ON THE FEKETE-SZEGÖ PROBLEM
AND ARGUMENT INEQUALITY FOR
STRONGLY QUASI-CONVEX FUNCTIONS

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Abstract. Let $Q(\beta)$ be the class of normalized strongly quasi-
convex functions of order $\beta$ in the open unit disk. Sharp Fekete-
Szegö inequalities are obtained for functions belonging to the class
$Q(\beta)$. We also consider the integral preserving properties in a sec-
tor.

1. Introduction

Let $A$ denote the class of functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$
and let $S$ be the subclass of $A$ consisting of all univalent functions. We
also denote by $S^*$, $K$ and $C$ the subclasses of $A$ consisting of functions
which are, respectively, starlike, convex and close-to-convex in $U$ (see,
e.g., Srivastava and Owa [17]).

For analytic functions $g$ and $h$ with $g(0) = h(0)$, $g$ is said to be
subordinate to $h$ if there exists an analytic function $w(z)$ such that
$w(0) = 0, |w(z)| < 1$ ($z \in U$), and $g(z) = h(w(z))$. We denote this
subordination by $g \prec h$ or $g(z) \prec h(z)$.

A classical result of Fekete and Szegö [5] determines the maximum
value of $|a_3 - \mu a_2^2|$, as a function of the real parameter $\mu$, for functions

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belonging to $S$. There are now several results of this type in the literature, each of them dealing with $|\alpha_3 - \mu a_2^2|$ for various classes of functions (see, e.g., [1, 7, 9]).

Denote by $Q(\beta)$ the class of strongly quasi-convex functions of order $\beta (\beta \geq 0)$. Thus $f \in Q(\beta)$ if and only if there exists $g \in K$ such that for $z \in U$,

$$\left| \arg \left\{ \left( \frac{zf'(z)}{g'(z)} \right)' \right\} \right| \leq \frac{\pi}{2} \beta.$$

In particular, $Q(1)$ is the class of quasi-convex functions introduced by Noor [13]. We also note that every quasi-convex function is close-to-convex and hence univalent in $U$.

In the present paper, we derive sharp Fekete-Szegö inequalities for functions belonging to the class $Q(\beta)$. Furthermore, the integral preserving properties are considered for functions in the class $Q(\beta)$.

2. Results

To prove our main results, we need the following lemmas.

**Lemma 2.1.** Let $p$ be analytic in $U$ and satisfy $\Re \{p(z)\} > 0$ for $z \in U$, with $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$. Then

\[(2.1) \quad |p_n| \leq 2 \quad (n \geq 1)\]

and

\[(2.2) \quad \left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2}.\]

**Lemma 2.2.** Let $h$ be convex (univalent) function in $U$ and $\omega$ be an analytic function in $U$ with $\Re \{\omega(z)\} \geq 0$. If $p$ is analytic in $U$ and $p(0) = h(0)$, then

$$p(z) + \omega(z)zp'(z) \prec h(z) \quad (z \in U)$$

implies

$$p(z) \prec h(z) \quad (z \in U).$$
**Lemma 2.3.** Let \( p \) be analytic in \( \mathcal{U} \) with \( p(0) = 1 \) and \( p(z) \neq 0 \) in \( \mathcal{U} \). Suppose that there exists a point \( z_0 \in \mathcal{U} \) such that

\[
\left| \arg \{ p(z) \} \right| < \frac{\pi}{2} \eta \quad \text{for} \quad |z| < |z_0|,
\]

and

\[
\left| \arg \{ p(z_0) \} \right| = \frac{\pi}{2} \eta (0 < \eta \leq 1).
\]

Then

\[
\frac{z_0 p'(z_0)}{p(z_0)} = ik\eta,
\]

where

\[
k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg \{ p(z_0) \} = \frac{\pi}{2} \eta,
\]

\[
k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg \{ p(z_0) \} = -\frac{\pi}{2} \eta,
\]

and

\[
\{ p(z_0) \}^{\frac{1}{a}} = \pm ia \quad (a > 0).
\]

The inequality (2.1) was first proved by Carathéodory [3] (also, see Duren [4, p. 41]) and the inequality (2.2) can be found in [15, p. 166]. Lemma 2.2 are the result proved by Miller and Mocanu [11], which has a number of important applications in the theory of univalent functions. Also Lemma 2.3 was proved by Nunokawa [14] as a new modification of well known Jack's Lemma [6].

