REMARKS ON SOME VARIATIONAL INEQUALITIES

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1. Introduction and Preliminaries

This is a continuation of the author's previous work [17]. In this paper, we consider mainly variational inequalities for single-valued functions.

We first obtain a generalization of the variational type inequality of Juberg and Karamardian [10] and apply it to obtain strengthened versions of the Hartman-Stampacchia inequality and the Brouwer fixed point theorem. Next, we obtain fairly general versions of Browder's variational inequality [5] and its subsequent generalizations due to Brezis et al. [4], Takahashi [23], Shih and Tan [19], Simons [20], and others. Finally, in this paper, we obtain a variational inequality for non-real locally convex t.v.s. which generalizes a result of Shih and Tan [19].

For terminology and notations, we follow [17]. For a subset $X$ of a vector space $E$ and $x \in E$, the inward and outward sets of $X$ at $x$, $I_K(x)$ and $O_K(x)$, are defined as follows:

$$I_X(x) = \{ x + r(u-x) \in E : u \in X, \ r > 0 \},$$
$$O_X(x) = \{ x - r(u-x) \in E : u \in X, \ r > 0 \}.$$

We begin with the following form of [17, Theorem 1], which can be deduced from a generalized Fan-Browder fixed point theorem in [15], [16] as in [17].

**Theorem 0.** Let $X$ be a convex space, $p, q : X \times X \rightarrow \mathbb{R} \cup \{ +\infty \}$ and $h : X \rightarrow \mathbb{R} \cup \{ +\infty \}$ functions satisfying

(i) $q(x, y) \leq p(x, y)$ for $(x, y) \in X \times X$ and $p(x, x) \leq 0$ for all $x \in X$;

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(ii) for each $y \in X$, $\{x \in X : p(x,y) + h(y) > h(x)\}$ is convex or empty;

(iii) for each $x \in X$, $\{y \in X : q(x,y) + h(y) > h(x)\}$ is compactly open; and

(iv) there exist a nonempty compact subset $K$ of $X$ and, for each finite subset $N$ of $X$, a compact convex subset $L_N$ of $X$ containing $N$ such that $y \in L_N \setminus K$ implies $q(x,y) + h(y) > h(x)$ for some $x \in L_N$.

Then there exists a point $y_0 \in K$ such that

$$q(x,y_0) + h(y_0) \leq h(x) \quad \text{for all } x \in X.$$ 

Moreover, the set of all such solutions $y_0$ is a compact subset of $K$.

2. Main results

Let $E$ be a real vector space, $F$ a nonempty set, and $\langle \cdot , \cdot \rangle : E \times F \to \mathbb{R}$ a real-valued function which is linear in the first variable in the sense: for each given $y \in F$, $\langle \cdot , y \rangle$ maps $E$ linearly into $\mathbb{R}$.

**Theorem 1.** Let $X$ be a convex space in $E$, $h : X \to \mathbb{R} \cup \{+\infty\}$ and $f, g : X \to F$ functions satisfying

(i) $\langle x - y, gy \rangle \leq \langle x - y, fy \rangle$ for $(x,y) \in X \times X$;

(ii) for each $y \in X$, $\{x \in X : \langle x - y, fy \rangle + h(y) > h(x)\}$ is convex or empty;

(iii) for each $x \in X$, $\{y \in X : \langle x - y, gy \rangle + h(y) > h(x)\}$ is compactly open; and

(iv) there exist a nonempty compact subset $K$ of $X$ and, for each finite subset $N$ of $X$, a compact convex subset $L_N$ of $X$ containing $N$ such that $y \in L_N \setminus K$ implies $(x - y, gy) + h(y) > h(x)$ for some $x \in L_N$.

Then there exists a $y_0 \in K$ such that

$$\langle x - y_0, gy_0 \rangle + h(y_0) \leq h(x) \quad \text{for all } x \in X.$$ 

Moreover, if $h : E \to \mathbb{R} \cup \{+\infty\}$ is convex, then the inequality holds for all $x \in I_X(y_0)$. 

