THE RANGE OF DERIVATIONS ON BANACH ALGEBRAS

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ABSTRACT. In this paper we show that if $D$ is a continuous linear Jordan derivation on a Banach algebra $A$ satisfying $||D(x^n), x^n||, x^n \in \text{rad}(A)$ for a positive integer $n$ and for all $x \in A$, then $D$ maps $A$ into $\text{rad}(A)$.

1. Introduction

Throughout this paper $R$ will represent an associative ring with center $Z$ and $A$ an associative algebra over a complex field $C$. $Z$ will represent the set of all integers and $Z^+$ the set of all positive integers. The (Jacobson) radical of $A$ is the intersection of all primitive ideals of $A$ and denoted by $\text{rad}(A)$. A ring $R$ is said to be $n$-torsion free if $nx = 0, x \in R$ implies $x = 0$. The commutator $xy - yx$ will be denoted by $[x, y]$, and we make extensive use of the basic identities $[xy, z] = [x, z]y + x[y, z], [x, yz] = [x, y]z + y[x, z]$. Recall that a ring $R$ is prime if $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$, and is semiprime if $aRa = \{0\}$ implies that $a = 0$. An additive mapping $D$ from $R$ to $R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. A derivation $D$ is inner if there exists an $a \in R$ such that $D(x) = [a, x]$ holds for all $x \in R$. An additive mapping $D$ from $R$ to $R$ is called a Jordan derivation if $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. A mapping $F$ from $R$ to $R$ is said to be commuting on $R$ if $[F(x), x] = 0$ holds for all $x \in R$, and is said to be centralizing on $R$ if $[F(x), x] \in Z$ holds for all $x \in R$. Obviously, every derivation is a Jordan derivation. The converse is in general not true. Brešar showed that every Jordan derivation on
a 2-torsion free semiprime ring is a derivation [1]. There has been con-
siderable interest in commuting, centralizing, and related mappings in
prime and semiprime rings. K. W. Jun and B. D. Kim [7] have obtained
the algebraic condition that every derivation on a Banach algebra maps
into its radical. In this paper we shall give the various algebraic condi-
tions on prime ring that every derivation on the ring is zero and using
these results, we show that every continuous linear Jordan derivation
with some conditions on a Banach algebra maps into its radical.

2. Preliminaries

We list a few more or less well-known results which will be needed in
the sequel.

**Remark.** $R$ will represent a prime ring with center $Z$ and extended
centroid $C$.

1. Suppose that the elements $a_i, b_i$ in the central closure of $R$ satisfy
   $\Sigma a_i y b_i = 0$. If $b_i \neq 0$ for some $i$ then the $a_i$'s are $C$-dependent.
2. The elements $a, b$ in the central closure of $R$ are $C$-dependent if
   and only if $ayb = bya$ holds for all $y \in R$.

The explanation of the notions of the extended centroid and the cen-
tral closure of a prime ring, as well as the proof of Remark, can be found
in [5, pp. 20–31].

The following lemma is due to L. O. Chung and J. Luh [3].

**Lemma 2.1.** Let $R$ be a $n!$-torsion free ring. Suppose that $t_1, t_2, \cdots, t_n \in R$ satisfy $kt_1 + k^2t_2 + \cdots + k^nt_n = 0$ for $k = 1, 2, \cdots, n$. Then $t_i = 0$ for all $i$.

**Lemma 2.2 ([8]).** Let $R$ be a noncommutative prime ring of $(n + 1)!$-
torsion free for a positive integer $n$. If a derivation $D$ satisfies either
$[D(x), x]x^n = 0$ or $x^n[D(x), x] = 0$ for all $x \in R$, then in both cases we
have $D = 0$.

Posner [10] proved that the existence of a nonzero centralizing deriva-
tion on a prime ring forces the ring to be commutative. Recently,
Vukman [12] has proved that in case there exists a nonzero derivation
$D : R \to R$, where $R$ is a prime ring of characteristic different from 2 and 3, such that the mapping $x \mapsto [D(x), x]$ is commuting on $R$, $R$ is commutative. We are going to generalize the result of Vukman [12, Theorem 1] as follows.

**Lemma 2.3.** Let $R$ be a noncommutative prime ring of $(n + 3)!$-torsion free for a positive integer $n$. Suppose that there exists a Jordan derivation $D : R \to R$ such that either $[[D(x), x], x]x^n = 0$ or $x^n[[D(x), x], x] = 0$ holds for all $x \in R$. Then we have $D = 0$ on $R$.

