STATICAL HALF LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE KAHLER MANIFOLD OF A QUASI-CONSTANT CURVATURE

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Abstract. In this paper, we study half lightlike submanifolds $M$ of an indefinite Kaehler manifold $\bar{M}$ of quasi-constant curvature such that the characteristic vector field $\zeta$ of $\bar{M}$ is tangent to $M$. First, we provide a new result for such a half lightlike submanifold. Next, we investigate a statical half lightlike submanifold $M$ of $\bar{M}$ subject such that (1) the screen distribution $S(TM)$ is totally umbilical or (2) $M$ is screen conformal.

1. Introduction

In the theory of Riemannian geometry, Chen-Yano [2] introduced the notion of a Riemannian manifold of a quasi-constant curvature as a Riemannian manifold $({\bar{M}}, g)$ equipped with a curvature tensor $\bar{R}$ of the following form:

$$\bar{R}(X, Y)Z = f_1\{g(Y, Z)X - g(X, Z)Y\}
+f_2\{\theta(Y)\theta(Z)X - \theta(X)\theta(Z)Y
+ g(Y, Z)\theta(X)\zeta - g(X, Z)\theta(Y)\zeta\}$$

for any vector fields $X, Y$ and $Z$ on $\bar{M}$, where $f_1$ and $f_2$ are smooth functions, $\zeta$ is a unit vector field which is called the characteristic vector field of $\bar{M}$, and $\theta$ is a 1-form associated with $\zeta$ by $\theta(X) = g(X, \zeta)$. It is well known that if $f_2 = 0$, then $\bar{M}$ is reduced to a space of constant curvature.

The theory of lightlike submanifolds is an important topic of research in differential geometry due to its application in mathematical physics. Half lightlike submanifold $M$ is a lightlike submanifold of codimension 2 such that $\text{rank}\{\text{Rad}(TM)\} = 1$, where $\text{Rad}(TM) = TM \cap TM^\perp$ is the radical distribution of $M$. It is a special case of general $r$-lightlike submanifold [4] such that $r = 1$. Its geometry is more general than that of lightlike hypersurface or coisotropic submanifold which is lightlike submanifolds $M$ of codimension...
2 such that \( \text{rank}\{\text{Rad}(TM)\} = 2 \). Much of its theory will be immediately
generalized in a formal way to arbitrary \( r \)-lightlike submanifolds.

In this paper, we study half lightlike submanifolds \( M \) of an indefinite Kaehler
manifold \( \bar{M} \) of quasi-constant curvature such that the characteristic vector
field \( \zeta \) of \( \bar{M} \) is tangent to \( M \). First, we provide a new result for such a half
lightlike submanifold. Next, we investigate a statical half lightlike submanifold
\( M \) of such an indefinite Kaehler manifold \( M \) subject such that (1) the screen
distribution \( S(TM) \) is totally umbilical or (2) \( M \) is screen conformal.

2. Preliminaries

Let \((M,\bar{g})\) be a half lightlike submanifold of a semi-Riemannian manifold
\((\bar{M},\bar{g})\) with the tangent bundle \( TM \), the normal bundle \( TM^\perp \), the radical
distribution \( \text{Rad}(TM) = TM \cap TM^\perp \), a screen distribution \( S(TM) \), and a
co-screen distribution \( S(TM^\perp) \) such that

\[
TM = \text{Rad}(TM) \oplus_{\text{orth}} S(TM), \quad TM^\perp = \text{Rad}(TM) \oplus_{\text{orth}} S(TM^\perp),
\]

where \( \oplus_{\text{orth}} \) denotes the orthogonal direct sum. We follow Duggal-Jin [5] for
notations and structure equations used in this article. Denote by \( H(M) \) the algebra of smooth functions on \( M \) and by \( \Gamma(E) \) the \( H(M) \) module of smooth
sections of a vector bundle \( E \). Also denote by (2.6) the first equation of
the two equations in (2.6). We use same notations for any others. Choose
\( L \in \Gamma(S(TM^\perp)) \) as a unit spacelike vector field, \( i.e., \ g(L,L) = 1 \), without loss
of generality. Consider the orthogonal complementary distribution \( S(TM^\perp) \)
to \( S(TM) \) in \( TM \), of rank 3. Certainly the vector fields \( \xi \) and \( L \) belong to
\( \Gamma(S(TM^\perp)) \). Hence we have the following orthogonal decomposition

\[
S(TM)^\perp = S(TM^\perp) \oplus_{\text{orth}} S(TM^\perp)^\perp,
\]

where \( S(TM^\perp)^\perp \) is the orthogonal complementary to \( S(TM^\perp) \) in \( S(TM^\perp) \), of
rank 2. It is known [5] that, for any null section \( \xi \) of \( \text{Rad}(TM) \), there exists a
uniquely defined null vector field \( N \) in \( S(TM^\perp)^\perp \) satisfying

\[
\bar{g}(\xi,N) = 1, \quad \bar{g}(N,N) = \bar{g}(N,X) = \bar{g}(N,L) = 0, \quad \forall X \in \Gamma(S(TM)).
\]

Denote by \( \text{ltr}(TM) \) the subbundle of \( S(TM^\perp)^\perp \) locally spanned by \( N \). We see that
\( S(TM^\perp)^\perp = \text{Rad}(TM) \oplus \text{ltr}(TM) \). Let \( \text{tr}(TM) = S(TM^\perp) \oplus_{\text{orth}} \text{ltr}(TM) \). We call \( N, \text{ltr}(TM) \) and \( \text{tr}(TM) \) the lightlike transversal vector
field, lightlike transversal vector bundle and transversal vector bundle of \( M \)
with respect to the screen distribution \( S(TM) \) respectively.

