_1}^{x_2}x_3 \mid a = x_1x_2x_3, x_1, x_2, x_3 \in S \right\} \]
\[ = A_a, \]
and so \( G \subseteq A \). Therefore \((H) = G\). \( \square \)

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