GENERALIZATION OF THE FEJÉR–HADAMARD’S INEQUALITY FOR CONVEX FUNCTION ON COORDINATES

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ABSTRACT. In this paper, we give generalization of the Fejér–Hadamard inequality by using definition of convex functions on n-coordinates. Results given in [8, 12] are particular cases of results given here.

1. Introduction

Convex functions are important and provide a base to build literature of mathematical inequalities. A function $f : I \to \mathbb{R}$, where $I$ is an interval in $\mathbb{R}$ is called convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y),$$

where $\lambda \in [0, 1]$, $x, y \in I$.

A bundle of inequalities in literature, are due to convex functions or functions related to convex functions see [4, 9, 15]. A classical inequality for convex functions is Hadamard inequality, this is given as follows:

$$f(a + b) \leq \frac{1}{b - a} \int_a^b f(t) \, dt \leq \frac{f(a) + f(b)}{2},$$

where $f : I \to \mathbb{R}$ is a convex function $a, b \in I, a < b$ (see [17, p. 137]).

In many areas of analysis, application of the Hadamard inequality appear for different classes of functions (see [1, 3, 6, 10, 18] for convex functions). Some useful mappings connected to this inequality are also defined by many authors, for example, see [2, 5, 10, 14]. In recent years, the concept of convexity has been extended and generalized in various directions. In this regards, very novel and innovative techniques are used by different authors (see, [11, 16]).
In 1906, Fejér (see [13] and [17, p. 138]) established the following weighted generalization of the Hadamard inequality. The inequalities
\[
\int_a^b f(x)g(x)dx \leq \frac{f(a) + f(b)}{2} \int_a^b g(x)dx
\]
hold for every convex function \( f : I \rightarrow \mathbb{R} \), \( a, b \in I \), and \( g : [a, b] \rightarrow \mathbb{R}^+ \) is symmetric about \( (a + b)/2 \).


**Definition 1.1.** Let \( \Delta^2 := [a, b] \times [c, d] \subset \mathbb{R}^2 \) with \( a < b \) and \( c < d \). A function \( f : \Delta^2 \rightarrow \mathbb{R} \) will be called convex on coordinates if the partial mapping \( f_y : [a, b] \rightarrow \mathbb{R}, f_y(u) := f(u, y) \) and \( f_x : [c, d] \rightarrow \mathbb{R}, f_x(v) := f(x, v) \) are convex, where defined for all \( y \in [c, d] \) and \( x \in [a, b] \).

**Theorem 1.2.** Let \( f : \Delta^2 \rightarrow \mathbb{R} \) be a convex mapping on coordinates in \( \Delta^2 \). Also let \( g_1 : [a, b] \rightarrow \mathbb{R}^+ \) and \( g_2 : [c, d] \rightarrow \mathbb{R}^+ \) be two integrable and symmetric functions about \( (a + b)/2 \) and \( (c + d)/2 \) respectively. Then one has the following inequalities
\[
\int_a^b f\left(\frac{a + b + c + d}{2}, c + d/2\right) \leq \frac{1}{2} \left[ \frac{1}{G_1} \int_a^b f\left(\frac{a + b}{2}, y\right)g_1(x)dx + \frac{1}{G_2} \int_c^d f\left(\frac{a + b}{2}, y\right)g_2(y)dy \right]
\]
\[
\leq \frac{1}{G_1G_2} \int_a^b \int_c^d f(x, y)g_1(x)g_2(y)dydx
\]
\[
\leq \frac{1}{4} \left[ \frac{1}{G_1} \int_a^b g_1(x)f(x, c)dx + \frac{1}{G_1} \int_c^d g_1(x)f(x, d)dx + \frac{1}{G_2} \int_c^d g_2(y)f(a, y)dy + \frac{1}{G_2} \int_c^d g_2(y)f(b, y)dy \right]
\]
\[
\leq \frac{1}{4} [f(a, c) + f(a, d) + f(b, c) + f(b, d)],
\]
where
\[
G_1 = \int_a^b g_1(x)dx \quad \text{and} \quad G_2 = \int_c^d g_2(y)dy.
\]

There in [12] some mappings connected to above inequality are also considered and their properties are discussed.