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Proof. Putting \( p(x, y) \equiv (x - y, fy) \) and \( q(x, y) \equiv (x - y, gy) \) in Theorem 0, we have a \( y_0 \in K \) satisfying
\[
(x - y_0, gy_0) + h(y_0) \leq h(x) \quad \text{for all} \quad x \in X.
\]
Moreover, suppose that \( h : E \to \mathbb{R} \cup \{+\infty\} \) is convex. If \( x \in I_X(y_0) \setminus X \), then there exist \( u \in X \) and \( r > 1 \) such that \( x = y_0 + r(u - y_0) \). Hence
\[
u - y_0 = \frac{1}{r}(x - y_0) \quad \text{and} \quad u = \frac{1}{r}x + (1 - \frac{1}{r})y_0 \in X.
\]
Since \( (u - y_0, gy_0) + h(y_0) \leq h(u) \), we have
\[
\frac{1}{r}(x - y_0, gy_0) + h(y_0) \leq h(u) \leq \frac{1}{r}h(x) + (1 - \frac{1}{r})h(y_0)
\]
or
\[
(x - y_0, gy_0) + h(y_0) \leq h(x) \quad \text{for all} \quad x \in I_X(y_0).
\]
This completes our proof.

**Corollary 1.1.** Let \( X \) be a convex space in \( E, h : X \to \mathbb{R} \cup \{+\infty\} \) a l.s.c. convex function, and \( f : X \to F \) a function such that
(a) for each \( x \in X \), \( y \mapsto (x - y, fy) \) is l.s.c. on compact subsets of \( X \), and
(b) the condition (iv) of Theorem 1 holds with \( f \equiv g \).

Then there exists a \( y_0 \in K \) such that
\[
(x - y_0, fy_0) + h(y_0) \leq h(x) \quad \text{for all} \quad x \in X.
\]
Moreover, if \( h : E \to \mathbb{R} \cup \{+\infty\} \) is a convex function which is l.s.c. on \( X \), then the inequality holds for all \( x \in I_X(y_0) \).

Proof. We use Theorem 1 with \( f \equiv g \). Since, for each \( y \in X \), \( x \mapsto (x - y, fy) \) is linear and \( x \mapsto h(x) \) is convex, the set \( \{x \in X : (x - y, fy) + h(y) > h(x)\} \) is convex or empty. This shows that the condition (ii) in Theorem 1 holds. Since \( h \) is l.s.c., the condition (a) implies (iii). Therefore, by Theorem 1, the conclusion follows.

For \( h \equiv 0 \), we have the following:
**Corollary 1.2.** Let $X$ be a convex space in $E$, and $f : X \to F$ a function.

(1) If, for each $x \in X$, $y \mapsto \langle x - y, fy \rangle$ is l.s.c. on compact subsets of $X$, and if there exist $K$ and $L_N$ as in (iv) of Theorem 0 such that $y \in L_N \setminus K$ implies $\langle x - y, fy \rangle > 0$ for some $x \in L_N$, then there exists a $y_0 \in K$ such that

$$\langle x - y_0, fy_0 \rangle \leq 0 \quad \text{for all} \quad x \in I_X(y_0).$$

(2) If, for each $x \in X$, $y \mapsto \langle y - x, fy \rangle$ is l.s.c. on compact subsets of $X$, and if there exist $K$ and $L_N$ as in (iv) of Theorem 0 with $y \in L_N \setminus K$ implies $\langle y - x, fy \rangle > 0$ for some $x \in L_N$, then there exists a $y_0 \in K$ such that

$$\langle x - y_0, fy_0 \rangle \leq 0 \quad \text{for all} \quad x \in O_X(y_0).$$

**Proof.** The case (1) is a direct consequence of Corollary 1.1 with $h \equiv 0$.

For (2), considering $\langle y - x, fy \rangle$ instead of $\langle x - y, fy \rangle$ in (1), we obtain a $y_0 \in K$ such that

$$\langle y_0 - x', fy_0 \rangle \leq 0 \quad \text{for all} \quad x' \in I_X(y_0).$$

For any $x \in O_X(y_0)$, let $x' = 2y_0 - x \in I_X(y_0)$. Then

$$\langle x - y_0, fy_0 \rangle \leq 0 \quad \text{for all} \quad x \in O_X(y_0).$$

**Remarks.**

1. If $E$ is a t.v.s. and if $x \mapsto \langle x, y \rangle$ is continuous on $E$ for each fixed $y \in F$, then the inward [resp. outward] set in Corollary 3.2 can be replaced by its closure.

