AN INTEGRAL FORMULA AND ITS APPLICATIONS

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Abstract. In this paper, we obtain an integral formula relating the measure of great spheres $S^{n-2}$ and arc length of a curve on the unit sphere $S^{n-1}$. As an application of the formula, we develop a geometric inequality for a spherical curve and prove generalized version of Fenchel’s theorem in $R^n$.

1. Introduction

Fenchel’s theorem states that $\int k ds \geq 2\pi$, with equality if and only if the curve is a convex plane curve. J. W. Milnor [4] reproved the result with a different method and many other mathematicians generalized the result to $R^n$.

The main purpose of this paper is to provide a simple and shorter proof than those previously known. Also, we present a geometric inequality for a spherical curve. The proofs are based on the Crofton’s formula on the measure of great spheres $S^{n-2}$ on the unit sphere $S^{n-1}$.

2. Preliminaries

Let $\alpha = \{ (c_1(s), c_2(s), \ldots, c_n(s)) | 0 \leq s \leq l \}$ be a closed curve in $R^n$ with arc length parameters.

For a Frenet frame

$$(\alpha(s), e_1(s), \cdots, e_n(s))$$

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of the curve $\alpha$, the Frenet equations are

$$\frac{d\alpha}{ds} = e_1,$$

$$\frac{de_i}{ds} = -\kappa_{i-1}(s)e_{i-1} + \kappa_i(s)e_{i+1}.$$ 

Here, $\kappa_i$'s are curvatures. Then the unit tangent vector $e_1(s) = \frac{d}{ds} (c_1(s), c_2(s), \cdots, c_n(s)) = (T_1(s), T_2(s), \cdots, T_n(s))$ defines a tangent curve on the unit sphere $S^{n-1}$. The total curvature of $\alpha$ is defined to be the quantity $\int_0^l \kappa_1(s)ds$, $\kappa_1(s) = \left\|\frac{de_1(s)}{ds}\right\|$. Thus the total curvature of $\alpha$ is the length of its tangent curve.

The following lemma shows how tangent curve of a closed curve ranges.

**Lemma 1.** Tangent curve $e_1(s)$ of a closed curve is not contained in any open hemisphere. $e_1$ is contained in a closed hemisphere if and only if $\alpha$ is in a hyperplane.

**Proof.** If $e_1$ were contained in a hemisphere, we may assume $T_n(s) \geq 0$ for all $0 \leq s \leq l$. Since $\alpha = (c_1(s), c_2(s), \cdots, c_n(s))$ is a closed curve,

\begin{equation}
0 = c_n(l) - c_n(0) = \int_0^l T_n(s)ds.
\end{equation}

Thus $T_n(s)$ cannot be strictly positive. Hence, $e_1$ cannot lie in an open hemisphere. From (1), since $T_n(s)$ is nonnegative, it must vanish identically, that is, $0 = T_n(s) = \frac{d}{ds}c_n(s)$. Hence $\alpha$ must be in a hyperplane $c_n(s) = \text{constant}$. Conversely, if $\alpha$ is in a hyperplane, then $e_1$ lies on a great sphere $S^{n-2}$ and hence is contained in a closed hemisphere. \qed

Every oriented great sphere determines uniquely a pole, the endpoint of the unit vector normal to the unit sphere $S^{n-2}$. So the following definition is meaningful.

**Definition 1.** The area of the domain of their poles is meant by the measure of a set of great spheres $S^{n-2}$ on the unit sphere $S^{n-1}$.

**3. Result**

Now we will give the measure of a set of great spheres $S^{n-2}$ on the unit sphere $S^{n-1}$ to develop an integral formula concerning an arc on the unit sphere $S^{n-1}$.
THEOREM 1. Let $\gamma$ be a smooth arc on the unit sphere $S^{n-1}$. The measure of the oriented great spheres $S^{n-2}$ of $S^{n-1}$ which meet $\gamma$, each counted a number of times equal to the number of its common points with $\gamma$, is equal to \( \frac{\text{Val}(S^{n-1})}{\pi} \) times the length of $\gamma$.

