ITERATIVE APPROXIMATIONS OF FIXED POINTS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN BANACH SPACES

S. S. Chang, J. Y. Park and Y. J. Cho

ABSTRACT. The purpose of this paper is to study the iterative approximations of fixed points for asymptotically pseudo-contractive mappings and asymptotically nonexpansive mappings in Banach spaces. The results presented in this paper extend and improve the main results in Geobel-Kirk [4], Liu [5] and Schu [7].

1. Introduction and Preliminaries

Throughout this paper, we assume that $E$ is a real Banach space, $E^*$ is the topological dual space of $E$, $\langle \cdot, \cdot \rangle$ denotes the dual pair between $E$ and $E^*$ and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{ f \in E^* : \langle x, f \rangle = \| x \| \| f \|, \| x \| = \| f \| \}$$

for all $x \in E$.

**Definition 1.1.** Let $D$ be a nonempty subset of $E$ and $T : D \to D$ be a mapping.

(1) $T$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that

$$\| T^n x - T^n y \| \leq k_n \| x - y \|$$

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for all \( x, y \in D \) and \( n = 1, 2, \ldots \).

2. \( T \) is said to be uniformly \( L \)-Lipschitzian with the constant \( L \geq 1 \) if

\[
\|T^n x - T^n y\| \leq L \|x - y\|
\]

for all \( x, y \in D \) and \( n = 1, 2, \ldots \).

3. \( T \) is said to be asymptotically pseudo-contractive if there exists a sequence \( \{k_n\} \) in \([1, \infty)\) with \( \lim_{n \to \infty} k_n = 1 \) and, for any \( x, y \in D \), \( j(x - y) \in J(x - y) \) such that

\[
\langle T^n x - T^n y, j(x - y) \rangle \leq k_n \|x - y\|^2
\]

for all \( n = 1, 2, \ldots \).

Remark 1.1. (1) If \( T \) is a nonexpansive mapping, then \( T \) is an asymptotically nonexpansive mapping with a constant sequence \( \{k_n\} \) defined by \( k_n = 1 \) for all \( n = 1, 2, \ldots \).

(2) If \( T \) is an asymptotically nonexpansive mapping with a sequence \( \{k_n\} \) in \([1, \infty)\) and \( \lim_{n \to \infty} k_n = 1 \), then \( T \) is a uniformly \( L \)-Lipschitzian mapping with the constant \( L = \sup_{n \geq 1} k_n < \infty \).

(3) If \( T \) is an asymptotically nonexpansive mapping, then \( T \) is an asymptotically pseudo-contractive mapping. But the converse is not true in general.

This can be seen from the following example:

Example 1.1. (6) Let \( E = R, D = [0, 1] \) and the mapping \( T : D \to D \) is defined by

\[
Tx = (1 - x^\frac{2}{3})^{\frac{3}{2}}
\]

for all \( x \in D \). We can prove that \( T \) is not Lipschitzian and so it is not asymptotically nonexpansive. Since \( T \circ T = T \) and it is monotonically decreasing, it follows that

\[
(x - y)(T^n x - T^n y) = \begin{cases} |x - y|^2, & \text{if } n \text{ is even}, \\ (x - y)(Tx - Ty) \leq 0 \leq |x - y|^2, & \text{if } n \text{ is odd}. \end{cases}
\]

This implies that \( T \) is an asymptotically pseudo-contractive mapping with the constant sequence \( \{1\} \).
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DEFINITION 1.2. Let $D$ be a nonempty convex subset of $E$, $T : D \rightarrow D$ be a mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$.

(1) The sequence $\{x_n\}$ defined by

\[
\begin{cases}
  x_0 \in D, \\
  x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n, \\
  y_n = (1 - \beta_n)x_n + \beta_n T^n x_n
\end{cases}
\]

for all $n = 0, 1, 2, \ldots$ is called the modified Ishikawa iterative sequence.

(2) Taking $\beta_n = 0$ for all $n = 0, 1, 2, \ldots$ in (1.1), the sequence $\{x_n\}$ defined by

\[
\begin{cases}
  x_0 \in D, \\
  x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n
\end{cases}
\]

for all $n = 0, 1, 2, \ldots$ is called the modified Mann iterative sequence.