With the help of Lemma 2.1, we now derive

**Theorem 2.1.** Let \( f \in Q(\beta) \) and be given by (1.1). Then for \( \beta \geq 0 \), we have

\[
9 \left| a_3 - \mu a_2^2 \right| \leq \begin{cases} 
1 + \frac{(1+\beta)^2(8-9\mu)}{4} & \text{if} \quad \mu \leq \frac{8\beta}{9(1+\beta)}, \\
1 + 2\beta + \frac{2(8-9\mu)}{8-\beta(8-9\mu)} & \text{if} \quad \frac{8\beta}{9(1+\beta)} \leq \mu \leq \frac{8}{9}, \\
1 + 2\beta & \text{if} \quad \frac{8}{9} \leq \mu \leq \frac{8(2+\beta)}{9(1+\beta)}, \\
-1 + \frac{(1+\beta)(2\mu-8)}{4} & \text{if} \quad \mu \geq \frac{8(2+\beta)}{9(1+\beta)}.
\end{cases}
\]
For each \( \mu \), there is a function in \( \mathcal{Q}(\beta) \) such that equality holds in all cases.

**Proof.** Let \( f \in \mathcal{Q}(\beta) \). Then it follows from the definition that we may write

\[
\frac{(zf'(z))'}{g'(z)} = p^\beta(z),
\]

where \( g \) is convex and \( p \) has positive real part. Let \( g(z) = z + b_2 z^2 + b_3 z^3 + \cdots \) and let \( p \) be given as in Lemma 2.1. Then by comparing the coefficients of both sides of (2.9), we obtain

\[
4a_2 = \beta p_1 + 2b_2
\]

and

\[
9a_3 = \frac{\beta(\beta - 1)}{2} p_1^2 + \beta p_2 + 3b_3 + 2\beta p_1 b_2.
\]

So, with \( x = (8 - 9\mu)/4 \), we have

\[
9(a_3 - \mu a_2^2) = 3 \left( b_3 + \frac{1}{3}(x - 2)b_2^2 \right)
\]

\[
+ \beta \left( p_2 + \frac{1}{4}(\beta x - 2)p_1^2 \right) + \beta x p_1 b_2.
\]

(2.10)

Since rotations of \( f \) also belong to \( \mathcal{Q}(\beta) \), without loss of generality, we may assume that \( a_3 - \mu a_2^2 \) is positive. Thus we now estimate \( \text{Re} \ (a_3 - \mu a_2^2) \).

Since \( g \in \mathcal{K} \), there exists \( h(z) = 1 + k_1 z + k_2 z^2 + \cdots \quad (z \in \mathcal{U}) \) with positive real part, such that \( g'(z) + zg''(z) = g'(z)h(z) \). Hence, by equating coefficients, we get that \( b_2 = k_1/2 \) and \( b_3 = (k_2 + k_1^2)/6 \). Therefore, letting \( b_2 = re^{i\phi}(0 \leq \rho \leq 1) \) and \( p_1 = 2re^{i\theta}(0 \leq r \leq 1) \) in (2.10), and applying Lemma 2.1, we obtain

\[
9\text{Re}(a_3 - \mu a_2^2) \leq (1 - \rho^2) + (x + 1)\rho^2 \cos 2\phi
\]

\[
+ 2\beta(1 - r^2) + \beta^2 r^2 \cos 2\theta + 2\beta xr \rho \cos(\theta + \phi)
\]

\[
= \psi(x), \quad \text{say}.
\]