Let \( \bar{\nabla} \) be the Levi-Civita connection of \( \bar{M} \) and \( P \) the projection morphism of
\( TM \) on \( S(TM) \). Then the local Gauss-Weingarten formulas of \( M \) and \( S(TM) \)
given respectively by

\[
\begin{align*}
\bar{\nabla}_X Y &= \nabla_X Y + B(X,Y)N + D(X,Y)L, \\
\bar{\nabla}_X N &= -A_N X + \tau(X)N + \rho(X)L, \\
\bar{\nabla}_X L &= -A_L X + \phi(X)N;
\end{align*}
\]
where $\nabla$ and $\nabla^*$ are the induced connections on $TM$ and $S(TM)$ respectively. Using the local Gauss-Weingarten formulas, we have

\begin{align}
(2.12) & \quad \nabla_X P Y = \nabla_X^* P Y + C(X, PY) \xi, \\
(2.13) & \quad \nabla_X \xi = -A^*_X X - \tau(X) \xi,
\end{align}

where $\nabla$ and $\nabla^*$ are the induced connections on $TM$ and $S(TM)$ respectively. $B$ and $D$ are called the local second fundamental forms of $M$, $C$ is called the local screen second fundamental form on $S(TM)$. $A^*_X$ and $A^*_X$ are called the shape operators, and $\tau$, $\rho$ and $\phi$ are 1-forms on $TM$.

From now and in the sequel, let $X, Y, Z$ and $W$ be the vector fields on $M$, unless otherwise specified. Since $\nabla$ is irrotational, $\nabla$ is also torsion-free and the second fundamental forms $B$ and $D$ are symmetric. The above local second fundamental forms are related to their shape operators by

\begin{align}
(2.6) & \quad B(X, Y) = g(A^*_X X, Y), \quad \tilde{g}(A^*_X X, N) = 0, \\
(2.7) & \quad C(X, PY) = g(A^*_X X, PY), \quad \tilde{g}(A^*_X X, N) = 0, \\
(2.8) & \quad D(X, Y) = g(A^*_X X, Y) - \phi(X) \eta(Y), \quad \tilde{g}(A^*_X X, N) = \rho(X),
\end{align}

where $\eta$ is a 1-form given by $\eta(X) = \tilde{g}(X, N)$. From (2.6), (2.8), we get

\begin{align}
(2.9) & \quad B(X, \xi) = 0, \quad D(X, \xi) = -\phi(X).
\end{align}

$A^*_X$ and $A^*_X$ are $S(TM)$-valued, and $A^*_X$ is self-adjoint on $TM$ such that

\begin{align}
(2.10) & \quad A^*_X \xi = 0.
\end{align}

The induced connection $\nabla$ of $M$ is not metric and satisfies

\begin{align}
(2.11) & \quad (\nabla_X g)(Y, Z) = B(X, Y) \eta(Z) + B(X, Z) \eta(Y).
\end{align}

But the induced connection $\nabla^*$ on $S(TM)$ is a metric connection.

**Definition.** A half lightlike submanifold $M$ of a semi-Riemannian manifold $(\tilde{M}, \tilde{g})$ is said to be statistical [11, 12] if $\nabla_X \tilde{L} \in \Gamma(S(TM))$ for any $X \in \Gamma(TM)$.

From (2.3) and (2.8), we show that the above definition is equivalent to the conditions: $\phi = 0$ and $\rho = 0$. The condition $\phi = 0$ is equivalent to the conception: $M$ is irrotational, i.e., $\nabla_X \xi \in \Gamma(TM)$ [14]. The condition $\rho = 0$ is equivalent to the conception: $M$ is solenoidal, i.e., $A^*_X X \in \Gamma(S(TM))$ [13].