The concept of an asymptotically nonexpansive mapping was introduced by Goebel-Kirk [4] in 1972, which was closely related to the theory of fixed points of mappings in Banach spaces. An early fundamental result due to Browder [1] states that, if $E$ is a uniformly convex Banach space, $D$ is a nonempty bounded closed convex subset of $E$ and $T : D \rightarrow D$ is an asymptotically nonexpansive mapping, then $T$ has a fixed point in $D$.

On the other hand, the concept of an asymptotically pseudo-contractive mapping was introduced by Schu [7] in 1991.

The iterative approximation problems for asymptotically nonexpansive mappings and asymptotically pseudo-contractive mappings were studied extensively by Goebel-Kirk [4], Liu [5] and Schu [7] in Hilbert spaces and uniformly convex Banach spaces, respectively. The purpose of this paper is, by using a new iterative technique, to study the iterative approximative problem of fixed points for asymptotically nonexpansive mappings and asymptotically pseudo-contractive mappings in the setting of Banach spaces. The results represented in this paper extend and improve the main results in [4]-[5], [7] and the proof methods given in this paper is quite different from the methods given in [4]-[5], [7].

We first recall the following results for our main results:
Lemma 1.1 ([2], [3]). Let $E$ be a real Banach space. Then, for any $x, y \in E$,

$$
\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle
$$

for all $j(x + y) \in J(x + y)$, where $J : E \to 2^E^*$ is the normalized duality mapping.

Lemma 1.2 ([8]). Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative real sequences. If there exists a positive integer $n_0$ such that

$$
a_{n+1} \leq (1 - t_n)a_n + b_n
$$

for all $n \geq n_0$, where $0 \leq t_n < 1$, $\sum_{n=0}^{\infty} t_n = \infty$ and $b_n = o(t_n)$. Then $\lim_{n\to\infty} a_n = 0$.

2. Main Results

In this section, we give some iterative approximation theorems of fixed points for asymptotically pseudo-contraction mappings and asymptotically nonexpansive mappings. Let $F(T)$ denote the set of all fixed points of $T$.

Theorem 2.1. Let $D$ be a nonempty closed convex subset of $E$, $T : D \to D$ be a uniformly $L$-Lipschitzian asymptotically pseudo-contractive mapping with a real sequence $\{k_n\}$ in $[1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$ and the constant $L \geq 1$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$ satisfying the following conditions:

(i) $\alpha_n \to 0$, $\beta_n \to 0$ ($n \to \infty$),
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Let $\{x_n\}$ be the modified Ishikawa iterative sequence defined by (1.1). If $F(T) \neq \emptyset$, $T(D)$ is bounded, and, for any given $q \in F(T)$, there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ and $\phi(0) = 0$ such that

$$
\langle T^nx_{n+1} - q, j(x_{n+1} - q) \rangle \leq k_n\|x_{n+1} - q\|^2 - \phi(\|x_{n+1} - q\|)
$$

for all $n = 0, 1, 2, \cdots$, where $j(x_{n+1} - q) \in J(x_{n+1} - q)$ is such that

$$
\langle T^nx_{n+1} - T^nq, j(x_{n+1} - q) \rangle \leq k_n\|x_{n+1} - q\|^2
$$

for all $n = 0, 1, 2, \cdots$, then the sequence $\{x_n\}$ converges strongly to the fixed point $q$ of $T$. 

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Proof. It follows from Lemma 1.1 and (1.1) that

\[
\|x_{n+1} - q\|^2 \\
= \|(1 - \alpha_n)(x_n - q) + \alpha_n(T^ny_n - q)\|^2 \\
\leq (1 - \alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\langle T^ny_n - q, j(x_{n+1} - q)\rangle \\
\leq (1 - \alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\langle T^ny_n - T^n x_{n+1}, j(x_{n+1} - q)\rangle \\
+ 2\alpha_n\langle T^n x_{n+1} - q, j(x_{n+1} - q)\rangle.
\]

(2.2)

Now, we consider the third term on the right side of (2.2). By (2.1), we have

\[
2\alpha_n\langle T^n x_{n+1} - q, j(x_{n+1} - q)\rangle \\
\leq 2\alpha_n [k_n\|x_{n+1} - q\|^2 - \phi(\|x_{n+1} - q\|)]
\]

(2.3)