(2.11)
We consider first the case \(8\beta/(9(1 + \beta)) \leq \mu \leq 8/9\). In this case, we see that \(0 \leq x \leq 2/(1 + \beta)\). Then we obtain
\[
\psi(x) = 1 - \rho^2 + (x + 1)\rho^2 \cos 2\phi + \beta(2(1 - r^2) + \beta x r^2 \cos 2\theta + 2x r \rho \cos(\theta + \phi)) \\
\leq x + 1 + \beta(2 - 2r^2 + \beta x r^2 \cos 2\theta + 2x r).
\]
Since the expression \(-2t^2 + \beta xt^2 \cos 2\theta + 2xt\) is the largest when \(t = x/(2 - \beta x \cos 2\theta)\), we have
\[
-2t^2 + \beta xt^2 \cos 2\theta + 2xt \leq \frac{x^2}{2 - \beta x \cos 2\theta} \leq \frac{x^2}{2 - \beta x}.
\]
Thus
\[
\psi(x) \leq x + 1 + \beta \left(2 + \frac{x^2}{2 - \beta x}\right) = 1 + 2\beta + \frac{2(8 - 9\mu)}{8 - \beta(8 - 9\mu)},
\]
and from (2.11), we obtain the second inequality of the theorem. Equality occurs only if
\[
p_1 = \frac{2(8 - 9\mu)}{8 - \beta(8 - 9\mu)}, \quad p_2 = 2, \quad b_2 = b_3 = 1,
\]
and the corresponding function \(f\) is defined by
\[
(zf'(z))^c = \frac{1}{(1 - z)^2} \left(\frac{1 + z}{1 - z} + (1 - \lambda)\frac{1 - z}{1 + z}\right)^B, \quad f(0) = 0,
\]
where
\[
\lambda = \frac{8 + (1 - \beta)(8 - 9\mu)}{16 - 2\beta(8 - 9\mu)}.
\]
We now prove the first inequality. Let \(\mu \leq 8\beta/(9(1 + \beta))\). Then we obtain that \(x \geq 2/(1 + \beta) = x_0\), and
\[
\psi(x) = \psi(x_0) + (x - x_0)(\rho^2 \cos 2\phi + \beta^2 r^2 \cos 2\theta + 2\beta \rho r \cos(\theta + \phi)) \\
\leq \psi(x_0) + (x - x_0)(1 + \beta)^2 \\
\leq 1 + \frac{(1 + \beta)^2(8 - 9\mu)}{4},
\]
as required. Equality occurs only if \( c_1 = c_2 = 2, \ b_2 = b_3 = 1 \), and the corresponding function \( f \) is defined by

\[
(z f'(z))' = \frac{1}{(1 - z)^2} \left( \frac{1 + z}{1 - z} \right)^\beta, \quad f(0) = 0.
\]

Let \( x_1 = -2/(1 + \beta) \). At first, we will show that \( \psi(x_1) \leq 1 + 2\beta \). Then the remaining inequalities follow easily from this one. We have

\[
(-2 + \beta x_1 \cos 2\theta)t^2 + 2x_1 t \rho \cos(\theta + \phi) \leq \frac{x_1^2 \rho^2 \cos^2(\theta + \phi)}{2 - \beta x_1 \cos 2\theta}
\]

for all real \( t \). Hence we obtain

\[
\psi(x_1) - (1 + 2\beta) \\
\leq \rho^2 \left( -1 + (x_1 + 1) \cos 2\phi + \frac{\beta x_1^2 (1 + \cos 2(\theta + \phi))}{2(2 - \beta x_1 \cos 2\theta)} \right).
\]

Thus we consider the inequality

\[
\beta x_1^2 (1 + \cos 2(\theta + \phi)) + 2(2 - \beta x_1 \cos 2\theta)(-1 + (x_1 + 1) \cos 2\phi) \leq 0,
\]

which is true if

(2.12) \[ 2\beta^2 \cos^2 \theta \sin^2 \phi + 2\beta \cos \theta \sin \theta \cos \phi \sin \phi + \cos^2 \phi \geq 0. \]

Now, for all real \( t \),

\[
2t^2 + 2t \sin \theta \cos \phi + \cos^2 \phi \geq 0,
\]

so, by taking \( t = \beta \cos \theta \sin \phi \), we obtain (2.12). Thus \( \psi(x_1) \leq 1 + 2\beta \).

Next, we consider two possibilities. We suppose that \( x_1 \leq x \leq 0 \), that is, \( 8/9 \leq \mu \leq 8(2 + \beta)/(9(1 + \beta)) \). Note that for \( 0 \leq \lambda \leq 1 \),

\[
\psi(\lambda x_1) = \lambda \psi(x_1) + (1 - \lambda)\psi(0) = 1 + 2\beta.
\]

