FUZZY IDEALS IN NEAR-RINGS

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Abstract. In this paper, we give another proof of Theorem 2.13 of [4] without using the sup property. For the homomorphic image $f(\mu)$ and preimage $f^{-1}(\nu)$ of fuzzy left (resp. right) ideals $\mu$ and $\nu$ respectively, we establish the chains of level left (resp. right) ideals of $f(\mu)$ and $f^{-1}(\nu)$, respectively. Moreover, we prove that a necessary condition for a fuzzy ideal $\mu$ of a near-ring $R$ to be prime is that $\mu$ is two-valued.

1. Introduction

S. Abou-Zaid [1] introduced the notion of a fuzzy subnear-ring, and studied fuzzy left (resp. right) ideals of a near-ring, and gave some properties of fuzzy prime ideals of a near-ring. In [4], S. D. Kim and H. S. Kim proved that the homomorphic image of a fuzzy left (resp. right) ideal which has the “sup property” is a fuzzy left (resp. right) ideal. In this paper, we give another proof of Theorem 2.13 of [4] without using the sup property. For the homomorphic image $f(\mu)$ and preimage $f^{-1}(\nu)$ of fuzzy left (resp. right) ideals $\mu$ and $\nu$ respectively, we establish the chains of level left (resp. right) ideals of $f(\mu)$ and $f^{-1}(\nu)$, respectively. Moreover, we prove that a necessary condition for a fuzzy ideal $\mu$ of a near-ring $R$ to be prime is that $\mu$ is two-valued.

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2. Preliminaries

By a near-ring [8] we mean a non-empty set \( R \) with two binary operations "+" and "." satisfying the following axioms:

1. \((R,+)\) is a group,
2. \((R,\cdot)\) is a semigroup,
3. \(x \cdot (y + z) = x \cdot y + x \cdot z\) for all \(x, y, z \in R\).

Precisely speaking, it is a left near-ring because it satisfies the left distributive law. We will use the word "near-ring" in stead of "left near-ring". We denote \(xy\) instead of \(x \cdot y\). Note that \(x0 = 0\) and \(x(-y) = -xy\) but in general \(0x \neq 0\) for some \(x \in R\). Let \(R\) and \(S\) be near-rings. A map \(f : R \to S\) is called a (near-ring) homomorphism if \(f(x + y) = f(x) + f(y)\) and \(f(xy) = f(x)f(y)\) for any \(x, y \in R\). An ideal \(I\) of a near-ring \(R\) is a subset of \(R\) such that

4. \((I,+)\) is a normal subgroup of \((R,+)\),
5. \(RI \subseteq I\),
6. \((r + i)s - rs \in I\) for any \(i \in I\) and any \(r, s \in R\).

Note that \(I\) is a left ideal of \(R\) if \(I\) satisfies (4) and (5), and \(I\) is a right ideal of \(R\) if \(I\) satisfies (4) and (6).

We note that the intersection of a family of left (resp. right) ideals is a left (resp. right) ideal, and that the onto homomorphic image of a left (resp. right) ideal is also a left (resp. right) ideal.

We now review some fuzzy logic concepts (see [2], [9] and [10] for details). A fuzzy set \(\mu\) in a set \(R\) is a function \(\mu : R \to [0,1]\). Let \(\text{Im}(\mu)\) denote the image set of \(\mu\). Let \(\mu\) be a fuzzy set in a set \(R\). For \(\alpha \in [0,1]\), the set

\[R^\alpha_\mu := \{x \in R | \mu(x) \geq \alpha\}\]

is called a level subset of \(\mu\).

Let \(f\) be a mapping from a set \(R\) to a set \(S\) and let \(\mu\) and \(\nu\) be fuzzy sets in \(R\) and \(S\), respectively. Then \(f(\mu)\), the image of \(\mu\) under \(f\), is a fuzzy set in \(S\):

\[f(\mu)(y) := \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise}, \end{cases}\]
for all $y \in S$. $f^{-1}(\nu)$, the preimage of $\nu$ under $f$, is a fuzzy set in $R$:

$$f^{-1}(\nu)(x) := \nu(f(x))$$

for all $x \in R$.

We say that a fuzzy set $\mu$ in $R$ has the sup property if, for any subset $T$ of $R$, there exists $t_0 \in T$ such that

$$\mu(t_0) = \sup_{t \in T} \mu(t).$$

Let $f$ be a mapping from a set $R$ to a set $S$ and let $\mu$ be a fuzzy set in $R$. Then $\mu$ is said to be $f$-invariant if $f(x) = f(y)$ implies $\mu(x) = \mu(y)$ for all $x, y \in R$. Clearly, if $\mu$ is $f$-invariant then $f^{-1}(f(\mu)) = \mu$.