Next, we consider the second term on the right side of (2.2). Since $T$ is uniformly $L$-Lipschitzian, we have

\[
2\alpha_n\langle T^ny_n - T^n x_{n+1}, j(x_{n+1} - q)\rangle \\
\leq 2\alpha_nL\|y_n - x_{n+1}\|\|x_{n+1} - q\|.
\]

(2.4)

On the other hand, we have

\[
\|y_n - x_{n+1}\| \\
= \|(1 - \alpha_n)(x_n - y_n) + \alpha_n(T^ny_n - y_n)\| \\
\leq (1 - \alpha_n)\beta_n\|T^nx_n - x_n\| + \alpha_n\|T^ny_n - q + q - y_n\| \\
\leq (1 - \alpha_n)\beta_n\|T^nx_n - q + q - x_n\| + \alpha_n(1 + L)\|y_n - q\| \\
\leq (1 - \alpha_n)\beta_n\|x_n - q\| \\
+ \alpha_n(1 + L)(1 - \beta_n)(x_n - q) + \beta_n(T^nx_n - q) \\
\leq (1 - \alpha_n)\beta_n(1 + L)\|x_n - q\| + \alpha_n(1 + L)L\|x_n - q\| \\
= d_n\|x_n - q\|,
\]

where $d_n = (1 - \alpha_n)\beta_n(1 + L) + \alpha_n(1 + L)L$ and $d_n \to 0$ as $n \to \infty$ since $\alpha_n, \beta_n \to 0$ as $n \to \infty$. Thus, substituting (2.3)~(2.5) into (2.2)
and simplifying, we have
\begin{equation}
\|x_{n+1} - q\|^2 \\
\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n Ld_n \|x_n - q\| \|x_{n+1} - q\| \\
+ 2\alpha_n k_n \|x_{n+1} - q\|^2 - 2\alpha_n \phi(\|x_{n+1} - q\|).
\end{equation}

Since \(2\|x_n - q\| \|x_{n+1} - q\| \leq \|x_n - q\|^2 + \|x_{n+1} - q\|^2\), it follows from (2.6) that
\begin{equation}
\|x_{n+1} - q\|^2 \leq \frac{(1 - \alpha_n)^2 + \alpha_n Ld_n}{1 - 2\alpha_n k_n - \alpha_n Ld_n} \|x_n - q\|^2 \\
- \frac{(2\alpha_n)}{1 - 2\alpha_n k_n - \alpha_n Ld_n} \phi(\|x_{n+1} - q\|).
\end{equation}

Since \(\alpha_n \to 0\), \(\beta_n \to 0\) and \(k_n \to 1\) as \(n \to \infty\), there exists a positive integer \(n_0\) such that \(1 - 2\alpha_n k_n - \alpha_n Ld_n > 0\) for all \(n \geq n_0\). Hence, without loss of generality, we can assume that
\[1 - 2\alpha_n k_n - \alpha_n Ld_n > 0\]
for all \(n \geq 0\). Besides, if there exists a nonnegative integer \(m\) such that \(x_m = q\), then we have
\[y_m = (1 - \beta_m) x_m + \beta_m T^m x_m = q.
\]

By induction, we can prove that \(x_{m+i} = y_{m+i} = q\) for all \(i \geq 0\) and so \(x_n \to q\) as \(n \to \infty\). The conclusion of Theorem 2.1 is proved. Hence, without loss of generality, we can assume that \(x_n \neq q\) for all \(n \geq 0\). Let
\[\sigma = \inf_{n \geq 0} \frac{\phi(\|x_{n+1} - q\|)}{\|x_{n+1} - q\|^2}.
\]

1. If \(\sigma > 0\). Taking \(\gamma \in (0, \min\{1, \sigma\})\), it follows from (2.7) that
\[\|x_{n+1} - q\|^2 \leq \frac{(1 - \alpha_n)^2 + \alpha_n Ld_n}{1 - 2\alpha_n k_n - \alpha_n Ld_n} \|x_n - q\|^2 \\
- \frac{2\alpha_n \gamma}{1 - 2\alpha_n k_n - \alpha_n Ld_n} \|x_{n+1} - q\|^2,
\]
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which implies that

\[(2.8) \quad \|x_{n+1} - q\|^2 \leq \frac{(1 - \alpha_n)^2 + \alpha_n Ld_n}{1 - 2\alpha_n k_n - \alpha_n Ld_n + 2\alpha_n \gamma} \|x_n - q\|^2.\]