2. The coercivity assumption in (1) is implied by the following:

\[ (*) \quad \text{there exists a nonempty compact convex subset } L \text{ of } X \text{ such that, for each } y \in X \setminus L, \text{ there is an } x \in L \text{ satisfying } \langle x - y, fy \rangle > 0. \]
Corollary 1.2(1) with the assumption (*) improves the "variational type" inequality of Juberg and Karamardian [10, Theorem]. In fact, they assumed closedness of $X$ and local convexity of $E$, and obtained weaker conclusion.

3. For a compact $X$, the condition (*) holds automatically. Therefore, from Corollary 1.2, we have the following:

**Corollary 1.3.** Let $X$ be a compact convex subset in a t.v.s. $E$, $F$ a topological space, and $f : X \to F$ a function such that $(x, y) \mapsto \langle x, fy \rangle$ is continuous on $E \times X$. Then there exists a $y_0 \in X$ such that

$$\langle x - y_0, fy_0 \rangle \leq 0 \quad \text{for all } x \in W(y_0).$$

**Remark.** Here $W(y_0)$ denotes any of $\overline{I}_X(y_0)$ or $\overline{O}_X(y_0)$. Corollary 1.3 strengthens Juberg and Karamardian [10, Lemma]. They showed that Corollary 1.2 follows from Corollary 1.3 in a particular case.

Let $\langle \cdot, \cdot \rangle$ denote the inner product of a real inner product space. Then Corollary 1.3 reduces to the following:

**Corollary 1.4.** Let $X$ be a compact convex subset in an inner product space $E$ and $f : X \to E$ a continuous map. Then there exists an $x_0 \in X$ satisfying

$$\langle fx_0, y - x_0 \rangle \leq 0 \quad \text{for all } y \in W(x_0)$$

**Remark.** The origin of Corollary 1.4 goes back to Hartman and Stampacchia [9] in 1966 for $\mathbb{R}^n$. See also Stampacchia [22, Theorem 2.2] and Moré [13, Theorem 2.1].

We now show that Corollary 1.4 implies the following well-known generalization of the Brouwer fixed point theorem.

**Corollary 1.5.** Let $X$ be a compact convex subset in an inner product space $E$ and $g : X \to E$ a continuous map such that $gx \in W(x)$ for all $x \in \text{Bd} \, X$. Then $g$ has a fixed point.

**Proof.** For any $x \in X$ we have $gx \in W(x)$. In fact, for any $x \in \text{Int} \, X$, we have $gx \in E = I_X(x) = O_X(x)$. Define $f \equiv g - 1_X : X \to E$. Then by Corollary 1.4, there exists an $x_0 \in X$ such that

$$\langle gx_0 - x_0, y - x_0 \rangle \leq 0 \quad \text{for all } y \in W(x_0).$$

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Since \( gx_0 \equiv y \) lies in \( W(x_0) \), we must have \( x_0 = gx_0 \) as desired.

Let \( E \) be a real t.v.s., \( E^* \) its topological dual (i.e., the vector space of all continuous linear functionals \( E \to \mathbb{R} \)), and \( \langle \cdot, \cdot \rangle : E^* \times E \to \mathbb{R} \) denote the natural pairing.

**Theorem 2.** Let \( X \) be a convex space in \( E \) and let

\[
p(x, y) \equiv \langle fx, y - x \rangle + h(x) - h(y)
\]

where \( h : X \to \mathbb{R} \) is a l.s.c. convex function and \( f : X \to E^* \) is a function such that

(a) for each \( y \in X \), \( x \mapsto \langle fx, y - x \rangle \) is l.s.c. on compact subsets of \( X \), and

(b) there exist a nonempty compact subset \( K \) of \( X \) and, for any finite subset \( N \) of \( X \), a compact convex subset \( L_N \) of \( X \) containing \( N \) such that \( x \in L_N \setminus K \) implies \( p(x, y) > 0 \) for some \( y \in L_N \).