We need the following Gauss-Codazzi equations (for a full set of these equations see [5]). Denote by $\tilde{R}$, $\tilde{R}$ and $\tilde{R}^*$ the curvature tensors of $\nabla$, $\nabla$ and $\nabla^*$ respectively. Using the local Gauss-Weingarten formulas, we have

\begin{align}
(2.12) & \quad \tilde{R}(X, Y) Z = \tilde{R}(X, Y) Z + B(X, Z) A^*_Y Y - B(Y, Z) A^*_X X \\
& \quad + D(X, Z) A^*_Y Y - D(Y, Z) A^*_X X \\
& \quad + \{ (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\
& \quad + \tau(X) B(Y, Z) - \tau(Y) B(X, Z) \\
& \quad + \phi(X) D(Y, Z) - \phi(Y) D(X, Z) \} N \\
& \quad + \{ (\nabla_X D)(Y, Z) - (\nabla_Y D)(X, Z) \} + \tilde{R}(X, Z) B(Y, Z) \\
& \quad - \tilde{R}(Y) B(X, Z) \} L,
\end{align}
Denote by $\mathcal{R}$.

Due to [6], using (2.6) we get

Using (2.13) and the first Bianchi's identity, we obtain

This shows that $\mathcal{R}$ is called the induced Ricci tensor.

Let $\bar{\mathcal{R}}$ denote the induced tensor of type $(0, 2)$ on $\bar{M}$ such that

In the case $R = 0$, we say that $\bar{M}$ is flat. We set $\dim \bar{M} = n + 3$.

The Ricci tensor $\bar{\mathcal{R}}ic$ of $\bar{M}$ is defined by

Denote by $\bar{\mathcal{R}}ic^{(0, 2)}$ the induced tensor of type $(0, 2)$ on $\bar{M}$ such that

Due to [6], using (2.6)–(2.8) and the Gauss equation (2.12), we get

Using (2.13) and the first Bianchi's identity, we obtain

This shows that $\bar{\mathcal{R}}ic^{(0, 2)}$ is not symmetric. A tensor field $\bar{\mathcal{R}}ic^{(0, 2)}$ of $\bar{M}$ is called its induced Ricci tensor and denoted by $\bar{\mathcal{R}}ic$ if it is symmetric. In this case, $\bar{M}$ is called Ricci flat if $\bar{\mathcal{R}}ic = 0$. $\bar{M}$ is called an Einstein manifold if there exists a smooth function $\kappa$ such that

Let $\nabla_X N = \pi(\nabla_X N)$, where $\pi$ is the projection morphism of $TM$ on $\text{ltr}(TM)$. Then $\nabla^\ell$ is a linear connection on $\text{ltr}(TM)$. We say that $\nabla^\ell$ is a lightlike transversal connection. Define a curvature tensor $\mathcal{R}^\ell$ on $\text{ltr}(TM)$ by

$$\mathcal{R}^\ell(X, Y)N = \nabla^\ell_X \nabla^\ell_Y N - \nabla^\ell_Y \nabla^\ell_X N - \nabla^\ell_{[X,Y]}N.$$
If $R^\ell$ vanishes identically, then the lightlike transversal connection $\nabla^\ell$ is said to be flat. This definition comes from the definition of flat normal connection \[1\] in the theory of classical geometry of non-degenerate submanifolds. We quote the following result (see [9, 10]).

**Theorem 2.1.** Let $M$ be a half lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. The following assertions are equivalent:

1. The lightlike transversal connection of $M$ is flat, i.e., $R^\ell = 0$.
2. The 1-form $\tau$ is closed, i.e., $d\tau = 0$, on any $\mathcal{U} \subset M$.
3. The Ricci type tensor $R^{(\ell, 2)}$ is an induced Ricci tensor of $M$.

**Note 1.** $d\tau$ is independent to the choice of the section $\xi \in \Gamma(TM^\perp)$. Indeed, suppose $\tau$ and $\bar{\tau}$ are 1-forms with respect to the sections $\xi$ and $\bar{\xi}$, respectively. By directed calculation, it follows that $d\tau = d\bar{\tau}$ \[5\]. In case $d\tau = 0$, by the cohomology theory, there exists a smooth function $f$ such that $\tau = df$. Consequently we get $\tau(X) = X(f)$. If we take $\xi = \lambda \xi$, it follows that $\tau(X) = \bar{\tau}(X) + X(\ln \lambda)$. Setting $\lambda = \exp(f)$ in this equation, we get $\bar{\tau} = 0$. Thus if $d\tau = 0$, we can take a 1-form $\tau$ such that $\tau = 0$. We call the pair $\{\xi, N\}$ whose corresponding 1-form $\tau$ vanishes the canonical null pair of $M$.

3. Indefinite Kaehler manifolds

Let $\bar{M} = (\bar{M}, J, \bar{g})$ be a real even dimensional indefinite Kaehler manifold, where $\bar{g}$ is a semi-Riemannian metric of index $q = 2v$, $0 < v < \frac{1}{2}(\dim \bar{M})$, and $J$ is an almost complex structure on $\bar{M}$ such that, for all $X, Y \in \Gamma(TM)$,

\[(3.1) \quad J^2 = -I, \quad \bar{g}(JX, JY) = \bar{g}(X, Y), \quad (\bar{\nabla}_X J)Y = 0.\]