In [12] authors extended the definition of convex functions on coordinates to \( n \)-coordinates and gave the Hadamard’s inequality for \( n \)-coordinates and related results. In this paper we give Fejér–Hadamard’s inequality for convex functions on coordinates and show that results proved in [8, 12] are particular case of results in this paper.
2. Main results

For $n \geq 2$, let $a_i, b_i$ ($i = 1, 2, \ldots, n$) be real numbers such that $a_i < b_i$ for $i = 1, 2, \ldots, n$. We consider an $n$-dimensional interval $\Delta^n$ defined as $\Delta^n = \prod_{i=1}^{n}[a_i, b_i]$. In [12] the definition of a convex function on $n$-coordinates is given as follows:

Definition 2.1. Let $(x_1, \ldots, x_n) \in \Delta^n$. A mapping $f : \Delta^n \to \mathbb{R}$ is called convex on $n$-coordinates if the functions $f'_{x_n}$, where $f'_{x_n}(t) := f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)$, are convex on $[a_i, b_i]$ for $i = 1, 2, \ldots, n$.

Recall that a mapping $f : \Delta^n \to \mathbb{R}$ is convex in $\Delta^n$ if for $x = (x_1, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \Delta^n$ and $\alpha \in [0, 1]$, the following inequality holds:

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y).$$

It can be seen that every convex mapping $f : \Delta^n \to \mathbb{R}$ is convex on the $n$-coordinates but converse is not true.

Let $f : \Delta^n \to \mathbb{R}$ be convex in $\Delta^n$. Consider $f'_{x_n} : [a_i, b_i] \to \mathbb{R}$, defined by

$$f'_{x_n}(t) = f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n), \quad t \in [a_i, b_i].$$

Now for $x, y \in [a_i, b_i]$ and $\alpha \in [0, 1]$,

$$f'_{x_n}(\alpha x + (1 - \alpha)y) = f(x_1, \ldots, x_{i-1}, \alpha x + (1 - \alpha)y, x_{i+1}, \ldots, x_n) = f(\alpha x_1 + (1 - \alpha)x_1, \ldots, \alpha x + (1 - \alpha)y, \ldots, \alpha x_n + (1 - \alpha)x_n)
\leq \alpha f(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) + (1 - \alpha)f(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)
= \alpha f'_{x_n}(x) + (1 - \alpha)f'_{x_n}(y),$$

which implies $f'_{x_n}$ is convex on $[a_i, b_i]$, that is, $f$ is convex on $n$-coordinates. For converse we give the following counter example:

Example 2.2. Let us consider a mapping $f : [0, 1]^n \to \mathbb{R}$ defined as

$$f(x_1, \ldots, x_n) = x_1 \cdot x_2 \cdots x_n.$$

It is convex on $n$-coordinates as follows:

$$f'_{x_n}(\alpha x + (1 - \alpha)y) = x_1 \cdots x_{i-1} \cdot (\alpha x + (1 - \alpha)y) \cdot x_{i+1} \cdots x_n = \alpha(x_1 \cdots x_{i-1} \cdot x \cdot x_{i+1} \cdots x_n) + (1 - \alpha)(x_1 \cdots x_{i-1} \cdot y \cdot x_{i+1} \cdots x_n)
= \alpha f'_{x_n}(x) + (1 - \alpha)f'_{x_n}(y).$$

But for $x = (1, 1, \ldots, 1, 0), y = (0, 1, 1, \ldots, 1) \in [0, 1]^n$, we have

$$f(\alpha x + (1 - \alpha)y) = f(\alpha, 1, \ldots, 1 - \alpha) = \alpha(1 - \alpha)$$

and

$$\alpha f(x) + (1 - \alpha)f(y) = \alpha \cdot 0 + (1 - \alpha) \cdot 0 = 0.$$
This gives
\[ f(ax + (1 - a)y) > af(x) + (1 - a)f(y) \] for all \( a \in (0, 1) \),
that is, \( f \) is not convex on \([0, 1]^n\).