Hence we have \( \psi(x) \leq 1 + 2\beta \) and this proves the third inequality of the theorem. Equality occurs only if \( p_1 = b_2 = 0, \quad p_2 = 2, \quad b_3 = 1/3 \), and the corresponding function \( f \) is defined by

\[
(z f'(z))' = \frac{(1 + z^2)^\beta}{(1 - z^2)^{1+\beta}}, \quad f(0) = 0.
\]
Secondly, we suppose that $x \leq x_1$, that is, $\mu \geq (8(2 + \beta))/(9(1 + \beta))$. Then we have

$$
\psi(x_0) = \psi(x_1) + (x - x_1)(\rho^2 \cos 2\phi + \beta r^2 \cos 2\theta + 2\beta \rho r \cos(\theta + \phi)) \\
\leq \psi(x_1) + (x_1 - x)(1 + \beta)^2 \\
\leq -1 + \frac{(1 + \beta)^2(9\mu - 8)}{4},
$$

and this is the last inequality of the theorem. Equality occurs only if $p_1 = 2i$, $p_2 = -2$, $b_2 = i$, $b_3 = -1$, and the corresponding function $f$ is defined by

$$
(zf'(z))' = \frac{1}{(1 - iz)^2} \left(\frac{1 + iz}{1 - iz}\right)^\beta, \quad f(0) = 0.
$$

Therefore we complete the proof of Theorem 2.1.

For a function $f$ belonging to the class $A$, we define the integral operator $F_\gamma$ as follows:

$$
(2.13) \quad F_\gamma(f) := F_\gamma(f)(z) = \frac{\gamma + 1}{z^\gamma} \int_0^z t^{\gamma - 1}f(t)dt \quad (\gamma \geq 0; \quad z \in \mathcal{U}).
$$

Many authors have studied the integral operator of the form (2.13) where $\gamma$ is a real constant and $f$ belongs to some favored classes of functions. Various interesting developments involving the operator (2.13), for examples, can be found in [2, 8, 10]. We also denote the class $K[A,B]$ by

$$
K[A,B] = \left\{ f \in A : 1 + \frac{zf''(z)}{f'(z)} < \frac{1 + Az}{1 + Bz} \quad (z \in \mathcal{U}; \quad -1 \leq B < A \leq 1) \right\}.
$$

Next, we prove

**Theorem 2.2.** Let $f \in A$. If

$$
\left| \arg \left\{ \left(\frac{zf'(z)}{g'(z)}\right)' \right\} \right| < \frac{\pi}{2} \delta \quad (0 < \delta \leq 1; \quad z \in \mathcal{U})
$$

for some $g \in K[A,B]$, then

$$
\left| \arg \left\{ \left(\frac{zf'_\gamma(f)}{F'_\gamma(g)}\right)' \right\} \right| < \frac{\pi}{2} \eta,
$$
where \( F_\gamma \) is given by (2.13) and \( \eta(0 < \eta \leq 1) \) is the solution of the equation:

(2.14)

\[
\delta = \begin{cases} 
\eta + \frac{2}{\pi} \tan^{-1} \left( \frac{\eta \sin \frac{\eta}{2} (1-t(A,B,c))}{(1+\frac{A^2}{2}+c)+\eta \cos \frac{\eta}{2} (1-t(A,B,c))} \right) & \text{for } B \neq -1, \\
\eta & \text{for } B = -1,
\end{cases}
\]

when

(2.15)

\[
t(A, B, c) = \frac{2}{\pi} \sin^{-1} \left( \frac{A - B}{1 - AB + c(1 - B^2)} \right).
\]

Proof. Let

\[
p(z) = \frac{z(F'_\gamma(f))'}{F'_\gamma(g)} \quad \text{and} \quad q(z) = 1 + \frac{zF''_\gamma(g)}{F'_\gamma(g)}.
\]

From the assumption for \( g \) and an application of Briot-Bouquet differential subordination [12, p. 81], we see that \( F_\gamma(g) \in K[A, B] \). Using the equation

\[
zF'_\gamma(f)(z) + \gamma F_\gamma(f)(z) = (1 + \gamma)f(z)
\]

and simplifying, we obtain

\[
\frac{(zf'(z))'}{g'(z)} = p(z) + \frac{zp'(z)}{q(z) + c}.
\]

Since \( q \in K[A, B] \), we note [16] that

(2.16)

\[
\left| \frac{(zF'_\gamma(g))'}{F'_\gamma(g)} - \frac{1 - AB}{1 - B^2} \right| < \frac{A - B}{1 - B^2} \quad (z \in \mathcal{U} ; \ B \neq -1)
\]

and

(2.17)

\[
\text{Re} \left\{ \frac{(zF'_\gamma(g))'}{F'_\gamma(g)} \right\} > \frac{1 - A}{2} \quad (z \in \mathcal{U} ; \ B = -1).
\]