Proof. We suppose $\gamma$ is defined by a unit vector $e_1(s)$ expressed as a function of its arc length $s$. In a certain neighborhood of $s$, we take a frame field \( \{e_2(s), \ldots, e_n(s)\} \) as follows; we set \( \{e_2(s), e_3(s)\} \) such that \( \frac{de}{ds} = a_{12}e_2 + a_{13}e_3 \), \( \{e_2(s), \ldots, e_n(s)\} \) satisfies $e_i \cdot e_j = \delta_{ij}, 1 \leq i, j \leq n$ and $\det(e_1, e_2, \ldots, e_n) = +1$. From differentiation of $e_1 \cdot e_j = \delta_{ij}$ and \( \{e_2(s), e_3(s)\} \), we obtain the skew-symmetric matrix of the coefficients as follows;

\[
\begin{pmatrix}
\frac{de_1}{ds} \\
\frac{de_2}{ds} \\
\frac{de_3}{ds} \\
\vdots \\
\frac{de_n}{ds}
\end{pmatrix} = \begin{pmatrix}
0 & a_{12} & a_{13} & 0 & \cdots & 0 \\
-a_{12} & 0 & a_{23} & a_{24} & \cdots & a_{2n} \\
-a_{13} & -a_{23} & 0 & a_{34} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & -a_{2i} & -a_{3i} & -a_{4i} & \cdots & a_{in} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & -a_{2n} & -a_{3n} & -a_{4n} & \cdots & 0
\end{pmatrix} \begin{pmatrix}
e_1 \\
e_2 \\
e_3 \\
\vdots \\
e_i \\
\vdots \\
e_n
\end{pmatrix}.
\]

If an oriented great sphere $S^{n-2}$ meets $\gamma$ at the point $e_1(s)$, its pole is of the form

\[
\phi(s, \theta_1, \ldots, \theta_{n-2}) = (\sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2})e_2(s) + \cdots + (\cos \theta_{n-2} \sin \theta_{n-1} \cdots \sin \theta_{n-2})e_n(s) + \cdots + \cos \theta_{n-2}e_n(s),
\]

where $0 \leq \theta_1 \leq 2\pi, 0 \leq \theta_2, \ldots, \theta_{n-2} \leq \pi$.

Thus \( (s, \theta_1, \ldots, \theta_{n-2}) \) serve as local coordinates in the domain of these poles, we wish to find an expression for the element of area of this
\[ \phi_s = (\sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2}) e'_0(s) + \cdots \\
+ (\cos \theta_{n-2} \sin \theta_1 \cdots \sin \theta_{n-2}) e'_1(s) + \cdots + \cos \theta_{n-2} e'_{n-1}(s) \\
= -(a_{12} \sin \theta_1 \cdots \sin \theta_{n-2} + a_{13} \cos \theta_1 \cdots \sin \theta_{n-2}) e_1 + \\
(a_{23} \sin \theta_1 \cdots \sin \theta_{n-2} + \cdots - a_{2n} \cos \theta_{n-2}) e_2 + \cdots \\
+ (a_{2n} \sin \theta_1 \cdots \sin \theta_{n-2} + \cdots - a_{(n-1)n} \cos \theta_{n-2}) e_n. \]

\[ \phi_{\theta_1} = \cos \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} e_2 - \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} e_3 \]

\[ \vdots \]

\[ \phi_{\theta_{n-2}} = \sin \theta_1 \sin \theta_2 \cdots \cos \theta_{n-2} e_2 + \cdots - \sin \theta_{n-2} e_n. \]