It is interesting to note that if \( f : \Delta^n \to \mathbb{R} \) is a convex mapping on \( n \)-coordinates, then \( f_{x^n}^i : [a_i, b_i] \to \mathbb{R} \) is a convex function on \([a_i, b_i]\) for each \( i = 1, 2, \ldots, n \). Also, if \( g_i : [a_i, b_i] \to \mathbb{R} \) is a symmetric function about \( \frac{a_i + b_i}{2} \), then from Fejér–Hadamard’s inequality, we have
\[ f_{x^n}^i \left( \frac{a_i + b_i}{2} \right) \leq \frac{1}{G_i} \int_{a_i}^{b_i} f_{x^n}^i(x_i)g_i(x_i)dx_i, \quad i = 1, 2, \ldots, n, \]
where
\[ G_i = \int_{a_i}^{b_i} g_i(x_i)dx_i. \]

This gives us
\[ \sum_{k=1}^{n} f_{x^n}^k \left( \frac{a_k + b_k}{2} \right) \leq \sum_{k=1}^{n} \frac{1}{G_k} \int_{a_k}^{b_k} f_{x^n}^k(x_k)g_k(x_k)dx_k. \] (4)

**Theorem 2.3.** Let \( (x_1, \ldots, x_n) \in \Delta^n \) and \( f : \Delta^n \to \mathbb{R} \) be a convex mapping on \( n \)-coordinates. Also, let \( g_i : [a_i, b_i] \to \mathbb{R} \) be an integrable and symmetric function about \( \frac{a_i + b_i}{2} \) for each \( i = 1, \ldots, n \). Then we have
\[ \sum_{k=1}^{n} \frac{1}{G_k} \int_{a_k}^{b_k} f_{x^n}^{k+1} \left( \frac{a_{k+1} + b_{k+1}}{2} \right)g_k(x_k)dx_k \]
\[ \leq \sum_{k=1}^{n} \frac{1}{G_k G_{k+1}} \int_{a_k}^{b_k} \int_{a_{k+1}}^{b_{k+1}} f(x)g_k(x_k)g_{k+1}(x_{k+1})dx_{k+1}dx_k \]
\[ \leq \frac{1}{2} \sum_{k=1}^{n} \left[ \frac{1}{G_k} \int_{a_k}^{b_k} \left( f_{x^n}^{k+1}(a_{k+1}) + f_{x^n}^{k+1}(b_{k+1}) \right) g_k(x_k)dx_k \right], \]
where
\[ G_k = \int_{a_k}^{b_k} g_k(x_k)dx_k, \]
and with \( n + 1 \to 1 \). These inequalities are sharp.

**Proof.** By applying the Fejér–Hadamard’s inequality for convex function \( f_{x^n}^{k+1} \) on interval \([a_{k+1}, b_{k+1}]\) we have
\[ f_{x^n}^{k+1} \left( \frac{a_{k+1} + b_{k+1}}{2} \right) G_{k+1} \leq \int_{a_{k+1}}^{b_{k+1}} f_{x^n}^{k+1}(x_{k+1})g_{k+1}(x_{k+1})dx_{k+1} \]
\[ \leq \left( \frac{f_{x^n}^{k+1}(a_{k+1}) + f_{x^n}^{k+1}(b_{k+1})}{2} \right) G_{k+1}. \] (6)
Multiplying (6) by \( g_k(x_k) \) we have
\[
f_{x_n}^{k+1}\left(\frac{a_{k+1} + b_{k+1}}{2}\right) g_k(x_k)G_{k+1} \leq \int_{a_{k+1}}^{b_{k+1}} f_{x_n}^{k+1}(x_{k+1})g_k(x_k)g_{k+1}(x_{k+1})dx_{k+1}
\]
\[
\leq \left(\frac{f_{x_n}^{k+1}(a_{k+1}) + f_{x_n}^{k+1}(b_{k+1})}{2}\right) g_k(x_k)G_{k+1}.
\]

Now by integrating on \([a_k, b_k]\) we get
\[
G_{k+1} \int_{a_k}^{b_k} f_{x_n}^{k+1}\left(\frac{a_{k+1} + b_{k+1}}{2}\right) g_k(x_k)dx_k
\]
\[
\leq \int_{a_k}^{b_k} \int_{a_{k+1}}^{b_{k+1}} f_{x_n}^{k+1}(x_{k+1})g_k(x_k)g_{k+1}(x_{k+1})dx_{k+1}dx_k
\]
\[
\leq G_{k+1} \int_{a_k}^{b_k} f_{x_n}^{k+1}\left(\frac{a_{k+1} + b_{k+1}}{2}\right) g_k(x_k)dx_k.
\]