Then there exists an \( x_0 \in K \) such that

\[
p(x_0, y) \leq 0 \quad \text{for all} \quad y \in X.
\]

Moreover, if \( h : E \to \mathbb{R} \) is a convex function which is l.s.c. on \( X \), then the conclusion holds for all \( y \in I_X(x_0) \).

**Proof.** In Corollary 1.1, interchange \( x \) and \( y \) and put \( F = E^* \).

**Remarks.**

1. Note that Brézis, Nirenberg, and Stampacchia [4, Application 3] obtained Theorem 2 under the stronger assumption that \( f \) is pseudo-monotone and continuous with a much stronger condition than (b). Theorem 2 improves Brézis [3, Corollary 29] and Hartman and Stampacchia [9, Theorems 1.1 and 5.1].

2. Theorem 2 also improves Allen [1, Corollary 1]. In fact, he assumed the following particular form of (b):

(b)' let \( L \) be a nonempty compact convex subset of \( X \) and suppose that for each \( x \in X \setminus L \) there exists \( y \in L \) such that \( p(x, y) > 0 \).
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From now on, let $E^*$ have any topology such that a continuous function $f : X \to E^*$ satisfies the requirement (a) of Theorem 2. For example, we equip $E^*$ with the topology of uniform convergence on bounded subsets of $E$.

**Corollary 2.1.** Let $X$ be a convex subset of $E$, and $f : X \to E^*$ continuous.

1. If there exist $K$ and $L_N$ as in (b) of Theorem 2 such that $x \in L_N \setminus K$ implies $\langle fx, y - x \rangle > 0$ for some $y \in L_N$, then there exists an $x_0 \in K$ such that

   \[ \langle fx_0, y - x_0 \rangle \leq 0 \quad \text{for all} \quad y \in \overline{I}_X(x_0). \]

2. If there exist $K$ and $L_N$ as in (b) of Theorem 2 such that $x \in L_N \setminus K$ implies $\langle fx, x - y \rangle > 0$ for some $y \in L_N$, then there exists an $x_0 \in K$ such that

   \[ \langle fx_0, y - x_0 \rangle \leq 0 \quad \text{for all} \quad y \in \overline{O}_X(x_0). \]

**Proof.** (1) By putting $h \equiv 0$ in Theorem 2, we know that there exists an $x_0 \in K$ such that

   \[ \langle fx_0, y - x_0 \rangle \leq 0 \quad \text{for all} \quad y \in I_X(x_0). \]

Since $fx_0 \in E^*$, this implies the conclusion.

(2) By the case for $\langle fx, y - x \rangle$ in Theorem 2, we know that there exists a point $x_0 \in K$ such that $\langle fx_0, x_0 - y' \rangle \leq 0$ for all $y' \in I_X(x_0)$ as in (1). For any $y \in O_X(x_0)$, let $y' = 2x_0 - y \in I_X(x_0)$. Then

   \[ \langle fx_0, y - x_0 \rangle = \langle fx_0, x_0 - y' \rangle \leq 0 \]

for all $y \in O_X(x_0)$. Hence, $\langle fx_0, y - x_0 \rangle \leq 0$ holds for all $y \in \overline{O}_X(x_0)$.

**Remarks.**

1. In case $X$ is compact, Corollary 2.1 reduces to Park [14, Theorem 2], which strengthens Browder [5, Theorem 2].

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2. In case $X$ is a closed convex subset of a t.v.s. $E$, if there exists a compact convex subset $L$ of $X$ such that

$$K \equiv \{ x \in X : \langle fx, y - x \rangle \leq 0 \text{ for all } y \in L \} \subset L,$$

is compact, then the same conclusion holds. This improves Takahashi [23, Theorem 3].

3. Instead of the continuity of $f$, it suffices to assume the condition (a) of Theorem 2. Hence, Corollary 2.1 improves Allen [1, Corollary 2].