Since \(\alpha_n \to 0, d_n \to 0\) and \(k_n \to 1\) as \(n \to \infty\), there exists a positive integer \(n_1\) such that, for all \(n \geq n_1\),

\[
(1 - \alpha_n)^2 + \alpha_n Ld_n - (1 - \frac{\gamma}{2} \alpha_n)(1 - 2\alpha_n k_n - \alpha_n Ld_n + 2\alpha_n L) \\
= \alpha_n [2(k_n - 1) + \alpha_n + 2Ld_n - \alpha_n k_n L - \frac{\gamma}{2} \alpha_n Ld_n + \alpha_n \gamma^2 - \frac{3}{2} \gamma] \\
\leq 0.
\]

This implies that

\[
\frac{(1 - \alpha_n)^2 + \alpha_n Ld_n}{1 - 2\alpha_n k_n - \alpha_n Ld_n + 2\alpha_n \gamma} \leq (1 - \frac{\gamma}{2} \alpha_n)
\]

for all \(n \geq n_1\). Therefore, (2.8) can be written as follows:

\[
\|x_{n+1} - q\|^2 \leq (1 - \frac{\gamma}{2} \alpha_n) \|x_n - q\|^2
\]

for all \(n \geq n_1\). Taking \(a_n = \|x_n - q\|^2, b_n = 0, t_n = \frac{\gamma}{2} \alpha_n\), it follows from Lemma 1.2 that \(a_n \to 0\), i.e., \(x_n \to q\) as \(n \to \infty\).

2. If \(\sigma = 0\). By the strictly increasing property of \(\phi\) and the definition of \(\sigma\), there exists a subsequence \(\{x_{n_j+1}\}\) of \(\{x_n\}\) such that \(\|x_{n_j+1} - q\| \to 0\) as \(j \to \infty\). In fact, since \(\delta = 0\), there exists a subsequence \(\{x_{n_j+1}\}\) of \(\{x_n\}\) such that

\[
\frac{\phi(\|x_{n_j+1} - q\|)}{\|x_{n_j+1} - q\|^2} \to 0
\]

as \(n_j \to \infty\). Letting \(M = \text{sup}\{\|y\| : y \in T(D)\}\). For any \(\epsilon > 0\), there exists a positive integer \(n_{j_0}\) such that

\[
\frac{\phi(\|x_{n_{j_0}+1} - q\|)}{\|x_{n_{j_0}+1} - q\|^2} \leq \frac{\phi^{-1}(\epsilon)}{M}
\]

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for \( n_j \geq n_{j_0} \), and so \( \phi(\|x_{n_j+1} - q\|) \leq \phi^{-1}(\epsilon) \), i.e., \( \{x_{n_j+1} - q\| \leq \epsilon \) for \( n_j \geq n_{j_0} \). Since \( \epsilon \) is arbitrary, we have

\[
\|x_{n_j+1} - q\| \to 0 \quad \text{as} \quad n_j \to \infty.
\]

Since \( \alpha_n \to 0 \), \( k_n \to 1 \) and \( d_n \to 0 \) as \( n \to \infty \), for any given \( \epsilon \in (0, 1) \), there exists a positive integer \( n_{j_0} \) such that, for all \( n \geq n_{j_0} \),

\[
\begin{align*}
\|x_{n_{j_0}+1} - q\| < \epsilon, \\
\alpha_n < \frac{1}{4} \phi(\epsilon), \quad Ld_n < \frac{1}{4} \phi(\epsilon), \\
k_n < 1 + \frac{1}{4} \phi(\epsilon).
\end{align*}
\]

Next, we prove that

\[
\|x_{n_{j_0}+i} - q\| \leq \epsilon
\]

for all \( i \geq 1 \). Indeed, for \( i = 1 \), the conclusion follows from (2.9). For \( i = 2 \), if

\[
\|x_{n_{j_0}+2} - q\| > \epsilon,
\]

then, by the strictly increasing property of \( \phi \), we have

\[
\phi(\|x_{n_{j_0}+2} - q\|) > \phi(\epsilon).
\]

Letting

\[
h_n = \frac{1}{1 - 2\alpha_n k_n - \alpha_n Ld_n}.
\]