As \( G_k > 0, G_{k+1} > 0 \), then divide by \( G_kG_{k+1} \) we get
\[
\frac{1}{G_k} \int_{a_k}^{b_k} f_{x_n}^{k+1}\left(\frac{a_{k+1} + b_{k+1}}{2}\right) g_k(x_k)dx_k
\]
\[
\leq \frac{1}{G_kG_{k+1}} \int_{a_k}^{b_k} \int_{a_{k+1}}^{b_{k+1}} f_{x_n}^{k+1}(x_{k+1})g_k(x_k)g_{k+1}(x_{k+1})dx_{k+1}dx_k
\]
\[
\leq \frac{1}{G_k} \int_{a_k}^{b_k} f_{x_n}^{k+1}\left(\frac{a_{k+1} + b_{k+1}}{2}\right) g_k(x_k)dx_k.
\]

Taking summation from 1 to \( n \) we get (5).

If we consider \( f(x_1, \ldots, x_n) = x_1 \ldots x_n \), then inequalities in (5) become equality, which shows these are sharp.

\[\Box\]

**Theorem 2.4.** Let \((x_1, \ldots, x_n) \in \Delta^n\) and \( f : \Delta^n \to \mathbb{R} \) be a convex mapping on \( n \)-coordinates. Also, let \( g_i : [a_i, b_i] \to \mathbb{R} \) be an integrable and symmetric function about \( \frac{a_i + b_i}{2} \) for each \( i = 1, \ldots, n \). Then we have
\[
\sum_{k=1}^{n} \frac{1}{G_k} \int_{a_k}^{b_k} \left(f_{a_n}^{k}(x_k) + f_{b_n}^{k}(x_k)\right) dx_k
\]
\[
\leq \frac{n}{2} \left(f(a) + f(b)\right) + \frac{1}{2} \sum_{k=1}^{n} \left(f_{a_n}^{k}(b_k) + f_{b_n}^{k}(a_k)\right),
\]

where \( a = (a_1, a_2, \ldots, a_n) \) and \( b = (b_1, b_2, \ldots, b_n) \). The above inequality is sharp.

**Proof.** As \( f : \Delta^n \to \mathbb{R} \) is a convex mapping on \( n \)-coordinates, therefore \( f_{x_n}^i : [a_i, b_i] \to \mathbb{R} \) is convex on \([a_i, b_i]\) for each \( i = 1, 2, 3, \ldots, n \). From Féjér inequality
for each $i = 1, 2, 3, \ldots, n$ we have,

$$(8) \quad \frac{1}{G_i} \int_{a_i}^{b_i} f_{a_i}^i(x_i) dx_i g_i(x_i) \leq \frac{f(a_i) + f_{a_i}^i(b_i)}{2}$$

and

$$(9) \quad \frac{1}{G_i} \int_{a_i}^{b_i} f_{b_i}^i(x_i) dx_i g_i(x_i) \leq \frac{f_{b_i}^i(a_i) + f(b_i)}{2}.$$

Adding (8) and (9) we get,

$$(10) \quad \frac{1}{G_i} \int_{a_i}^{b_i} \left( f_{a_i}^i(x_i) + f_{b_i}^i(x_i) \right) dx_i g_i(x_i) \leq \frac{1}{2} (f(a) + f(b)) + \frac{1}{2} \left( f_{a_i}^i(b_i) + f_{b_i}^i(a_i) \right),$$

where $i = 1, 2, \ldots, n$. Taking sum from 1 to $n$ we get (7).

If we consider $f(x_1, \ldots, x_n) = x_1 \cdots x_n$, then inequalities in (7) become equality, which shows these are sharp. □

A special case of inequalities (4), (5), and (7) is stated in the following, which is main result of [12, Theorem 1].

**Corollary 2.5.** Let $\Delta^2 = [a, b] \times [c, d]$ and $f : \Delta^2 \to \mathbb{R}$ be a convex mapping on 2-coordinates. Also, let $g_i : [a_i, b_i] \to \mathbb{R}$ be an integrable and symmetric function about $a_i + b_i$ for each $i = 1, 2$. Then (3) is valid.

**Proof.** By putting $n = 2$ in Theorem 2.3 and Theorem 2.4, and taking $a_1 = a$, $b_1 = b$, $a_2 = c$, and $b_2 = d$, we get the required result. □

**Remark 2.6.** Further if we put $g_1(x) = 1$ and $g_2(x) = 1$, then we get main result of [8, Theorem 1].

### 3. Associated mappings

In this section we are interested to associate some mappings with the generalized Fejér–Hadamard inequality for a convex mapping on $n$-coordinates.