4. If $x_0 \in \text{Int } X$ or $X = E$ in Corollary 2.1, it is obvious that there exists $x^* \in E$ such that $fx^* = 0$. In fact, $\langle fx_0, y - x_0 \rangle \leq 0$ for all $y \in E = I_X(x_0)$ implies $fx_0 = 0$.

5. Corollary 2.1 has a very interesting interpretation when $X$ is a cone in $E$ as follows:

A nonempty closed subset $X$ is a cone in $E$ if $\alpha x + \beta y \in X$ for all $\alpha, \beta \geq 0$ and $x, y \in X$. The polar $X^*$ of a cone $X$ is the cone defined by

$$X^* \equiv \{ p \in E^* : \langle p, x \rangle \geq 0 \text{ for all } x \in X \}.$$

**Corollary 2.2.** Let $X$ be a cone in $E$ and $f : X \to E^*$ continuous. If there exist $K$ and $L_N$ as in (b) of Theorem 2 such that $x \in L_N \setminus K$ implies $\langle fx, x - y \rangle > 0$ for some $y \in L_N$, then there exists an $x_0 \in X$ such that

$$fx_0 \in X^* \text{ and } \langle fx_0, x_0 \rangle = 0.$$

**Proof.** By Corollary 2.1(2), there exists $x_0 \in K$ such that $\langle fx_0, y - x_0 \rangle \geq 0$ for all $y \in X$. Since $\langle fx_0, \alpha y \rangle \geq \langle fx_0, x_0 \rangle$ for all $\alpha > 0$ and $y \in X$, we obtain $\langle fx_0, y \rangle \geq 0$ for all $y \in X$, i.e., $fx_0 \in X^*$. Since $\langle fx_0, 0 - x_0 \rangle \geq 0$, we have $\langle fx_0, x_0 \rangle = 0$.

**Remarks.**

1. The problem of finding a vector $x_0 \in X$ satisfying the conclusion is known as the complementarity problem; several problems in mathematical programming, game theory, economics, operations research, and mechanics can be presented in this form.

**Corollary 2.3.** Let $X$ be a convex subset of $E$, and $T : X \to 2^{E^*}$ a multifunction having a continuous selection $f : X \to E^*$.

(1) If there exist $K$ and $L_N$ as in (b) of Theorem 2 such that $x \in L_N \setminus K$ implies $(fx, y - x) > 0$ for some $y \in L_N$, then there exist $x_0 \in K$ and $x_0^* \in E^*$ such that

$$x_0^* \in Tx_0 \quad \text{and} \quad \langle x_0^*, y - x_0 \rangle \leq 0 \quad \text{for all} \quad y \in \overline{I}_X(x_0).$$

(2) If there exist $K$ and $L_N$ as in (b) of Theorem 2 such that $x \in L_N \setminus K$ implies $(fx, x - y) > 0$ for some $y \in L_N$, then the same conclusion holds for all $y \in \overline{O}_X(x_0)$.

**Proof.** Put $x_0^* = fx_0$ in Corollary 2.1.

**Remark.** Corollary 2.3 is a particular form of the generalized quasi-variational inequalities. For related results, see, e.g., Shih and Tan [18].

The following is a simple consequence of Corollary 2.3.

**Corollary 2.4.** Let $X$ be a compact convex subset of $E$ and $T : X \to 2^{E^*}$ a multifunction satisfying

(i) $Tx$ is nonempty and convex for each $x \in X$; and

(ii) $T^{-1}y$ is open for each $y \in Y$.

Then there exist $x_0 \in X$ and $x_0^* \in E^*$ such that

$$x_0^* \in Tx_0 \quad \text{and} \quad \langle x_0^*, y - x_0 \rangle \leq 0 \quad \text{for all} \quad y \in W(x_0).$$

**Proof.** $T$ has a continuous selection by a result in [2].

**Remark.** Corollary 4.4 strengthens Simons [21, Theorem 4.5]. For another proof, see Komiya [12]. This generalizes and unifies fixed point theorems for multifunctions due to Browder [5], Fan [8], Takahashi [23], [25] and Cellina [7]. Simons [21] gave several comments on related results to Corollary 2.4 and deduced some fixed point theorems from Corollary 2.4.