Let $(M, g)$ be a half lightlike submanifold of an indefinite Kaeler manifold $\bar{M}$. Due to [7, 8], we choose a screen distribution $S(TM)$ such that $J(\text{Rad}(TM))$, $J(\text{ltr}(TM))$, and $J(S(TM^\perp))$ are vector subbundles of $S(TM)$. In this case, the screen distribution $S(TM)$ is expressed as follow:

\[S(TM) = \{J(\text{Rad}(TM)) \oplus J(\text{ltr}(TM))\} \oplus_{\text{orth}} J(S(TM^\perp)) \oplus_{\text{orth}} H_\alpha,\]

where $H_\alpha$ is a non-degenerate almost complex distribution with respect to $J$, i.e., $J(H_\alpha) = H_\alpha$. Denote $H' = J(\text{ltr}(TM)) \oplus_{\text{orth}} J(S(TM^\perp))$. Then

\[(3.2) \quad TM = H \oplus H',\]

where $H$ is a 2-lightlike almost complex distribution on $M$ such that

\[H = \text{Rad}(TM) \oplus_{\text{orth}} J(\text{Rad}(TM)) \oplus_{\text{orth}} H_\alpha.\]

Consider two lightlike and one spacelike vector fields $\{U, V\}$ and $W$ such that

\[(3.3) \quad U = -JN, \quad V = -J\xi, \quad W = -JL.\]

Denote by $S$ the projection morphism of $TM$ on $H$. By (3.2), for any vector field $X$ on $M$, $JX$ is expressed as follow

\[(3.4) \quad JX = FX + u(X)N + w(X)L,\]
where $u, v$ and $w$ are 1-forms locally defined on $M$ by
\begin{equation}
(3.5) \quad u(X) = g(X, V), \quad v(X) = g(X, U), \quad w(X) = g(X, W)
\end{equation}
and $F$ is a tensor field of type $(1, 1)$ globally defined on $M$ by $F = J \circ S$.
Applying $\nabla_X$ to (3.3) and using the Gauss-Weingarten formulas, we have
\begin{equation}
(3.6) \quad B(X, U) = C(X, V), \quad C(X, W) = D(X, U), \quad B(X, W) = D(X, V),
\end{equation}
\begin{equation}
(3.7) \quad \nabla_X U = F(A_\xi X) + \tau(X)U + \rho(X)W,
\end{equation}
\begin{equation}
(3.8) \quad \nabla_X V = F(A_\zeta X) - \tau(X)V - \phi(X)W,
\end{equation}
\begin{equation}
(3.9) \quad \nabla_X W = F(A_\alpha X) + \phi(X)U.
\end{equation}

**Theorem 3.1.** Let $M$ be a half lightlike submanifold of an indefinite Kaehler manifold $\tilde{M}$ of quasi-constant curvature such that $\zeta$ is tangent to $M$. Then
\begin{align*}
f_1 &= 0, \\
f_2 \theta(V) &= f_2 \theta(W) = 0, \\
f_2 \alpha &= 0.
\end{align*}
\begin{proof}
Comparing the tangential, lightlike transversal and co-screen components of the two equations (1.1) and (2.12), we get
\begin{align*}
(3.10) \quad R(X, Y)Z &= f_1(\bar{g}(Y, Z)X - \bar{g}(X, Z)Y) \\
&\quad + f_2(\bar{g}(Y, Z)\theta(X)\zeta - \bar{g}(X, Z)\theta(Y)\zeta) \\
&\quad + \theta(Y)\theta(Z)X - \theta(X)\theta(Z)Y
\end{align*}
\begin{align*}
&\quad + B(Y, Z)A_\alpha X - B(X, Z)A_\alpha Y \\
&\quad + D(Y, Z)A_\alpha X - D(X, Z)A_\alpha Y,
\end{align*}
\begin{align*}
(3.11) \quad (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) - \tau(Y)B(X, Z) \\
&\quad + \phi(X)D(Y, Z) - \phi(Y)D(X, Z) = 0,
\end{align*}
\begin{align*}
(3.12) \quad (\nabla_X D)(Y, Z) - (\nabla_Y D)(X, Z) + \rho(X)B(Y, Z) - \rho(Y)B(X, Z) = 0.
\end{align*}
Taking the scalar product with $N$ to (2.14), we have
\begin{align*}
g(R(X, Y)PZ, N) &= (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\
&\quad - \tau(X)C(Y, PZ) + \tau(Y)C(X, PZ).
\end{align*}
Substituting (3.10) into this equation and using (2.7)$_2$ and (2.8)$_2$, we obtain
\begin{align*}
(3.13) \quad (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\
&\quad - \tau(X)C(Y, PZ) + \tau(Y)C(X, PZ) \\
&\quad - \rho(X)D(Y, PZ) + \rho(Y)D(X, PZ) \\
&\quad = f_1\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\} \\
&\quad + f_2\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(PZ) \\
&\quad + \alpha f_2\{\theta(X)g(Y, PZ) - \theta(Y)g(X, PZ)\}.
\end{align*}
Applying $\nabla_X$ to (3.6)$_1$: $B(Y, U) = C(Y, V)$, we have
\begin{align*}
(\nabla_X B)(Y, U) &= (\nabla_X C)(Y, V) + g(A_\alpha Y, \nabla_X V) - g(A_\alpha Y, \nabla_X U).
\end{align*}
Using (3.1), (3.4) and (3.6)–(3.8), the last equation is reduced to
\[
(\nabla_X B)(Y, U) = (\nabla_X C)(Y, V) - 2\tau(X)C(Y, V) - \phi(X)D(Y, U) - \rho(X)D(Y, V) - g(A_\xi^2 X, F(A_\eta Y)) - g(A_\xi^2 Y, F(A_\eta X)).
\]
Substituting this equation into (3.11) such that \(Z = U\) and using (3.7), we get
\[
(\nabla_X C)(Y, V) - (\nabla_Y C)(X, V) - \tau(X)C(Y, V) + \tau(Y)C(X, V) - \rho(X)D(Y, V) + \rho(Y)D(X, V) = 0.
\]
Comparing this equation with (3.13) such that \(PZ = V\), we obtain
\[
(3.14) \quad f_1\{\eta(X)u(Y) - \eta(Y)u(X)\} + f_2\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(V) + f_2\alpha\{\theta(X)u(Y) - \theta(Y)u(X)\} = 0.
\]
Replacing \(Y\) by \(\xi\) to this equation and using the fact that \(\theta(\xi) = 0\), we have
\[
f_1u(X) + f_2\theta(X)\theta(V) = 0.
\]
Taking \(X = V\) and \(X = U\) to this equation by turns, we get
\[
\begin{align*}
f_2\theta(V) &= 0, \\
f_1 + f_2\theta(U)\theta(V) &= 0.
\end{align*}
\]
From these two equations, we see that \(f_1 = 0\). Taking \(X = \zeta\) and \(Y = U\) to (3.14) and using the facts that \(u(\zeta) = \theta(V)\) and \(f_2\theta(V) = 0\), we have \(f_2\alpha = 0\).

