ON SYMMETRIC BI-DERIVATIONS OF B-ALGEBRAS

Sıbel Altunbiçak Kayış and Şule Ayar Özbal

Abstract. In this paper, we introduce the notion of symmetric bi-derivations of a B-algebra and investigate some related properties. We study the notion of symmetric bi-derivations of a 0-commutative B-algebra and state some related properties.

1. Introduction

The notion of B-algebra was introduced by J. Neggers and H. S. Kim and some of its related properties in [10] were studied. This class of algebras is related to several classes of interest such as BCH/BCI/BCK-algebras. Later, the notion of a ranked trigroupoid as a natural followup on the idea of a ranked bigroupoid was given by N. O. Alshehri et al. in [2]. The notion of derivation in ring theory and near ring theory was applied to BCI-algebras by Y. B. Jun and Xin and some of its related properties were given by them [6]. Later, in [11] the notion of a regular derivation in BCI-algebras was applied to BCC-algebras by Prabpayak and Leerawat and also some of its related properties were investigated. In [1] the notion of derivation in B-algebra was given and some related properties were stated by N. O. Alshehri. Also, in [2] the notion of derivation on ranked bigroupoids was introduced and (X, ,)-self-(co)derivations were discussed by N. O. Alshehri, H. S. Kim and J. Neggers. The concept of symmetric bi-derivation was introduced by G. Maksa in [8] (see also [9]). J. Vukman proved some results concerning symmetric bi-derivation on prime and semiprime rings in [12, 13]. Later, the notion of left-right (resp. right-left) symmetric bi-derivation of BCI-algebras was introduced by S. Ilbira, A. Firat and Y. B. Jun in [5].

In this paper, we apply the notion of symmetric bi-derivation in rings, near rings and lattices to B-algebras. We introduce the concept of symmetric bi-derivation of a B-algebra. Additionally, this definition in 0-commutative B-algebra is studied and related properties are given.

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2. Preliminaries

Definition 2.1 ([10]). A $B$-algebra is a non-empty set $X$ with a constant $0$ and with a binary operation $\ast$ satisfying the following axioms for all $x, y, z \in X$:

(I) $x \ast x = 0$.

(II) $x \ast 0 = x$.

(III) $(x \ast y) \ast z = x \ast (z \ast (0 \ast y))$.

Proposition 2.2 ([10]). If $(X, \ast, 0)$ is a $B$-algebra, then for all $x, y, z \in X$:

(1) $(x \ast y) \ast (0 \ast y) = x$.

(2) $x \ast (y \ast z) = (x \ast (0 \ast z)) \ast y$.

(3) $x \ast y = 0$ implies $x = y$.

(4) $0 \ast (0 \ast x) = x$.

Theorem 2.3 ([10]). $(X, \ast, 0)$ is a $B$-algebra if and only if it satisfies the following axioms for all $x, y, z \in X$:

(5) $(x \ast z) \ast (y \ast z) = x \ast y$.

(6) $0 \ast (x \ast y) = y \ast x$.

Theorem 2.4 ([4]). In any $B$-algebra, the left and right cancellation laws hold.

Definition 2.5 ([7]). A $B$-algebra $(X, \ast, 0)$ is said to be 0-commutative if for all $x, y \in X$:

$x \ast (0 \ast y) = y \ast (0 \ast x)$.

Proposition 2.6 ([7]). If $(X, \ast, 0)$ is a 0-commutative $B$-algebra, then for all $x, y, z \in X$:

(7) $(0 \ast x) \ast (0 \ast y) = y \ast x$.

(8) $(z \ast y) \ast (z \ast x) = x \ast y$.

(9) $(x \ast y) \ast z = (x \ast z) \ast y$.

(10) $[x \ast (x \ast y)] \ast y = 0$.

(11) $(x \ast z) \ast (y \ast t) = (t \ast z) \ast (y \ast x)$.

From (11) and (3) we get that, if $(X, \ast, 0)$ is a 0-commutative $B$-algebra, then:

(12) $x \ast (x \ast y) = y$ for all $x, y \in X$.

For a $B$-algebra $X$, we denote $x \wedge y = y \ast (y \ast x)$ for all $x, y \in X$.

Definition 2.7. Let $X$ be a $B$-algebra. A mapping $D : X \times X \to X$ is called symmetric if $D(x, y) = D(y, x)$ holds for all $x, y \in X$.

Definition 2.8. Let $X$ be a $B$ algebra. A mapping $d : X \to X$ is said to be regular if $d(0) = 0$.