For reflexive Banach spaces, Theorem 2 reduces to the following:
Corollary 2.5. Let \( X \) be a convex subset of a real reflexive Banach space \( E \), \( f : X \to E^* \) is a weakly continuous function, and \( h : X \to \mathbb{R} \) a weakly l.s.c. convex function. If

\[ (*) \text{ there exist a bounded subset } K \text{ of } X \text{ and, for each finite subset } N \text{ of } X, \text{ a closed bounded convex subset } L_N \text{ of } X \text{ containing } N \text{ such that } x \in L_N \setminus K \text{ implies } \langle fx, x - y \rangle + h(x) > h(y) \text{ for some } y \in L_N, \]

then there exists an \( x_0 \in K \) such that

\[ \langle fx_0, x_0 - y \rangle + h(x_0) \leq h(y) \text{ for all } y \in X. \]

Moreover, if \( h \) is defined on \( E \), then the conclusion holds for all \( y \in I_X(x_0) \).

Proof. Switch to the weak topology.

Remark. Browder [6, Theorem 6] obtained Corollary 2.4 under stronger assumptions, i.e.,

1. \( f \) is pseudo-monotone in the sense in [6] (which implies \( f \) is continuous from any finite topology of \( X \) to the weak topology of \( X^* \)), and

2. for some \( y_0 \in X \), there exists an \( R_0 \in \mathbb{R} \) such that

\[ \langle fx, x - y_0 \rangle + h(x) > h(y_0) \]

for all \( x \in X \) with \( \|x\| > R_0 \).

Note that (2) implies (*). In fact,

\[ K \equiv \{ x \in X : \langle fx, x - y_0 \rangle + h(x) \leq h(y_0) \} \subset \{ x \in X : \|x\| \leq R_0 \} \]

is bounded.

Finally in this paper, we add a variational inequality for a non-real Hausdorff locally convex space (simply, l.c.s.).

Theorem 3. Let \( X \) be a nonempty bounded convex subset of a l.c.s. \( E \), and \( f : X \to E^* \) continuous from \( X \) to the strong topology of \( E^* \) such that

\[ (*) \text{ there exist a nonempty compact subset } K \text{ of } X \text{ and, for each finite subset } N \text{ of } X, \text{ a compact convex subset } L_N \text{ of } X \text{ containing } N \text{ such that } y \in L_N \setminus K \text{ implies } \text{Re} \langle fy, y - x \rangle > 0 \text{ for some } x \in L_N. \]
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Then there exists a point $y_0 \in K$ such that

$$\text{Re} \langle y_0, y_0 - x \rangle \leq 0 \quad \text{for all } x \in \overline{I}_X(y_0).$$

**Proof.** Define $p : X \times X \to \mathbb{R}$ by

$$p(x, y) \equiv \text{Re} \langle fy, y \rangle \quad \text{for all } x, y \in X.$$

Then, for each $x \in X$, $p(x, \cdot)$ is continuous by [19, Lemma 1]. By applying Theorem 0 with $h \equiv 0$, the conclusion follows.

**REMARKS.**

1. Theorem 3 generalizes Shih and Tan [19, Theorem 10] since they assumed the following stronger condition than ($*$):

   ($**$) there exists a compact convex subset $L$ of $X$ such that, for each $y \in X \setminus L$, there is an $x \in L$ with $\text{Re} \langle fy, y - x \rangle > 0$.

2. If $X$ is closed in Theorem 3, ($*$) is implied by the following:

   ($***$) for some nonempty compact subset $C$ of $E$ and $x_0 \in X \cap C$,

   $$\text{Re} \langle fy, y - x_0 \rangle > 0 \quad \text{for all } y \in X \setminus C.$$

Therefore, Theorem 3 generalizes Shih and Tan [19, Theorem 11].

3. For compact $X$, Theorem 5 improves Browder's variational inequality in [5], [6].

**References**