3. Fuzzy Ideals

Let $R$ be a near-ring and $\mu$ be a fuzzy set in $R$. We say that $\mu$ is a fuzzy subnear-ring of $R$ if, for all $x, y \in R$,

1. $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$,
2. $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$.

$\mu$ is called a fuzzy ideal of $R$ if $\mu$ is a fuzzy subnear-ring of $R$ and

3. $\mu(y + x - y) \geq \mu(x)$,
4. $\mu(xy) \geq \mu(y)$,
5. $\mu((x + z)y - xy) \geq \mu(z)$,

for any $x, y, z \in R$.

Note that $\mu$ is a fuzzy left ideal of $R$ if it satisfies (7), (8), (9) and (10), and $\mu$ is a fuzzy right ideal of $R$ if it satisfies (7), (8), (9) and (11) (see [1]).

**Lemma 1** ([1, Theorem 4.2]). Let $\mu$ be a fuzzy set in a near-ring $R$. Then the level subset $R^\alpha_\mu$ is a subnear-ring (resp. an ideal) of $R$ for all $\alpha \in [0, 1]$, $\alpha \leq \mu(0)$ if and only if $\mu$ is a fuzzy subnear-ring (resp. a fuzzy ideal).

The following proposition will be used in the sequel.
PROPOSITION 1. Let $f$ be a mapping from a set $R$ to a set $S$, and let $\mu$ be a fuzzy set in $R$. Then for every $\alpha \in (0,1]$,

$$S^\alpha_{f(\mu)} = \bigcap_{0<\beta<\alpha} f(R^\alpha_{\mu} - \beta).$$

Proof. Let $\alpha \in (0,1]$. For $y = f(x) \in S$, assume that $y \in S^\alpha_{f(\mu)}$. Then

$$\alpha \leq f(\mu)(y) = f(\mu)(f(x)) = \sup_{z \in f^{-1}(f(x))} \mu(z).$$

Hence for every real number $\beta$ with $0 < \beta < \alpha$, there exists $x_0 \in f^{-1}(y)$ such that $\mu(x_0) > \alpha - \beta$, and so $y = f(x_0) \in f(R^\alpha_{\mu} - \beta)$. Therefore $y \in \bigcap_{0<\beta<\alpha} f(R^\alpha_{\mu} - \beta)$.

Conversely, let $y \in \bigcap_{0<\beta<\alpha} f(R^\alpha_{\mu} - \beta)$. Then $y \in f(R^\alpha_{\mu} - \beta)$ for every $\beta$ with $0 < \beta < \alpha$, which implies that there exists $x_0 \in R^\alpha_{\mu} - \beta$ such that $y = f(x_0)$. It follows that $\mu(x_0) \geq \alpha - \beta$ and $x_0 \in f^{-1}(y)$, so that

$$f(\mu)(y) = \sup_{z \in f^{-1}(y)} \mu(z) \geq \sup_{0<\beta<\alpha} \{\alpha - \beta\} = \alpha.$$

Hence $y \in S^\alpha_{f(\mu)}$, and the proof is complete. \qed


THEOREM 1. ([4, Theorem 2.12]). A near-ring homomorphic preimage of a fuzzy left (resp. right) ideal is a fuzzy left (resp. right) ideal.

THEOREM 2 ([4, Theorem 2.13]). A near-ring homomorphic image of a fuzzy left (resp. right) ideal having the sup property is a fuzzy left (resp. right) ideal.

Now we give another proof of Theorem 2 without using the sup property.

THEOREM 3. Let $f : R \rightarrow S$ be an onto near-ring homomorphism and let $\mu$ be a fuzzy left (resp. right) ideal of $R$. Then $f(\mu)$ is a fuzzy left (resp. right) ideal of $S$. 

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Proof. In view of Lemma 1 it is sufficient to show that $S_{f(\mu)}^\alpha$, $\alpha \in [0, \mu(0)]$, is a left (resp. right) ideal of $S$. Note that $S_{f(\mu)}^0 = S$, and if $\alpha \in (0, 1]$ then $S_{f(\mu)}^\alpha = \bigcap_{0 < \beta < \alpha} f(R_\mu^{\alpha-\beta})$ by Proposition 1. Since $R_\mu^{\alpha-\beta}$ is a left (resp. right) ideal of $R$ and since $f$ is onto, $f(R_\mu^{\alpha-\beta})$ is a left (resp. right) ideal of $S$. Therefore $S_{f(\mu)}^\alpha$ is an intersection of a family of left (resp. right) ideals is also a left (resp. right) ideal of $S$, ending the proof. \qed

Theorem 4. Let $f$ and $\mu$ be as in Theorem 3. Then there is a one-to-one correspondence between the set of all $f$-invariant left (resp. right) fuzzy ideals of $R$ and the set of all left (resp. right) fuzzy ideals of $S$.