3. The symmetric bi-derivations of $B$-algebras

The following definition introduces the notion of symmetric bi-derivation for a $B$-algebra.
Definition 3.1. Let $X$ be a $B$-algebra. A map $D : X \times X \to X$ is said to be a left-right symmetric bi-derivation (briefly, an $(l, r)$ symmetric bi-derivation) of $X$, if it satisfies the identity $D(x \ast y, z) = (D(x, z) \ast y) \wedge (x \ast D(y, z))$ for all $x, y, z \in X$.

If $D$ satisfies the identity $D(x \ast y, z) = (x \ast D(y, z)) \wedge (D(x, z) \ast y)$ for all $x, y, z \in X$, then $D$ is said to be a right-left derivation (briefly, an $(r, l)$ symmetric bi-derivation) of $X$. Moreover, if $D$ is both $(l, r)$ and $(r, l)$ symmetric bi-derivations, then it is said that $D$ is a symmetric bi-derivation.

Example 3.1. Let $X = \{0, 1, 2\}$ be a 0-commutative $B$-algebra with Cayley table as follows.

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Define a mapping $D : X \times X \to X$

$$D(x, y) = \begin{cases} 0, & (x, y) = (2, 2) \text{ and } (x, y) = (0, 1) \text{ and } (x, y) = (1, 0) \\ 1, & (x, y) = (0, 2) \text{ and } (x, y) = (2, 0) \text{ and } (x, y) = (1, 1) \\ 2, & (x, y) = (2, 1) \text{ and } (x, y) = (1, 2) \text{ and } (x, y) = (0, 0). \end{cases}$$

Then it can be checked that $D$ is an $(l, r)$ symmetric bi-derivation on $X$.

Example 3.2. Let $X = \{0, 1, 2, 3\}$ be a $B$-algebra with Cayley table as follows.

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Define a mapping $D : X \times X \to X$

$$D(x, y) = \begin{cases} 0, & (x, y) = (0, 0) \text{ and } (x, y) = (1, 1) \text{ and } (x, y) = (2, 2) \text{ and } (x, y) = (3, 3) \\ 1, & (x, y) = (3, 2) \text{ and } (x, y) = (2, 3) \text{ and } (x, y) = (0, 1) \text{ and } (x, y) = (1, 0) \\ 2, & (x, y) = (2, 0) \text{ and } (x, y) = (0, 2) \text{ and } (x, y) = (1, 3) \text{ and } (x, y) = (3, 1) \\ 3, & (x, y) = (3, 0) \text{ and } (x, y) = (0, 3) \text{ and } (x, y) = (1, 2) \text{ and } (x, y) = (2, 1). \end{cases}$$

Then it can be checked that $D$ is a symmetric bi-derivation on $X$.

Definition 3.2. Let $X$ be a $B$-algebra. A mapping $d : X \to X$ defined by $d(x) = D(x, x)$ for all $x \in X$ is called a trace of $D$, where $D : X \times X \to X$ is a symmetric mapping.
Proposition 3.3. Let $D$ be an $(l, r)$ symmetric bi-derivation on a $B$-algebra $X$. Then the followings hold:

(i) $D(x, y) = D(x, y) \wedge (x \ast D(0, y))$ for all $x, y \in X$.
(ii) $D(0, y) = D(x, y) \ast x$ for all $x, y \in X$.
(iii) $D(0, x) = d(x) \ast x$ for all $x \in X$ where $d$ is the trace of $D$.

Proof. (i) Let $x, y \in X$. By using (II) and the definition of an $(l, r)$ symmetric bi-derivation we get $D(x, y) = D(x, y) \ast 0, y = (D(x, y) \ast 0) \wedge (x \ast D(0, y)) = D(x, y) \wedge (x \ast D(0, y))$. Hence we find that $D(x, y) = D(x, y) \wedge (x \ast D(0, y))$.

(ii) Let $x, y \in X$.

\[
D(0, y) = D(x, x, y) \\
= (D(x, y) \ast x) \wedge (x \ast D(x, y)) \\
= (x \ast D(x, y)) \ast ((x \ast D(x, y)) \ast (D(x) \ast x)) \\
= ((x \ast D(x, y)) \ast (0 \ast (D(x) \ast x))) \ast (x \ast D(x, y)) \quad \text{by (2)} \\
= ((x \ast D(x, y)) \ast (x \ast D(x, y))) \ast (x \ast D(x, y)) \quad \text{by (1)} \\
= 0 \ast (x \ast D(x, y)) \\
= D(x, y) \ast x \quad \text{by (6)}.
\]

Therefore, $D(0, y) = D(x, y) \ast x$.