Applying \(\nabla_X\) to (3.6)\quad \(D(Y, U) = C(Y, W)\), and using (2.7), (2.8) and (3.7), we have
\[
(\nabla_X D)(Y, U) = (\nabla_X C)(Y, W) + g(A_\eta Y, \nabla_X W) - g(A_\xi Y, \nabla_X U) + \phi(Y)C(X, U).
\]
Using (2.8)\quad (3.1), (3.4), (3.6), (3.7) and (3.9), we have
\[
(\nabla_X D)(Y, U) = (\nabla_X C)(Y, W) - \tau(X)C(Y, W) - \rho(X)D(Y, W) - \rho(X)B(Y, U) + \phi(X)C(Y, U) + \phi(Y)C(X, U) - g(A_\xi X, F(A_\eta Y)) - g(A_\xi Y, F(A_\eta X)).
\]
Substituting this equation into (3.11) such that \(Z = U\) and using (3.7), we get
\[
(\nabla_X C)(Y, W) - (\nabla_Y C)(X, W) - \tau(X)C(Y, W) + \tau(Y)C(X, W) - \rho(X)D(Y, W) + \rho(Y)D(X, W) = 0.
\]
Comparing this equation with (3.13) such that \(PZ = W\), we obtain
\[
f_2\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(W) = 0.
\]
Taking \(Y = \zeta\) and \(Y = \xi\) to this equation, we get \(f_2\theta(W) = 0\). \(\Box\)
4. Totally umbilical screen distribution

If $\bar{M}$ is an indefinite Kaehler manifold of quasi-constant curvature, using (1.1) and the fact that $f_1 = 0$, we see that $\bar{g}(R(\xi, X)Y, N) = f_2\theta(X)\theta(Y)$, $\bar{g}(\bar{R}(\xi, X)Y, N) = f_2\theta(X)\theta(Y)$ and $\bar{Ric}(X, Y) = f_2\{g(X, Y) + (n+1)\theta(X)\theta(Y)\}$. Thus (2.17) is reduced to

\[
R^{(0,2)}(X, Y) = f_2\{g(X, Y) + (n-1)\theta(X)\theta(Y)\} + \rho(X)\phi(Y)
+ B(X, Y)tr A_\perp + D(X, Y)tr A_\perp
- g(A_\perp X, A_\perp Y) - g(A_\perp X, A_\perp Y).
\]

**Definition.** A screen distribution $S(TM)$ is called totally umbilical [4, 5] in $M$ if there exists a smooth function $\gamma$ such that $A_\perp X = \gamma PX$, or equivalently,

\[
C(X, PY) = \gamma g(X, Y).
\]

In case $\gamma = 0$, we say that $S(TM)$ is totally geodesic in $M$.