Proof. Straightforward in view of Theorem 1, Theorem 3 and the following results:

(i) $f^{-1}(f(\mu)) = \mu$, where $\mu$ is any $f$-invariant left (resp. right) fuzzy ideal of $R$;

(ii) $f(f^{-1}(\nu)) = \nu$, where $\nu$ is any left (resp. right) fuzzy ideal of $S$. \qed

Theorem 5. Let $f : R \to S$ be an onto homomorphism of near-rings and let $\mu$ and $\nu$ be left (resp. right) fuzzy ideals of $R$ and $S$, respectively such that

$$\text{Im}(\mu) = \{\alpha_0, \alpha_1, ..., \alpha_n\} \text{ with } \alpha_0 > \alpha_1 > ... > \alpha_n, \text{ and}$$

$$\text{Im}(\nu) = \{\beta_0, \beta_1, ..., \beta_m\} \text{ with } \beta_0 > \beta_1 > ... > \beta_m.$$ 

Then

(i) $\text{Im}(f(\mu)) \subseteq \text{Im}(\mu)$ and the chain of level left (resp. right) ideals of $f(\mu)$ is

$$f(R_\mu^{\alpha_0}) \subseteq f(R_\mu^{\alpha_1}) \subseteq ... \subseteq f(R_\mu^{\alpha_n}) = S.$$ 

(ii) $\text{Im}(f^{-1}(\nu)) = \text{Im}(\nu)$ and the chain of level left (resp. right) ideals of $f^{-1}(\nu)$ is

$$f^{-1}(S_\nu^{\beta_0}) \subseteq f^{-1}(S_\nu^{\beta_1}) \subseteq ... \subseteq f^{-1}(S_\nu^{\beta_m}) = R.$$ 

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Proof. (i) Since \( f(\mu)(y) = \sup_{x \in f^{-1}(y)} \mu(x) \) for all \( y \in S \), obviously \( \text{Im}(f(\mu)) \subset \text{Im}(\mu) \). Note that for any \( y \in S \),

\[
y \in f(R^\alpha_\mu) \iff \text{there exists } x \in f^{-1}(y) \text{ such that } \mu(x) \geq \alpha_i
\]

\[
\iff \sup_{z \in f^{-1}(y)} \mu(z) \geq \alpha_i
\]

\[
\iff f(\mu)(y) \geq \alpha_i
\]

\[
\iff y \in S^\alpha_{f(\mu)}.
\]

Hence \( f(R^\alpha_\mu) = S^\alpha_{f(\mu)} \) for \( i = 0, 1, \ldots, n \), and therefore the chain of level left (resp. right) ideals of \( f(\mu) \) is

\[
f(R^\alpha_\mu) \subset f(R^\alpha_{f(\mu)}) \subset \cdots \subset f(R^\alpha_n) = S.
\]

(ii) Since \( f^{-1}(\nu)(x) = \nu(f(x)) \) for all \( x \in R \) and since \( f \) is onto, we have \( \text{Im}(f^{-1}(\nu)) = \text{Im}(\nu) \). Note that for all \( x \in R \),

\[
x \in f^{-1}(S^\beta_\nu) \iff f(x) \in S^\beta_\nu
\]

\[
\iff \nu(f(x)) \geq \beta_i
\]

\[
\iff f^{-1}(\nu)(x) \geq \beta_i
\]

\[
\iff x \in R^\beta_{f^{-1}(\nu)},
\]

so that \( f^{-1}(S^\beta_\nu) = R^\beta_{f^{-1}(\nu)} \) for all \( i = 0, 1, \ldots, m \). Hence the chain of level left (resp. right) ideals of \( f^{-1}(\nu) \) is

\[
f^{-1}(S^\beta_\nu) \subset f^{-1}(S^\beta_{f^{-1}(\nu)}) \subset \cdots \subset f^{-1}(S^\beta_{m}) = R.
\]

This completes the proof. \( \square \)

Lemma 2. Let \( \mu \) and \( \nu \) be fuzzy left (resp. right) ideals of \( R \) and \( f(R) \) respectively, where \( f : R \to S \) is a near-ring homomorphism. Then \( f(\mu)(0) = \mu(0) \) and \( f^{-1}(\nu)(0) = \nu(0) \).
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Proof. Straightforward. □

Let \( \rho \) and \( \delta \) be two fuzzy sets in a near-ring \( R \). The product \( \rho \circ \delta \) is defined by

\[
\rho \circ \delta(x) := \begin{cases} 
\sup \{ \min\{\rho(y), \delta(z)\} \}, & \text{if } x = yz \\
0 & \text{if } x \text{ is not expressible as } x = yz.
\end{cases}
\]

A fuzzy ideal \( \mu \) of a near-ring \( R \) is said to be prime [1] if \( \mu \) is not a constant function and for any fuzzy ideals \( \rho \) and \( \delta \) of \( R \), \( \rho \circ \delta \subseteq \mu \) implies \( \rho \subseteq \mu \) or \( \delta \subseteq \mu \).