Hence, the element of area of \( \phi \) is

\[ |dA| = (\det(E_i \cdot E_j))^{\frac{1}{2}} \]

\[ = \sin^2 \theta_2 \sin^3 \theta_3 \cdots \sin^{n-2} \theta_{n-2} |a_{12} \sin \theta_1 + a_{13} \cos \theta_1| ds \, d\theta_1 \cdots d\theta_{n-2}, \]

where \( 0 \leq \theta_1 \leq 2\pi, 0 \leq \theta_2, \ldots, \theta_{n-2} \leq \pi. \)

On the other hand, since \( s \) is the arc length of \( \gamma \), we have

\[ a_{12}^2 + a_{13}^2 = 1, \]

and we put \( a_{12} = \cos \rho(s), a_{13} = \sin \rho(s), \) for some \( \rho(s) \). Then

\[ |dA| = \sin^2 \theta_2 \cdots \sin^{n-2} \theta_{n-2} |a_{12} \sin \theta_1 + a_{13} \cos \theta_1| ds \, d\theta_1 \cdots d\theta_{n-2} \]

\[ = \sin^2 \theta_2 \cdots \sin^{n-2} \theta_{n-2} |\cos \rho(s) \sin \theta_1 + \sin \rho(s) \cos \theta_1| ds \, d\theta_1 \cdots d\theta_{n-2} \]

\[ = \sin^2 \theta_2 \cdots \sin^{n-2} \theta_{n-2} |\sin(\rho(s) + \theta_1)| ds \, d\theta_1 \cdots d\theta_{n-2}. \]

Let \( X \) be the oriented great sphere \( S^{n-2} \) with \( \phi \) as its pole, and let \( n(X) \) be the number of points common to \( X \) and \( \gamma \). Then the measure of oriented great spheres \( S^{n-2} \) meeting \( \gamma \) in our theorem is given by

\[ \int n(X) |dA| = \int_0^t ds \int_0^\pi \cdots \int_0^{2\pi} |\sin(\rho(s) + \theta_1)| \sin^2 \theta_2 \cdots \sin^{n-2} \theta_{n-2} \, d\theta_1 \cdots d\theta_{n-2} \]

\[ = 4t \int_0^\pi \cdots \int_0^\pi \sin^2 \theta_2 \cdots \sin^{n-2} \theta_{n-2} \, d\theta_2 \cdots d\theta_{n-2}. \]
But
\[ \text{Vol}(S^{n-1}) = \int_0^\pi \cdots \int_0^{2\pi} |\sin \tau_2| \sin^2 \tau_3 \sin^3 \tau_4 \cdots \sin^{n-2} \tau_{n-1} d\tau_1 \cdots d\tau_{n-1} \]
\[ = 2\pi \cdot 2 \int_0^\pi \cdots \int_0^{\pi} \sin^2 \tau_3 \sin^3 \tau_4 \cdots \sin^{n-2} \tau_{n-1} d\tau_3 \cdots d\tau_{n-1}. \]

Thus
\[ \int_0^\pi \cdots \int_0^\pi \sin^2 \tau_3 \sin^3 \tau_4 \cdots \sin^{n-2} \tau_{n-1} d\tau_3 \cdots d\tau_{n-1} = \frac{\text{Vol}(S^{n-1})}{4\pi}. \]

Therefore, \[ \int n(X)|dA| = \frac{\text{Vol}(S^{n-1})}{\pi} l. \]

\[ \Box \]

**Corollary 1.** Let \( \alpha \) be a closed curve in \( \mathbb{R}^n \). Then \( \int \kappa ds \geq 2\pi \), with equality if and only if the curve is a convex plane curve.

**Proof.** We have known that the total curvature of \( \alpha \) is the length of its tangent curve. Furthermore, from Lemma 1, the tangent curve of a closed curve meets every great sphere \( S^{n-2} \) at least two points. So \( n(X) \geq 2 \). It follows that its length is
\[ l = \int \kappa ds = \frac{\pi}{\text{Vol}(S^{n-1})} \int n(X)|dA| \geq \frac{\pi}{\text{Vol}(S^{n-1})} \cdot 2 \cdot \text{Vol}(S^{n-1}) = 2\pi. \]

It remains to prove the second part of theorem.