(iii) Let $x \in X$. By using (I) and the definition of an $(l, r)$ symmetric bi-derivation we get

\[
D(0, x) = D(x, x, x) = (D(x, x) \ast x) \wedge (x \ast D(x, x)) \\
= (d(x) \ast x) \wedge (x \ast d(x)) \\
= (x \ast d(x)) \ast ((x \ast d(x)) \ast (d(x) \ast x)).
\]

By using (3) and (6),

\[
= ((x \ast d(x)) \ast (0 \ast (d(x) \ast x))) \ast (x \ast d(x)) \\
= ((x \ast d(x)) \ast (x \ast d(x))) \ast (x \ast d(x)) \\
= 0 \ast (x \ast d(x)) \\
= d(x) \ast x.
\]

Hence $D(0, x) = d(x) \ast x$. \hfill \square

Proposition 3.4. Let $d$ be the trace of the $(l, r)$ symmetric bi-derivation $D$ on a $B$-algebra $X$. Then the followings hold:

(i) $d(0) = D(x, 0) \ast x$ for all $x \in X$.
(ii) If $D(x, 0) = D(y, 0)$ for all $x, y \in X$, then $d$ is 1 - 1.
(iii) If $d$ is regular, then $D(x, 0) = x$.
(iv) If there is an element $x \in X$ such that $D(x, 0) = x$, then $d$ is regular.
(v) If there exists $x \in X$ such that $d(y) \ast x = 0$ or $x \ast d(y) = 0$ for all $y \in X$, then $d(y) = x$. 
Proof. (i) Let $x$ be an element in $X$. Since $x * x = 0$ we have
\[
d(0) = D(0,0) = D(x * x,0)
\]
\[
= (D(x,0) * x) \land (x * D(x,0))
\]
\[
= (x * D(x,0)) * [(x * D(x,0)) * (D(x,0) * x)].
\]
By using (I) and (6),
\[
= ((x * D(x,0)) * (0 * (D(x,0) * x))) * (x * D(x,0))
\]
\[
= ((x * D(x,0)) * (x * D(x,0))) * (x * D(x,0))
\]
\[
= 0 * (x * D(x,0))
\]
\[
= D(x,0) * x.
\]
Therefore, $d(0) = D(x,0) * x$ for all $x \in X$.
(ii) Let $x, y \in X$ such that $d(x) = d(y)$. Then by (i), we have $d(0) = D(x,0) * x$ and $d(0) = D(y,0) * y$. Thus $D(x,0) * x = D(y,0) * y$. Using Theorem 2.4 we have $x = y$. Hence we get $d$ is $1 - 1$.
(iii) Let $d$ be a regular. By part (i) we have $d(0) = D(x,0) * x$. Since $d$ is regular we have $d(0) = D(x,0) * x = 0$ and by (3) we get $D(x,0) = x$.
(iv) Let $D(x,0) = x$ for some $x \in X$. By (1) we have $D(x,0) * x = 0$ then we can write by part (i) $d(0) = D(x,0) * x = 0$ therefore $d(0) = 0$.
Hence we get that $d$ is regular.
(v) Let $x$ be an element in $X$ such that $d(y) * x = 0$ or $x * d(y) = 0$ for all $y \in X$ then by (3) we get $d(y) = x$. □

Proposition 3.5. Let $d$ be the trace of an $(r,l)$ symmetric bi-derivation of a $B$-algebra $X$. Then the followings hold:
(i) $d(0) = x * D(x,0)$ for all $x \in X$.
(ii) $d(x) = (x * D(0,x)) \land d(x)$ for all $x \in X$.
(iii) If $D(x,0) = D(y,0)$ for all $x, y \in X$, then $d$ is $1 - 1$.
(iv) If $d$ is regular, then $D(x,0) = x$.
(v) If there is an element $x \in X$ such that $D(x,0) = x$, then $d$ is regular.
(vi) If there exists $x \in X$ such that $d(y) * x = 0$ or $x * d(y) = 0$ for all $y \in X$, then $d(y) = x$.

Proof. (i) Let $x$ be an element in $X$. Since $x * x = 0$ we have
\[
d(0) = D(0,0) = D(x * x,0)
\]
\[
= (x * D(x,0)) \land (D(x,0) * x)
\]
\[
= (D(x,0) * x) \land [(D(x,0) * x) * (x * D(x,0))].
\]
By using (I) and (6),
\[
= [(D(x,0) * x) * (0 * (x * D(x,0))) * (D(x,0) * x)
\]
\[
= [(D(x,0) * x) * (D(x,0) * x)] * (D(x,0) * x)
\]
\[
= 0 * (D(x,0) * x)
\]
Therefore, $d(0) = x \ast D(x, 0)$ for all $x \in X$.

(ii) Let $x$ be an element in $X$ then we have $x \ast 0 = x$ and

$$d(x) = D(x, x) = D(x \ast 0, x) = (x \ast D(0, x)) \wedge (D(x, x) \ast 0).$$

By using (II) and (i),

$$= (x \ast D(0, x)) \wedge D(x, x) = (x \ast D(0, x)) \wedge d(x).$$

Hence we get $d(x) = (x \ast D(0, x)) \wedge d(x)$.