For a fuzzy left (resp. right) ideal \( \delta \) of a near-ring \( R \), let

\[
\delta_0 := \{ x \in R | \delta(x) = \delta(0) \}.
\]

Lemma 3 ([1, Theorem 3.7]). Let \( \delta \) be a fuzzy prime ideal of a near-ring \( R \). Then \( \delta_0 \) is a prime ideal of \( R \).

Proposition 2. Let \( f : R \to S \) be a near-ring homomorphism and let \( \delta \) be a fuzzy left (resp. right) ideal of \( R \). Then \( f(\delta_0) \subseteq f(\delta)_0 \), with equality if \( \delta \) has the sup property.

Proof. Let \( x \in \delta_0 \). Then

\[
f(\delta)(f(0)) \geq f(\delta)(f(x)) \geq \delta(x) = \delta(0) = f(\delta)(f(0)),
\]

and so \( f(\delta)(f(x)) = f(\delta)(f(0)) = f(\delta)(0) \). Hence \( f(x) \in f(\delta)_0 \) or \( f(\delta_0) \subseteq f(\delta)_0 \). Assume that \( \delta \) has the sup property and let \( x \in R \) be such that \( f(x) \in f(\delta)_0 \). Then

\[
\delta(0) = f(\delta)(f(x)) = \sup\{ \delta(y) | f(y) = f(x) \} = \delta(y)
\]

for some \( y \in R \) such that \( f(y) = f(x) \) since \( \delta \) has the sup property. Thus \( y \in \delta_0 \), and so \( f(x) = f(y) \in \delta_0 \). This completes the proof. □

Theorem 6. Let \( \mu \) be a fuzzy prime ideal of a near-ring \( R \). Then \( |\text{Im}(\mu)| = 2 \), i.e., \( \mu \) is two-valued. In particular, \( \mu(0) = 1 \).
Proof. Note that $|\text{Im}(\mu)| \geq 2$ since $\mu$ is not constant. Assume that $|\text{Im}(\mu)| \geq 3$. Let $\mu(0) = \alpha$ and $\lambda = \text{glb}\{\mu(x) | x \in R\}$. Then there exist $\gamma, \beta \in \text{Im}(\mu)$ such that $\lambda \leq \gamma < \beta < \alpha$. Let $\rho$ and $\delta$ be fuzzy sets in $R$ such that $\rho(x) := \frac{1}{2}(\gamma + \beta)$ for all $x \in R$ and

$$
\delta(x) := \begin{cases} 
\lambda & \text{if } x \notin R_{\mu}^\beta, \\
\alpha & \text{otherwise.}
\end{cases}
$$

Clearly, $\rho$ is a fuzzy ideal of $R$. We now prove that $\delta$ is a fuzzy ideal of $R$. Let $x, y \in R$. If $x, y \in R_{\mu}^\beta$, then $x - y \in R_{\mu}^\beta$ and $\delta(x - y) = \alpha = \min\{\delta(x), \delta(y)\}$. If $x \in R_{\mu}^\beta$ and $y \notin R_{\mu}^\beta$ (or $x \notin R_{\mu}^\beta$ and $y \in R_{\mu}^\beta$) then $x - y \notin R_{\mu}^\beta$ and

$$
\delta(x - y) = \lambda = \min\{\delta(x), \delta(y)\},
$$

since

$$
\delta(x) \text{ (or } \delta(y)) \geq \alpha > \lambda = \delta(y) \text{ (or } \delta(x)).
$$