For $n \geq 2$, let $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_n)$ and $A_i$ denotes arithmetic means of numbers $a_i$ and $b_i$, that is,

$$A_i = A(a_i, b_i) = \frac{a_i + b_i}{2}.$$

Also for $x = (x_1, x_2, \ldots, x_n) \in \Delta^n := \prod_{i=1}^{n}[a_i, b_i]$ and $t = (t_1, t_2, \ldots, t_n) \in [0, 1]^n$, we consider $s_i$ be a point on a segment between $x_i$ and $A_i$, that is,

$$s_i = t_i x_i + (1 - t_i) A_i.$$
For the mapping \( f : \Delta^n \to \mathbb{R} \) defined in previous section, we can associate a mapping \( \hat{H} : [0; 1]^n \to \mathbb{R} \) given by
\[
\hat{H}(t) = \sum_{k=1}^{n} \frac{1}{G_k G_{k+1}} \int_{a_k}^{b_k} \int_{a_{k+1}}^{b_{k+1}} f(s) g_k(x_k) g_{k+1}(x_{k+1}) dx_{k+1} dx_k,
\]
where \( s = (s_1, s_2, \ldots, s_n) \).

Consider \( \hat{H}_{i_n}^i : [0, 1] \to \mathbb{R} \), defined by
\[
\hat{H}_{i_n}^i(t) = \hat{H}(t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_n).
\]

We can rewrite \( \hat{H}_{i_n}^i(t) \) as follows:
\[
\hat{H}_{i_n}^i(t) = \hat{H}(t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_n)
\]
\[= \sum_{k=1}^{n} \frac{1}{G_k G_{k+1}} \int_{a_k}^{b_k} \int_{a_{k+1}}^{b_{k+1}} f(s_1, \ldots, s_{i-1}, tx_i + (1-t)A_i, s_{i+1}, \ldots, s_n) \times g_k(x_k) g_{k+1}(x_{k+1}) dx_{k+1} dx_k
\]
\[= \sum_{k=1}^{n} \frac{1}{G_k G_{k+1}} \int_{a_k}^{b_k} \int_{a_{k+1}}^{b_{k+1}} f_{x_i}^i (tx_i + (1-t)A_i) g_k(x_k) g_{k+1}(x_{k+1}) dx_{k+1} dx_k
\]
\[= \sum_{k=1}^{n} \frac{1}{G_k G_{k+1}} \int_{a_k}^{b_k} \int_{a_{k+1}}^{b_{k+1}} f_{x_i}^i \hat{g}(x_k) g_{k+1}(x_{k+1}) dx_{k+1} dx_k,
\]
where \( \hat{t} = tx_i + (1-t)A_i \). We will use this notation throughout the paper. We also need a following lemma given by Levin and Stečkin in [17, p. 200] to get desired results.

**Lemma 3.1.** Let \( f \) be convex on \([a, b]\) and \( g \) be symmetric about \((a + b)/2\) and nonincreasing function on \([a, (a + b)/2]\). Then
\[
\int_a^b f(x)g(x)dx \geq \frac{1}{b - a} \int_a^b f(x)dx \int_a^b g(x)dx.
\]

**Theorem 3.2.** Let \( f : \Delta^n \to \mathbb{R} \) be a convex mapping on \( n\)-coordinates on \( \Delta^n \). Then the mapping \( \hat{H} \) is convex on \( n\)-coordinates on \([0, 1]^n\). We also have
\[
\hat{H}(t) \geq \sum_{k=1}^{n} f(s_1, \ldots, s_{i-1}, A_k, A_{k+1}, \ldots, s_n)
\]
and
\[
\hat{H}(t) \leq \sum_{k=1}^{n} \frac{tk + t_{k+1}(1 - t_k)}{G_k G_{k+1}} \int_{a_k}^{b_k} \int_{a_{k+1}}^{b_{k+1}} f(s_1, \ldots, s_{k-1}, x_k, x_{k+1}, \ldots, s_n) \times g_k(x_k) g_{k+1}(x_{k+1}) dx_{k+1} dx_k
\]
Since given that $f$ is convex on the $A_k$, we have

\[ \hat{H}^i_n(\alpha u + \beta v) \]