If \( \alpha \) is a plane convex curve, then \( c_1 \) is contained in a closed hemisphere. Furthermore, \( c_1 \) lies on a great circle. Since \( \alpha \) is convex, \( c_1 \) is a great circle. Thus \( n(X) \) is equal to 2. If \( \alpha \) is not a plane convex curve, \( c_1 \) is not contained in any closed hemisphere from Lemma 1. So for some position of \( S^{n-1} \), \( n(X) > 2 \). Hence \( \int \kappa ds > 2\pi \). This completes the proof of our corollary. \[ \Box \]

A non-oriented geodesic \( C \) on \( S^2 \) can be determined by one of its poles, that is, by either of the extremities of the diameter perpendicular to it. We consider a fixed geodesic \( C_0 \) and a fixed point \( P \) on it. The geodesic \( C \) can be determined for the abscissa \( t \) of one of the intersection points from \( C \) and \( C_0 \) and the angle \( \phi \) between the two circles. From [5], we have the density for measuring sets of geodesics on \( S^2 \) as follows;

\[ dC = \sin \phi d\phi dt. \]

Let \( K \) be a spherical oriented closed curve and \( C_0 \) be a fixed geodesic. Let \( \tau(s) \) be the angle between \( C_0 \) and the geodesic tangent to \( K \) at
$K(s)$ parametrized by arc length $s$ of $K$. Then the curvature of $K$ at $s$ is defined by $\kappa = \frac{d\tau}{ds}$ and the absolute total curvature is defined by the integral

$$c_a = \int_K |\kappa| ds = \int_K |d\tau|.$$  

The curvature is assigned a magnitude, measuring the rate of deviation from geodesic-ahedness. The absolute total curvature is a quantity which measures the total turning of the tangent geodesic.

The breadth corresponding to a point $A$ of an oriented closed curve $K$ on $S^2$ is equal to the length of arc $AB$ (Figure 1) of the geodesic orthogonal to $K$ at the point $A$ which is comprehended between $A$ and the point of intersection with another geodesic tangent to $K$ also orthogonal to $AB$ and contain $K$ between two tangent geodesics.

The following inequality is a generalization to the sphere of the inequality by Fáry [3] obtained for plane curves.

**Corollary 2.** If a closed curve of length $L$ on the sphere $S^2$ with absolute total curvature $c_a$ can be enclosed by a spherical circle of radius $\rho$, then $L \leq \rho c_a$.

**Proof.** Let $K$ be a spherical oriented closed curve, so that there is a prescribed sense of rotation. Let $C_0$ be a fixed geodesic. Let $s$ denote the
arc length of $K$ and let $\tau(s)$ be the angle between the tangent geodesic to $K$ and a fixed geodesic. Let $v(\tau)$ denote the number of unoriented tangents to $K$ that has the direction $\tau$. Since each direction $\tau$ appears $v(\tau)$ times, the equation (3) can be written

$$c_a = \int_0^\pi v(\tau) |d\tau|.$$  

On the other hand, if a geodesic $C$ has the direction $\tau$ and meets $K$ in $n$ points $p_i$, then there are at least $n$ tangents to $K$ that has the direction $\tau$ (one for each of the arcs $p_1p_2, p_2p_3, \cdots, p_{n-1}p_n, p_np_1$) and thus $n(\tau) \leq v(\tau)$. From our theorem for non-directed geodetics and (2), we have

$$2L = \int_{C \cap K \neq \emptyset} n dt C = \int dt \int n |\sin \tau | d\tau$$

$$\leq \Delta_m \int_0^\pi v |d\tau| = \Delta_m c_a$$

where $L$ is the length of $K$ which lies inside a spherical circle of radius $\rho$. Since $\Delta_m$ is the maximal breadth of $K$, $\Delta_m \leq 2\rho$. So $L \leq \rho c_a$. \qed

References