If $x \notin R_{\mu}^\beta$ and $y \notin R_{\mu}^\beta$ then $\delta(x) = \delta(y) = \lambda$ and so

$$
\delta(x - y) \geq \lambda = \min\{\delta(x), \delta(y)\}.
$$

Hence $\delta(x - y) \geq \min\{\delta(x), \delta(y)\}$ for all $x, y \in R$. Similarly, we know that

$$
\delta(xy) \geq \min\{\delta(x), \delta(y)\} \text{ for all } x, y \in R.
$$

Hence $\delta$ is a fuzzy subnear-ring of $R$. For any $y \in R$, if $y \in R_{\mu}^\beta$ then $xy \in R_{\mu}^\beta$ for all $x \in R$, and so $\delta(xy) = \alpha = \delta(y)$. If $y \notin R_{\mu}^\beta$, then $\delta(xy) \geq \lambda = \delta(y)$. Hence $\delta(xy) \geq \delta(y)$ for all $x, y \in R$. Let $x, y \in R$. If $x \in R_{\mu}^\beta$ then $y + x - y \in R_{\mu}^\beta$ and $\delta(y + x - y) = \alpha = \delta(x)$. If $x \notin R_{\mu}^\beta$, then $\delta(y + x - y) \geq \lambda = \delta(x)$. This proves that $\delta$ is a fuzzy left ideal of $R$. Let $x, y, z \in R$. If $z \in R_{\mu}^\beta$, then $(x + z)y - xy \in R_{\mu}^\beta$ and $\delta((x + z)y - xy) = \alpha = \delta(z)$. If $z \notin R_{\mu}^\beta$, then $\delta(z) = \lambda \leq \delta((x + z)y - xy)$. Hence $\delta((x + z)y - xy) \geq \delta(z)$ for all $x, y, z \in R$, and therefore $\delta$ is a fuzzy ideal of $R$. Now we show that $\rho \circ \delta \subseteq \mu$. Consider the following cases:
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Case (i) $x = 0$. Then

$$
\rho \circ \delta(x) = \sup_{x=yz} \{\min\{\rho(y), \delta(z)\}\} \leq \frac{1}{2}(\gamma + \beta) < \alpha = \mu(0).
$$

Case (ii) $0 \neq x \in R^\beta_\mu$. Then $\mu(x) \geq \beta$, and

$$
\rho \circ \delta(x) = \sup_{x=yz} \{\min\{\rho(y), \delta(z)\}\} \leq \frac{1}{2}(\gamma + \beta) < \beta \leq \mu(x).
$$

Case (iii) $0 \neq x \notin R^\beta_\mu$. For any $y, z \in R$ such that $x = yz$, we have $z \notin R^\beta_\mu$. Thus $\delta(z) = \lambda$ and so

$$
\rho \circ \delta(x) = \sup_{x=yz} \{\min\{\rho(y), \delta(z)\}\} = \lambda \leq \mu(x).
$$

Thus in each case, $\rho \circ \delta(x) \leq \mu(x)$ or $\rho \circ \delta \subseteq \mu$.

Next we show that neither $\rho \subseteq \mu$ nor $\delta \subseteq \mu$. We can find $x \in R$ such that $\mu(x) = \gamma$. Then

$$
\rho(x) = \frac{1}{2}(\gamma + \beta) > \gamma = \mu(x).
$$

Hence $\rho \nsubseteq \mu$. We also know that $\mu(y) = \beta$ for some $y \in R$. It follows that $y \in R^\beta_\mu$ and $\delta(y) = \alpha > \beta = \mu(y)$. Therefore $\delta \nsubseteq \mu$. This shows that $\mu$ is not a fuzzy prime ideal of $R$, which is a contradiction. Hence $|\text{Im}(\mu)| = 2$. Now let $|\text{Im}(\mu)| = \{\alpha, \gamma\}$ and $\gamma < \alpha$. Then $\mu(0) = \alpha$ since $\mu(0) \geq \mu(x)$ for all $x \in R$. Assume that $\alpha \neq 1$. Then there exists $\beta \in [0, 1]$ such that $\alpha < \beta \leq 1$. Let $\rho$ and $\delta$ be fuzzy sets in $R$ such that $\rho(x) := \frac{1}{2}(\alpha + \gamma)$ for all $x \in R$ and

$$
\delta(x) := \begin{cases} 
\beta & \text{if } x \in \mu_0, \\
\gamma & \text{otherwise.}
\end{cases}
$$

Clearly $\rho$ is a fuzzy ideal of $R$. Since $\mu_0$ is an ideal of $R$, $\delta$ is a fuzzy ideal of $R$. It can be easily checked that $\rho \circ \delta \subseteq \mu$. Since $\mu(0) = \alpha < \beta = \delta(0)$, we have $\delta \nsubseteq \mu$. Note that there exists $x \in R$ such that $\mu(x) = \gamma < \frac{1}{2}(\alpha + \gamma) = \rho(x)$, so that $\rho \nsubseteq \mu$. This is a contradiction to the hypothesis. Hence $\mu(0) = 1$, ending the proof.

\[\square\]

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