Now

\[ \alpha u + \beta v = (\alpha u + \beta v)x_i + (1 - \alpha u - \beta v)A_i \]

\[ = \alpha (ux_i + (1 - u)A_i) + \beta (vx_i + (1 - v)A_i) \]

\[ = \alpha \hat{u} + \beta \hat{v}. \]

This gives us

\[ \hat{H}^i_n(\alpha u + \beta v) \]

\[ = \sum_{k=1}^n \frac{1}{G_k G_{k+1}} \int_{a_k}^{b_{k+1}} f^i_n(\alpha \hat{u} + \beta \hat{v})g_k(x_k)g_{k+1}(x_{k+1})dx_{k+1} dx_k. \]

Since given that $f^i_n$ is convex, therefore we have

\[ \hat{H}^i_n(\alpha u + \beta v) \]

\[ \leq \sum_{k=1}^n \frac{1}{G_k G_{k+1}} \int_{a_k}^{b_{k+1}} (\alpha f^i_n(\hat{u}) + \beta f^i_n(\hat{v}))g_k(x_k)g_{k+1}(x_{k+1})dx_{k+1} dx_k \]

\[ = \alpha \hat{H}^i_n(\hat{u}) + \beta \hat{H}^i_n(\hat{v}). \]

Which implies $\hat{H}^i_n$ is convex, that is, $\hat{H}$ is convex on $n$-coordinates.

To prove inequality (11), we consider

\[ \hat{H}(t) = \sum_{k=1}^n \frac{1}{G_k G_{k+1}} \int_{a_k}^{b_{k+1}} f^i_n(s_i)g_k(x_k)g_{k+1}(x_{k+1})dx_{k+1} dx_k \]

\[ = \sum_{k=1}^n \frac{1}{G_k + 1} \int_{a_k}^{b_{k+1}} \left[ \frac{1}{G_k} \int_{a_k}^{b_k} f^i_n(s_i)g_k(x_k)g_{k+1}(x_{k+1})dx_k \right] dx_{k+1}. \]

Since $f$ is convex on the $k$th coordinate and $\frac{1}{G_k} \int_{a_k}^{b_k} g_k(x_k)dx_k = 1$, we apply Jensen’s inequality for integrals on $k$th coordinate to get

\[ H(t) \geq \sum_{k=1}^n \frac{1}{G_k + 1} \int_{a_k}^{b_{k+1}} \left[ f^k_{s_n} \left( \frac{1}{G_k} \int_{a_k}^{b_k} (s_i)g_k(x_k)dx_k \right) \right] g_{k+1}(x_{k+1})dx_{k+1}. \]
Now it follows from Lemma 3.1, that
\[ \hat{H}(t) \geq \sum_{k=1}^{n} \frac{1}{G_{k+1}} \int_{a_{k+1}}^{b_{k+1}} f_{k}^{k} \left( \frac{a_{k} + b_{k}}{2} \right) g_{k+1}(x_{k+1})dx_{k+1}. \]

Now using convexity of \( f \) on \((k+1)\)th coordinate. Again applying Jensen’s inequality and Lemma 3.1 on \((k+1)\)th coordinate we get inequality in (11).

Now to prove inequality (12), we first use convexity of \( f \) on \( k \)th coordinate, then on \((k+1)\)th coordinate, we have
\[
\hat{H}(t) \leq \frac{t_{k}}{G_{k}G_{k+1}} \int_{a_{k}}^{b_{k}} \int_{a_{k+1}}^{b_{k+1}} f_{k}^{k}(x_{k})g_{k}(x_{k})g_{k+1}(x_{k+1})dx_{k+1}dx_{k} \\
+ \frac{1 - t_{k}}{G_{k}G_{k+1}} \int_{a_{k}}^{b_{k}} \int_{a_{k+1}}^{b_{k+1}} \left( \frac{a_{k} + b_{k}}{2} \right) g_{k}(x_{k})g_{k+1}(x_{k+1})dx_{k+1}dx_{k} \\
\leq \frac{t_{k}(1 - t_{k+1})}{G_{k}G_{k+1}} \int_{a_{k}}^{b_{k}} f(s_{1}, \ldots, s_{i-1}, x_{k}, x_{k+1}, s_{k+2}, \ldots, s_{n}) \\
\times g_{k}(x_{k})g_{k+1}(x_{k+1})dx_{k+1}dx_{k} \\
+ \frac{t_{k}(1 - t_{k+1})}{G_{k}G_{k+1}} \int_{a_{k}}^{b_{k}} f(s_{1}, \ldots, s_{i-1}, x_{k}, \frac{a_{k} + b_{k+1}}{2}, s_{k+2}, \ldots, s_{n})g_{k}(x_{k})dx_{k} \\
+ \frac{t_{k+1}(1 - t_{k})}{G_{k+1}} \int_{a_{k+1}}^{b_{k+1}} f(s_{1}, \ldots, s_{i-1}, A_{k}, x_{k+1}, s_{k+2}, \ldots, s_{n})g_{k}(x_{k})dx_{k} \\
+ \frac{(1 - t_{k})(1 - t_{k+1})}{G_{k}G_{k+1}} \int_{a_{k}}^{b_{k}} \int_{a_{k+1}}^{b_{k+1}} f(s_{1}, \ldots, s_{k-1}, A_{k}, A_{k+1}, s_{k+2}, \ldots, s_{n}) \\
\times g_{k}(x_{k})g_{k+1}(x_{k+1})dx_{k+1}dx_{k}.
\]