**Note 2.** If $M$ is irrotational and $S(TM)$ is totally umbilical, then (4.1) reduces

\[
R^{(0,2)}(X, Y) = f_2\{g(X, Y) + (n-1)\theta(X)\theta(Y)\}
+ B(X, Y)tr A_\perp + D(X, Y)tr A_\perp
- \gamma g(X, A_\perp Y) - g(A_\perp X, A_\perp Y).
\]

As $A_\perp$ is self-adjoint, it follows that $R^{(0,2)}$ is symmetric, i.e., $R^{(0,2)}$ is the induced Ricci tensor $\bar{Ric}$ of $M$. Therefore, $d\tau = 0$ and the transversal connection is flat by Theorem 2.1. As $d\tau = 0$, we can take $\tau = 0$ by Note 1.

**Theorem 4.1.** Let $M$ be a tactial half lightlike submanifold of an indefinite Kaehler manifold $M$ of a quasi-constant curvature such that $\zeta$ is tangent to $M$. If $S(TM)$ is totally umbilical, then we have the following results:

1. $S(TM)$ is totally geodesic and parallel distribution,
2. $M$ is locally a product manifold $C_\zeta \times M^*$, where $C_\zeta$ is a null geodesic tangent to $TM^\perp$, and $M^*$ is a leaf of $S(TM)$,
3. $f_1 = f_2 = 0$, i.e., $M$ is flat, and the curvature tensor $R$ is given by

\[
R(X, Y)Z = D(Y, Z)A_\perp X - D(X, Z)A_\perp Y.
\]
4. Moreover, if $M$ is an Einstein manifold, then $M$ is Ricci flat.

**Proof.** As $M$ is statical, the 1-forms $\phi$ and $\rho$ are satisfied $\phi = \rho = 0$. Applying $\nabla_X$ to $C(Y, PZ) = \gamma g(Y, PZ)$ and using (2.11), we have

\[
(\nabla_X C)(Y, PZ) = (X\gamma)g(Y, PZ) + \gamma B(X, PZ)\eta(Y).
\]

Substituting this and (4.2) into (3.13) with $f_1 = f_2\alpha = \rho = 0$, we obtain

\[
(X\gamma)g(Y, Z) - (\gamma g(X, Z) + \gamma (B(X, Z)\eta(Y) - B(Y, Z)\eta(X))
= f_2\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(Z).
\]
Taking \( Y = U, Z = V \) and \( Y = V, Z = U \) to (4.4) by turns and using (3.9), (4.2) and the facts that \( f_2 \theta(V) = 0 \) and \( \eta(V) = 0 \), we obtain
\[
X \gamma = (U \gamma) u(X), \quad X \gamma = (V \gamma) v(X).
\]
From these equations, we get \( X \gamma = 0 \). Thus \( \gamma \) is a constant. (4.4) reduces
\[
\{ \gamma B(X, Z) + f_2 \theta(X) \theta(Z) \} \eta(Y) = \{ \gamma B(Y, Z) + f_2 \theta(Y) \theta(Z) \} \eta(X).
\]
Taking \( Y = \xi \) to this equation and using (2.9), we have
\[
(4.5) \quad \gamma B(X, Y) = -f_2 \theta(X) \theta(Y).
\]
Taking \( Y = U \) to this equation and using (3.3), (3.5) and (4.2), we have
\[
(4.6) \quad \gamma^2 u(X) = -f_2 \theta(X) \theta(U).
\]
Assume that \( f_2 \neq 0 \). Taking \( X = \zeta \) to (4.6), we have
\[
\gamma^2 \theta(V) = -f_2 \theta(U).
\]
As \( f_2 \neq 0 \), if we product with \( f_2 \) to the last equation and use the fact that \( f_2 \theta(V) = 0 \), then we obtain \( f_2 \theta(U) = 0 \). Taking \( X = U \) to (4.6) and using the fact that \( f_2 \theta(U) = 0 \), we get \( \gamma = 0 \). As \( \gamma = 0 \), taking \( X = Y = \zeta \) to (4.5), we have \( f_2 = 0 \). It is a contradiction. Therefore, \( f_2 = 0 \). As \( f_2 = 0 \), from (4.6), we obtain \( \gamma = 0 \).

(1) As \( \gamma = C = 0 \), \( S(TM) \) is totally geodesic and, from (2.4) we see that \( S(TM) \) is a parallel distribution.

(2) As \( S(TM) \) is a parallel distribution, \( \text{Rad}(TM) \) is also an auto-parallel distribution due to (2.5) and (2.10). As \( TM = \text{Rad}(TM) \oplus S(TM) \), by the decomposition theorem of de Rham [3], \( M \) is locally a product manifold \( \mathbb{C} \times M^* \), where \( \mathbb{C} \) is a null geodesic tangent to \( \text{Rad}(TM) \) and \( M^* \) is a leaf of \( S(TM) \).

