ON STABLE MINIMAL SURFACES IN THREE DIMENSIONAL
MANIFOLDS OF NONNEGATIVE SCALAR CURVATURE

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1. Introduction
The following is the basic problem about the stability in Riemannian
Geometry; given a Riemannian manifold N, find all stable complete
minimal submanifolds of N. As answers of this problem, do Carmo-
Peng [1] and Fischer-Colbrie and Schoen [3] showed that the stable
minimal surfaces in $R^3$ are planes and Schoen-Yau [5] and Fischer-
Colbrie and Schoen [3] gave a solution for the case where the ambient
space is a three dimensional manifold with nonnegative scalar curvature.
In this paper we will remove the assumption of finite absolute total
curvature in [3, Theorem 3].

2. The main result
We prove the following theorem.

THEOREM. Let N be a complete oriented 3–manifold of nonnegative
scalar curvature S. Let M be an oriented complete, stable minimal sur-
face in N. If M is noncompact, then M is conformally equivalent to the
complex plane C or the cylinder A. If the latter case occurs, then M is
flat and totally geodesic and the scalar curvature of N is zero along M.
Therefore if the scalar curvature of N is everywhere positive, then M
cannot be a cylinder.

If the Ricci curvature of N is nonnegative, then M is conformally
equivalent to the complex plane C or M is a flat and totally geodesic
cylinder.

In order to prove this theorem, we need following lemmas.

LEMMA 1. If $u$ is a positive function on M satisfying $\Delta u + (S - K +
\frac{1}{2} \|B\|^2)u = 0$ on M, where $K$ is the Gaussian curvature of M and $\|B\|^2$

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is the square of the length of the second fundamental form of \( M \), then 
\( d\bar{s}^2 = u^2 ds^2 \) is a complete metric on \( M \) with nonnegative Gaussian curvature where \( ds^2 \) is the original metric on \( M \).

**Proof.** Use the method of the proof in [2, Theorem 2.1].

**Lemma 2([4]).** Let \( M \) be a finitely connected, open Riemann surface on which a complete conformal metric \( e^{\varphi(x)} |dz| \) is defined and \( K \) a Gaussian curvature of \( M \). Suppose that either \( \int_M K^- dv < \infty \) or \( \int_M K^+ dv < \infty \) where \( K^+(x) = \max \{ K(x), 0 \} \), \( K^-(x) = -\min \{ K(x), 0 \} \) and \( dv \) is the volume element of \( M \). Then \( \int_M K dv \leq 2\pi \kappa(M) \) where \( \kappa(M) \) denotes the Euler–Poincaré characteristic.

**Remark 3.** This is a result of S. Cohn–Vossen in the extended form.

The results of Theorem are proved in [3, Theorem 3] with the exception of the assertion that if \( M \) is conformally equivalent to the cylinder \( A \), then \( M \) is flat and totally geodesic and the scalar curvature of \( N \) is zero along \( M \). (This follows from [3, Theorem 3] only if \( M \) is assumed to have finite absolute total curvature).

**Proof of Theorem.** Let \( M \) be conformally equivalent to a cylinder. Since \( M \) is stable, there exists a positive solution \( u \) on \( M \) satisfying

\[
\Delta u + (S - K + \frac{1}{2} \|B\|^2) u = 0.
\]

Then by Lemma 1, \( d\bar{s}^2 = u^2 ds^2 \) is a complete metric on \( M \) with nonnegative Gaussian curvature \( \bar{K} \). Since \( M \) is conformally equivalent to a cylinder, the Euler–Poincare characteristic \( \kappa(M) = 0 \). By Lemma 2, \( \bar{K} \equiv 0 \) on \( M \). Hence \( -u \Delta u - \|\nabla u\|^2 \equiv 0 \). So \( S + \frac{1}{2} \|B\|^2 + \|\nabla u\|^2 \equiv 0 \) on \( M \). Since \( S \geq 0 \) and \( u \) satisfies \( \Delta u + (S - K + \frac{1}{2} \|B\|^2) u = 0 \), we have \( S \equiv 0 \), \( \|B\|^2 \equiv 0 \) and \( K \equiv 0 \) on \( M \). Hence \( M \) is flat and totally geodesic and the scalar curvature of \( N \) is zero along \( M \). This completes the proof.

**Remark 4.** Schoen and Yau [6] proved that the cylinder is totally geodesic without assumption of finite absolute total curvature.
References


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