Now by (4), we can have
\[
\sum_{k=1}^{n} \frac{1}{G_{k+1}} \int_{a_{k+1}}^{b_{k+1}} f(s_{1}, \ldots, A_{k}, x_{k+1}, \ldots, s_{n})g_{k}(x_{k+1})dx_{k+1} \\
\leq \sum_{k=1}^{n} \frac{1}{G_{k}G_{k+1}} \int_{a_{k}}^{b_{k}} \int_{a_{k+1}}^{b_{k+1}} f(s_{1}, \ldots, x_{k}, x_{k+1}, \ldots, s_{n})g_{k}(x_{k})g_{k+1}(x_{k+1})dx_{k+1}dx_{k}
\]
and from the first inequality in Theorem 2.3
\[
\sum_{k=1}^{n} \frac{1}{G_{k}} \int_{a_{k}}^{b_{k}} f(s_{1}, \ldots, x_{k}, A_{k+1}, \ldots, s_{n})g_{k}(x_{k+1})dx_{k+1}
\]
\[ \leq \sum_{k=1}^{n} \frac{1}{G_k G_{k+1}} \int_{a_k}^{b_k} \int_{a_{k+1}}^{b_{k+1}} f(s_1, \ldots, x_k, x_{k+1}, \ldots, s_n) g_k(x_k) g_{k+1}(x_{k+1}) dx_{k+1} dx_k. \]

Now using the inequalities (14) and (15) in (13) we get (12). □

The particular case of above theorem is the following result, which is Theorem 2.4 given in [12].

**Corollary 3.3.** Let \( f : \Delta^2 \to \mathbb{R} \) be a convex function on 2-coordinates. Then the mapping \( \hat{H} \), defined as
\[
\hat{H}(t, s) = \frac{1}{G_1 G_2} \int_a^b \int_c^d f \left( (tx + (1-t)a + b) \frac{2}{2}, sy + (1-s)c + d \right) g_1(x) g_2(y) dy dx,
\]
is convex on the coordinates on \([0,1]^2\). Further if \( g_1 \) is nonincreasing on \([a, (a+b)/2]\) and \( g_2 \) is nonincreasing on \([c, (c+d)/2]\), then
\[
\inf_{(t, s) \in [0,1]^2} \hat{H}(t, s) = f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) = \hat{H}(0, 0)
\]
and
\[
\sup_{(t, s) \in [0,1]^2} \hat{H}(t, s) = \frac{1}{G_1 G_2} \int_a^b \int_c^d f(x, y) g_1(x) g_2(y) dy dx = \hat{H}(1, 1).
\]