(3) As \( f_1 = f_2 = 0 \), \( M \) is flat. As \( f_1 = f_2 = A_{\zeta} = 0 \), from (3.10), \( R \) is given by
\[
R(X, Y)Z = D(Y, Z)A_\zeta X - D(X, Z)A_\zeta Y.
\]

(4) As \( C = 0 \), using (2.6), (2.8) and (3.6)1,2, we have
\[
B(X, U) = 0, \quad D(X, U) = 0, \quad A_\zeta^2 U = 0, \quad A_\zeta X = 0.
\]
Substituting (2.18) into (4.1) such that \( f_2 = 0 \) and \( Y = U \) and then, using the last equations, we obtain \( \kappa = 0 \). Therefore, \( M \) is Ricci flat. \( \Box \)

**Theorem 4.2.** Let \( M \) be an Einstein statical half lightlike submanifold of an indefinite Kaehler manifold \( M \) of a quasi-constant curvature such that \( \zeta \) is tangent to \( M \). If \( S(TM) \) is totally umbilical, then \( M \) is Ricci flat.

**Proof.** As \( C = 0 \), from (3.6)2 and the facts that \( \phi = 0 \) and \( \rho = 0 \), we obtain
\[
(4.7) \quad D(X, U) = 0, \quad A_\zeta U = 0.
\]
As \( f_2 = \gamma = A_{\zeta} = 0 \), from (4.3), the induced Ricci tensor \( R^{(0, 2)} \) is given by
\[
(4.8) \quad R^{(0, 2)}(X, Y) = D(X, Y)tr A_\zeta - g(A_\zeta X, A_\zeta Y),
\]
where $\ell = \text{tr } A_L$. As $M$ is Einstein, substituting (2.18) into (4.8), we have
\[
g(A_L X, A_L Y) - \ell g(A_L X, Y) + \kappa g(X, Y) = 0.
\]
Taking $X = U$ and $Y = V$ to this equation and using (4.7), we obtain $\kappa = 0$. Therefore $M$ is Ricci flat.

Denote by $G = J(\text{Rad}(TM)) \oplus \text{orth} J(S(TM)) \oplus \text{orth} H_0$. Then $G$ is a complementary vector subbundle to $J(\text{ltr}(TM))$ in $S(TM)$ and we have
\[
S(TM) = J(\text{ltr}(TM)) \oplus G.
\]

Theorem 4.3. Let $M$ be a statical half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of quasi-constant curvature such that $\zeta$ is tangent to $M$. If $S(TM)$ is totally umbilical, then $M$ is locally a product manifold $C_\xi \times C_U \times M^\#$, where $C_\xi$ and $C_U$ are null geodesics tangent to $\text{Rad}(TM)$ and $J(\text{ltr}(TM))$ respectively and $M^\#$ is a leaf of $G$.

Proof. As $M$ is statical and $S(TM)$ is totally umbilical, we have
\[
(4.9) \quad \nabla_X U = 0,
\]
due to $A_N = \tau = \rho = 0$. Thus $J(\text{ltr}(TM))$ is a parallel distribution on $M$. From (2.5) and (2.10), $\text{Rad}(TM)$ is also a parallel distribution on $M$. Using (4.9), we derive
\[
g(\nabla_X Y, U) = 0, \quad g(\nabla_X V, U) = 0, \quad g(\nabla_X W, U) = 0,
\]
for all $X \in \Gamma(G)$ and $Y \in \Gamma(H_0)$. Thus $G$ is also parallel. By the decomposition theorem of de Rham [3], $M$ is locally a product manifold $C_\xi \times C_U \times M^\#$, where $C_\xi$ and $C_U$ are null geodesics tangent to $\text{Rad}(TM)$ and $J(\text{ltr}(TM))$ respectively and $M^\#$ is a leaf of $G$. \qed

5. Screen conformal lightlike hypersurfaces

Definition. A half lightlike submanifold $M$ is called screen conformal [6, 7] if there exists a non-vanishing function $\varphi$ such that $A_N = \varphi A_L^*$, or equivalently,
\[
C(X, PY) = \varphi B(X, Y).
\]
If $\varphi$ is a non-zero constant, then we say that $M$ is screen homothetic.

Note 3. If $M$ is irrotational and screen conformal, then (4.1) is reduced to
\[
R^{(0,2)}(X, Y) = f_2\{g(X, Y) + (n-1)\theta(X)\theta(Y)\}
+ B(X, Y)\text{tr } A_N + D(X, Y)\text{tr } A_L
- \varphi g(A_L^2 X, A_L^2 Y) - g(A_L X, A_L Y).
\]
Thus $R^{(0,2)}$ is symmetric, $d\tau = 0$ and the transversal connection is flat by Theorem 2.1. In this section, since $d\tau = 0$, we also take $\tau = 0$ as Section 4.
Proposition 5.1. Let $M$ be a half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of a quasi-constant curvature such that $\zeta$ is tangent to $M$. If $M$ is irrotational and screen conformal, then the curvature function $f_2$ is satisfied $f_2\theta(U) = 0$. Moreover, $M$ is statical and screen homothetic, then $f_2 = 0$.