(iii) Let $x, y \in X$ such that $d(x) = d(y)$. Then by (i), we have $d(0) = x \ast D(x, 0)$ and $d(0) = y \ast D(y, 0)$. Thus $x \ast D(x, 0) = y \ast D(y, 0)$. Using Theorem 2.4 we have $x = y$. Hence we get $d$ is 1 − 1.

(iv) Let $d$ be a regular. By part (i) we have $d(0) = x \ast D(x, 0)$. Since $d$ is regular we have $d(0) = x \ast D(x, 0)$ and $d(0) = y \ast D(y, 0)$. Thus $x \ast D(x, 0) = y \ast D(y, 0)$. Using Theorem 2.4 we have $x = y$. Hence we get $d$ is regular.

(v) Let $D(x, 0) = x$ for some $x \in X$. By (1) we have $x \ast D(x, 0) = 0$ then we can write by part (i) $d(0) = x \ast D(x, 0) = 0$ therefore $d(0) = 0$.

Hence we get that $d$ is regular.

(vi) Let $x$ be an element in $X$ such that $d(y) \ast x = 0$ or $x \ast d(y) = 0$ for all $y \in X$ then by (3) we get $d(y) = x$. \qed

**Proposition 3.6.** Let $(X, \ast, 0)$ be a 0-commutative $B$-algebra and $d$ be the trace of an $(l, r)$ symmetric bi-derivation $D$ of $X$. Then the followings hold for all $x, y \in X$:

(i) $d(x \ast y) = (d(x) \ast y) \ast y$.

(ii) $D(x, 0) \ast D(y, 0) = x \ast y$.

**Proof.** (i) Let $x, y \in X$, then we have

$$d(x \ast y) = D(x \ast y, x \ast y) = (D(x, x \ast y) \ast y) \wedge ((x \ast D(y, x \ast y)) \wedge (D(x, x \ast y)) \ast (D(x, x \ast y) \ast y)).$$

Then by (12) we have,

$$= D(x, x \ast y) \ast y = [(D(x, x) \ast y) \wedge (x \ast D(x, y))] \ast y.$$

Then by (12) we have,

$$= (D(x, x) \ast y) \ast y.$$

Hence we get $d(x \ast y) = (d(x) \ast y) \ast y$.

(ii) If $x, y \in X$, then from Proposition 3.4(i) we can write $d(0) = D(x, 0) \ast x$ and $d(0) = D(y, 0) \ast y$. From here we get $D(y, 0) \ast y = D(x, 0) \ast x$ and we have
\((D(y,0) \ast y) \ast (D(x,0) \ast x) = 0\) and we can also write by (11) \((x \ast y) \ast (D(x,0) \ast D(y,0)) = 0\). Hence, by (3) we have \(D(x,0) \ast D(y,0) = x \ast y\).

**Proposition 3.7.** Let \((X, \ast, 0)\) be a 0-commutative B-algebra and \(d\) be the trace of an \((r, l)\) symmetric bi-derivation of \(X\). Then the followings hold for all \(x, y \in X\):

(i) \(d(x \ast y) = d(y)\).

(ii) \(d\) is constant.

(iii) \(D(y,0) \ast D(x,0) = y \ast x\).

**Proof.**

(i) 

\[
d(x \ast y) = D(x \ast y, x \ast y) \\
= (x \ast D(y, x \ast y)) \land (D(x, x \ast y) \ast y) \\
= (D(x, x \ast y) \ast y) \ast ((D(x, x \ast y) \ast y) \ast (x \ast D(y, x \ast y))) \\
= (x \ast D(y, x \ast y)) \\
= x \ast [(x \ast D(y, y)) \land (D(y, x) \ast y)] \\
= x \ast (x \ast D(y, y)) = x \ast (x \ast d(y)) \\
= d(y) \quad \text{by (12)}.
\]

Hence we get \(d(x \ast y) = d(y)\).

(ii) Let \(x \in X\). By using (11) and (ii) we have \(d(x) = d(x \ast 0) = d(0)\). Hence \(d\) is constant.

(iii) Let \(x, y \in X\), then from Proposition 3.5(i) we can write \(d(0) = x \ast D(x,0)\) and \(d(0) = y \ast D(y,0)\). From here we get \(x \ast D(x,0) = y \ast D(y,0)\) and we have \((x \ast D(x,0)) \ast (y \ast D(y,0)) = 0\) and we can also write by (13) \((D(y,0) \ast D(x,0)) \ast (y \ast x) = 0\). Hence, by (3) we have \(D(y,0) \ast D(x,0) = y \ast x\). \qed

**References**


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