**Proof.** By putting \( n = 2 \) in Theorem 3.2, we get required result. □

**Remark 3.4.** Further if we take \( g_1(x) = 1 \) and \( g_2(x) = 1 \), then we get Theorem 2 in [8].

**Theorem 3.5.** Let \( f : \Delta^n \to \mathbb{R} \) be a convex mapping on \( \Delta^n \). Then the mapping \( \hat{H} \) is convex on \([0,1]^n\). Also the mapping \( \hat{h} : [0,1] \to \mathbb{R} \), defined by \( \hat{h}(t) = \hat{H}(t, \ldots, t) \) is convex and one has the bounds
\[
\hat{h}(t) \geq \sum_{k=1}^{n} f(s_1, \ldots, s_{i-1}, A_k, A_{k+1}, \ldots, s_n)
\]
and
\[
\hat{h}(t) \leq \sum_{k=1}^{n} \frac{t(2-t)}{G_k G_{k+1}} \int_{a_k}^{b_k} \int_{a_{k+1}}^{b_{k+1}} f(s_1, \ldots, s_{k-1}, x_k, x_{k+1}, \ldots, s_n) \\
\times g_k(x_k) g_{k+1}(x_{k+1}) dx_{k+1} dx_k \\
+ \sum_{k=1}^{n} (1-t)^2 f(s_1, \ldots, s_{k-1}, A_k, A_{k+1}, \ldots, s_n).
\]

**Proof.** Let \( u, v \in [0,1]^n \) and \( \alpha, \beta \in [0,1] \) such that \( \alpha + \beta = 1 \). Then
\[
\hat{H}(\alpha u + \beta v) = \sum_{k=1}^{n} \frac{1}{G_k G_{k+1}} \int_{a_k}^{b_k} \int_{a_{k+1}}^{b_{k+1}} f \left( \alpha u_1 + \beta v_1, \ldots, \alpha u_n + \beta v_n \right) \\
\times g_k(x_k) g_{k+1}(x_{k+1}) dx_{k+1} dx_k.
\]
For each \(i = 1, 2, \ldots, n\), we have
\[
\alpha u_i + \beta v_i = (\alpha u_i + \beta v_i)x_i + (1 - \alpha u_i - \beta v_i)A_i = \alpha(u_i x_i + (1 - u_i)A_i) + \beta(v_i x_i + (1 - v_i)A_i) = \alpha \hat{u}_i + \beta \hat{v}_i.
\]
This gives
\[
H(\alpha u + \beta v) = \sum_{k=1}^{n} \frac{1}{G_k G_{k+1}} \int_{a_k}^{b_k} \int_{a_{k+1}}^{b_{k+1}} f(\alpha \hat{u}_1 + \beta \hat{v}_1, \ldots, \alpha \hat{u}_n + \beta \hat{v}_n)
\times g_k(x_k) g_{k+1}(x_{k+1}) dx_{k+1} dx_k
\leq \sum_{k=1}^{n} \frac{1}{G_k G_{k+1}} \int_{a_k}^{b_k} \int_{a_{k+1}}^{b_{k+1}} (\alpha f(\hat{u}_1, \ldots, \hat{u}_n) + \beta f(\hat{v}_1, \ldots, \hat{v}_n))
\times g_k(x_k) g_{k+1}(x_{k+1}) dx_{k+1} dx_k
= \alpha H(u) + \beta H(v).
\]
This shows that \(\hat{H}\) is convex on \([0, 1]^n\). Similar to above, we can show that \(\hat{h}\) is convex on \([0, 1]\) and using bounds of mapping \(\hat{H}\) in Theorem 3.2 we can get bounds of mapping \(h\). □

The particular case of above theorem is the following result, which is Theorem 2.6 in [12].

**Corollary 3.6.** Suppose that \(f : \Delta^2 \rightarrow \mathbb{R}\) is a convex mapping on 2-coordinates. Let \(h : [0, 1] \rightarrow \mathbb{R}\) be the mapping defined as \(\hat{h}(t) = \hat{H}(t, t)\), then \(\hat{h}\) is convex on coordinates on \(\Delta\). Also if \(g_1\) is nonincreasing on \([a, (a+b)/2]\) and \(g_2\) is nonincreasing on \([c, (c+d)/2]\), then
\[
\inf_{t \in [0,1]} \hat{h}(t) = f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) = \hat{H}(0,0)
\]
and
\[
\sup_{t \in [0,1]} h_{g_1 g_2}(t) = \frac{1}{G_1 G_2} \int_{a}^{b} \int_{c}^{d} f(x, y)g_1(x)g_2(y)dydx = H_{g_1 g_2}(1,1).
\]

**Proof.** By putting \(n = 2\) in Theorem (3.5), we get (3.6). □

**Remark 3.7.** Further if we take \(g_1(x) = 1\) and \(g_2(x) = 1\), the we obtain Theorem 3 in [8].

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