Proof. Applying $\nabla_X$ to $C(Y, PZ) = \varphi B(Y, PZ)$, we have

$$(\nabla_X C)(Y, PZ) = (X \varphi) B(Y, PZ) + \varphi(\nabla_X B)(Y, PZ).$$

Substituting this into (3.13) such that $\tau = 0$ and using (3.11), we obtain

$$(X \varphi) B(Y, PZ) - (Y \varphi) B(X, PZ) - \rho(X) D(Y, PZ) + \rho(Y) D(X, PZ) = f_2\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(PZ).$$

Replacing $Y$ by $\xi$ to this and using (2.9) and the fact that $\theta(\xi) = 0$, we get

$$(\xi \varphi) B(X, Y) - \rho(\xi) D(X, Y) = f_2\theta(X)\theta(Y).$$

Taking $Y = V$ to (5.3) and using (3.6) and the fact that $f_2\theta(V) = 0$, we have

$$(\xi \varphi) B(X, V) - \rho(\xi) B(X, W) = 0.$$ 

Replacing $Y$ by $U$ to (5.3) and using (3.6) and the fact that $f_2\theta(U) = 0$, we have

$$(\xi \varphi) C(X, V) - \rho(\xi) C(X, W) = f_2\theta(U)\theta(U).$$

From the last two equations and (5.1), we obtain $f_2\theta(X)\theta(U) = 0$. Replacing $X$ by $\zeta$, we get $f_2\theta(U) = 0$. If $M$ is statical and screen homothetic, then $\xi \varphi = 0$ and $\rho(\xi) = 0$. Therefore, taking $X = Y = \zeta$ to (5.3), we get $f_2 = 0$. \qed

Theorem 5.2. Let $M$ be an Einstein half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of a quasi-constant curvature such that $\zeta$ is tangent to $M$. If $M$ is irrotational and screen conformal, then the function $\kappa$, given by (2.18), is satisfied $\kappa = f_2$. Moreover, $M$ is statical and screen homothetic, then it is Ricci flat, i.e., $\kappa = 0$.

Proof. As $\{U, V\}$ is a null basis of $J(Rad(TM)) \oplus J(ltr(TM))$, the vector fields $\mu = U - \varphi V, \nu = U + \varphi V$

form an orthogonal basis of $J(Rad(TM)) \oplus J(ltr(TM))$. From (3.5) and (5.1), we obtain

$$B(X, \mu) = 0, \quad A_\mu^\nu \mu = 0.$$ 

From (2.8), (3.6) and the fact that $\phi = 0$, we also obtain

$$D(X, \mu) = 0, \quad A_\nu^\mu \mu = \rho(\mu)\xi.$$ 

As $f_2\theta(V) = 0$ and $f_2\theta(U) = 0$, we also have

$$f_2\theta(\mu) = 0, \quad f_2\theta(\nu) = 0.$$ 

Taking $X = Y = \mu$ to (5.2) and using (5.4) $\sim$ (5.6), we have $\kappa = f_2$. If $M$ is statical and screen homothetic, then $\kappa = 0$ as $f_2 = 0$. \qed
Let $\mathcal{H}' = \text{Span}\{\mu\}$. Then $\mathcal{H} = H_o \oplus_{\text{orth}} \text{Span}\{\nu, W\}$ is a complementary vector subbundle to $\mathcal{H}'$ in $S(TM)$ and we have the following decomposition

$$(5.7) \quad S(TM) = \mathcal{H}' \oplus_{\text{orth}} \mathcal{H}.$$ 

**Theorem 5.3.** Let $M$ be a statical and screen homothetic half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of quasi-constant curvature such that $\zeta$ is tangent to $M$. Then $M$ is locally a product manifold $\mathcal{C}_\xi \times \mathcal{C}_\mu \times M^3$, where $\mathcal{C}_\xi$ and $\mathcal{C}_\mu$ are null and non-null geodesics tangent to $\text{Rad}(TM)$ and $\mathcal{H}'$, respectively and $M^3$ is a leaf of $\mathcal{H}$. 

**Proof.** As $M$ is statical and screen homothetic, using (3.7), (3.8) and the fact that $F$ is linear operator, we have

$$(5.8) \quad \nabla_X \mu = 0.$$ 

This implies that $\mathcal{H}'$ is a parallel distribution on $M$. From (2.5) and (2.10), $\text{Rad}(TM)$ is also a parallel distribution on $M$. Using (5.8), we derive

$$g(\nabla_X Y, \mu) = 0, \quad g(\nabla_X \nu, \mu) = 0, \quad g(\nabla_X W, \mu) = 0,$$

for all $X \in \Gamma(\mathcal{H})$ and $Y \in \Gamma(H_o)$. Thus $\mathcal{H}$ is also parallel. By the decomposition theorem of de Rham [3], $M$ is locally a product manifold $\mathcal{C}_\xi \times \mathcal{C}_\mu \times M^3$, where $\mathcal{C}_\xi$ and $\mathcal{C}_\mu$ are null and non-null geodesics tangent to $\text{Rad}(TM)$ and $\mathcal{H}'$, respectively and $M^3$ is a leaf of $\mathcal{H}$. □

**References**


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