Then, from (2.16) and (2.17), we have

\[
q(z) + c = \rho e^{i \frac{\pi}{2} \phi},
\]
where

\[
\begin{align*}
\{ & \frac{1-A}{1-B} + c < \rho < \frac{1-A}{1+B} + c \\
& -t(A, B, c) < \phi < t(A, B, c) \text{ for } B \neq -1,
\end{align*}
\]

where \( t(A, B, c) \) is given by (2.15), and

\[
\begin{align*}
\{ & \frac{1-A}{2} + c < \rho < \infty \\
& -1 < \phi < 1 \text{ for } B = -1.
\end{align*}
\]

Here, we note that \( p \) is analytic in \( U \) with \( p(0) = 1 \) and \( \text{Re} \, p(z) > 0 \) in \( U \) by applying the assumption and Lemma 2.2 with \( \omega(z) = 1/(q(z) + c) \). Hence \( p(z) \neq 0 \) in \( U \).

If there exists a point \( z_0 \in U \) such that the conditions (2.3) and (2.4) are satisfied, then (by Lemma 2.3) we obtain (2.5) under the restrictions (2.6-8).

At first, we suppose that

\[
\{ p(z_0) \} \frac{1}{\eta} = ia \quad (a > 0).
\]

For the case \( B \neq -1 \), we then obtain

\[
\begin{align*}
\arg & \left\{ \frac{(z_0f''(z_0))'}{g'(z_0)} \right\} \\
= & \arg \left\{ p(z_0) \left( 1 + \frac{1}{q(z_0) + c} \frac{z_0p'(z_0)}{p(z_0)} \right) \right\} \\
= & \arg \{ p(z_0) \} + \arg \left\{ 1 + \eta k (\rho e^{i\frac{\pi}{2}})^{-1} \right\} \\
= & \frac{\pi}{2} \eta + \tan^{-1} \left( \frac{\eta k \sin[\frac{\pi}{2}(1 - \phi)]}{\rho + \eta k \cos[\frac{\pi}{2}(1 - \phi)]} \right) \\
\geq & \frac{\pi}{2} \eta + \tan^{-1} \left( \frac{\eta \sin[\frac{\pi}{2}(1 - t(A, B))] \left( \frac{1+A}{1+B} + c \right) + \eta \cos[\frac{\pi}{2}(1 - t(A, B))]}{\eta \sin[\frac{\pi}{2}(1 - t(A, B))] + \eta \cos[\frac{\pi}{2}(1 - t(A, B))] \left( \frac{1+A}{1+B} + c \right)} \right) \\
= & \frac{\pi}{2} \delta,
\end{align*}
\]

where \( \delta \) and \( t(A, B) \) are given by (2.14) and (2.15), respectively. Similarly, for the case \( B = -1 \), we have

\[
\arg \left\{ \frac{(z_0f''(z_0))'}{g'(z_0)} \right\} \geq \frac{\pi}{2} \eta = \frac{\pi}{2} \delta.
\]
These evidently contradict the assumption of the theorem.

Next, in the case $p(z_0)^{\delta} = -ia$ ($a > 0$), applying the same method as the above, we also can prove the theorem easily. Therefore we complete the proof of Theorem 2.2.

**Remark.** From Theorem 2.2, we see easily that every function in $Q(\delta)$ ($0 < \delta \leq 1$) preserves the angles under the integral operator defined by (2.13).

By letting $g(z) = z$ and $B \to A$ ($A < 1$) in Theorem 2.2, we have

**Corollary.** If $f \in \mathcal{A}$ and

$$|\arg \{(zf'(z))'\}| < \frac{\pi}{2} \delta \ (0 < \delta \leq 1 \ ; \ z \in \mathcal{U})$$

then

$$|\arg \{(zF_\gamma'(f))'\}| < \frac{\pi}{2} \eta$$

where $F_\gamma$ is given by (2.13) and $\eta(0 < \eta \leq 1)$ is the solution of the equation:

$$\delta = \eta + \frac{2}{\pi} \tan^{-1}\left(\frac{\eta}{1+c}\right).$$

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**References**


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