Then the first term on the right side of (2.7) can be written as follows:

\[
\frac{(1 - \alpha_n)^2 + \alpha_n Ld_n}{1 - 2\alpha_n k_n - \alpha_n Ld_n} \|x_n - q\|^2
= \|x_n - q\|^2 + h_n \alpha_n (\alpha_n - 2 + 2Ld_n + 2k_n) \|x_n - q\|^2.
\]

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It follows from (2.7) and (2.9), we have

$$
\|x_{n_j+1} - q\|^2
\leq \|x_{n_j+1} - q\|^2 + h_{n_j+1}\alpha_{n_j+1}[(\alpha_{n_j+1} - 2 + 2Ld_{n_j+1}
+ 2k_{n_j+1})\|x_{n_j+1} - q\|^2 - 2\phi(\epsilon)]
\leq \epsilon^2 + h_{n_j+1}\alpha_{n_j+1}\left[\frac{\phi(\epsilon)}{4} - 2 + \frac{\phi(\epsilon)}{2} + 2\left(1 + \frac{1}{4}\phi(\epsilon)\right)\epsilon^2 - 2\phi(\epsilon)\right]
\leq \epsilon^2 + h_{n_j+1}\alpha_{n_j+1}\left[\frac{5}{4}\phi(\epsilon) - 2\phi(\epsilon)\right]
\leq \epsilon^2,
$$

which contradicts (2.11) and so we have

$$
\|x_{n_j+1} - q\|^2 \leq \epsilon.
$$

By induction, we can prove that (2.10) is true. By the arbitrariness of $\epsilon \in (0, 1)$, we have $x_n \rightarrow q$ as $n \rightarrow \infty$. This completes the proof. 

From Theorem 2.1, we can obtain the following result:

**Theorem 2.2.** Let $D$ be a nonempty closed convex subset of $E$ and $T : D \rightarrow D$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ in $[1, \infty)$ such that $\lim_{n \rightarrow \infty} k_n = 1$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$ satisfying the conditions (i) and (ii) in Theorem 2.1. Let $\{x_n\}$ be the modified Ishikawa iterative sequence defined by (1.1). If $F(T) \neq \emptyset$ and, for any given $q \in F(T)$, there exists a strictly increasing function $\phi : [0, \infty) \rightarrow [0, \infty)$ and $\phi(0) = 0$ such that the condition (2.1) in Theorem 2.1 is satisfied. Then the sequence $\{x_n\}$ converges strongly to the fixed point $q$ of $T$.

**Proof.** Since $T$ is an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ in $[1, \infty)$ and $\lim_{n \rightarrow \infty} k_n = 1$, $T$ is a uniformly $L$-Lipschitz asymptotically pseudo-contractive mapping with the sequence $\{k_n\}$ in $[1, \infty)$, $\lim_{n \rightarrow \infty} k_n = 1$ and the constant $L = \sup_{n \geq 1} k_n$. Therefore, the conclusion can be obtained from Theorem 2.1 immediately. 

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Theorem 2.3. Let $D$ be a nonempty closed convex subset of $E$ and $T : D \to D$ be a uniformly $L$-Lipschitzian asymptotically pseudo-contractive mapping with a sequence $\{k_n\}$ in $[1, \infty)$ such that

$$\lim_{n \to \infty} k_n = 1.$$ 

Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ satisfying the following conditions:
(i) $\alpha_n \to 0$ as $n \to \infty$,
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Let $\{x_n\}$ be the modified Mann iterative sequence defined by (1.2). If $F(T) \neq \emptyset$ and, for any given $q \in F(T)$, there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ and $\phi(0) = 0$ such that the condition (2.1) in Theorem 2.1 is satisfied. Then the sequence $\{x_n\}$ converges strongly to the fixed point $q$ of $T$.

Proof. Taking $\beta_n = 0$ for all $n = 0, 1, 2, \ldots$ in Theorem 2.1, the conclusion can be obtained immediately. \hfill \Box

Remark 2.1. Theorems 2.1, 2.2 and 2.3 extend and improve the main results in Goebel-Kirk [4], Liu [5] and Schu [7]. Moreover, the methods of proof given in this paper are more simple and quite different from the proof methods given in [4]-[6].

References

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S. S. Chang, Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, People’s Republic of China

J. Y. Park, Department of Mathematics, Pusan National University, Pusan 609-735, Korea

Y. J. Cho, Department of Mathematics, Gyeongsang National University, Chinju 660-701, Korea