**Proof.** We introduce a symmetric biadditive mapping $F : R \times R \to R$ by the relation $F(x, y) = [D(x), y] + [D(y), x]$ for all $x, y \in R$. A routine calculation shows that the relations $F(xy, z) = F(x, z)y + xF(y, z) + D(x)[y, z] + [x, z]D(y)$ is fulfilled for all $x, y, z \in R$. Let us write $f(x)$ for $F(x, x)$. Thus $f(x) = 2[D(x), x]$ for all $x \in R$. The mapping $f$ satisfies the relation $f(x + \lambda y) = f(x) + \lambda^2 f(y) + 2\lambda F(x, y)$ for all $x, y \in R$ and $\lambda \in \mathbb{Z}$. We prove the lemma under the assumption $[[D(x), x], x]x^n = 0$. The other case is similarly proved. Now the assumption of the lemma can be written in the form

$$[f(x), x]x^n = 0, \quad x \in R. \tag{2.1}$$

Replacing $x$ by $x + \lambda y$ in (2.1), we get

$$0 = \lambda a_1(x, y) + \lambda^2 a_2(x, y) + \cdots + \lambda^{n+2} a_{n+2}(x, y)$$

for all $x, y \in R$, $\lambda \in \mathbb{Z}$, where $a_i(x, y)$ denotes the sum of these terms in which $y$ appears as a term in the product $i$ times. Applying Lemma 2.1, we have $a_1(x, y) = 0$ for all $x, y \in R$. That is,

$$0 = [f(x), x](x^{n-1}y + x^{n-2}yx + \cdots + xyx^{n-2} + yx^{n-1}) + ([f(x), y] + 2[F(x, y), x])x^n, \quad x, y \in R. \tag{2.2}$$

Let us replace $y$ by $yx$ in (2.2). Then, by (2.1) and (2.2) we get

$$0 = 3[y, x]f(x)x^n + 2[[y, x], x]D(x)x^n, \quad x, y \in R, \tag{2.3}$$

which reduces to

$$0 = y(3xD(x)x^{n+1} - 2x^2D(x)x^n) + xD(x)x^n - 3D(x)x^{n+1}) + x^2yD(x)x^n$$
for all $x, y \in R$. The substitution $x^ky$ for $y$ in (2.3) leads to

$$
0 = x^k y (3xD(x)x^{n+1} - 2x^2D(x)x^n) + x^{k+1}y(xD(x)x^n
- 3D(x)x^{n+1}) + x^{k-2}yD(x)x^n, \quad x, y \in R, \quad k \in \mathbb{Z}^+.
$$

Substituting $xy$ for $y$ in (2.2) and using the above relation (2.2), we obtain that

$$
0 = \left[ f(x), x \right] (x^n y + x^{n-1} y x + \cdots + xy x^{n-1})
+ \left( \left[ f(x), xy \right] + 2\left[ f(x), xy, x \right] \right) x^n
$$

$$
= \left[ f(x), x \right] (x^n y + x^{n-1} y x + \cdots + xy x^{n-1})
+ 3f(x)[y, x]x^n + 3[f(x), x]yx^n + 2D(x)[y, x]x^n
- x[f(x), x](x^{n-1} y + x^{n-2} y x + \cdots + xy x^{n-1})
$$

for all $x, y \in R$. Putting $y = x^ny$ in (2.5), we obtain by assumption,

$$
0 = 3f(x)x^n[y, x]x^n + 2D(x)x^n[y, x]x^n, \quad x, y \in R,
$$

which leads to

$$
0 = (3xD(x)x^{n+1} - 2D(x)x^{n+2})yx^n + (D(x)x^{n+1} - 3xD(x)x^n)yx^{n+1} + D(x)x^n y x^{n+2}
$$

for all $x, y \in R$.

Suppose $D \neq 0$. Then there exists an $x \in R$ such that $D(x)x^n \neq 0$.
Let $D(x)x^n \neq 0$. We set conveniently

$$
a = 3xD(x)x^{n+1} - 2x^2D(x)x^n, \quad b = xD(x)x^n - 3D(x)x^{n+1},
$$

$$
c = D(x)x^n, \quad a_k = x^k, \quad k \in \mathbb{Z}^+.
$$

First, we claim that if $a_n, a_{n+1}$ are $C$-dependent, then $[D(x), x]x^{n+2} = 0$. Observe that from (2.6) $a_n, a_{n+1}, a_{n+2}$ are $C$-dependent by Remark 1. Thus we have by setting $n - 1$ instead of $k$ in (2.4),

$$
0 = a_{n-1}ya + a_nyb + a_{n+1}yc, \quad y \in R.
$$

Substituting $za_n y$ for $y$ in (2.7), we obtain that $0 = a_{n-1}za_n y a + a_n za_n y b + a_{n+1} za_n y c$ for all $y, z \in R$. But on the other hand we see from (2.7) that $a_n za_n y b = -a_n za_{n-1} y a - a_n za_{n+1} y c$. Comparing the last two relations, we arrive at

$$
0 = (a_{n-1} za_n - a_n za_{n-1}) ya + (a_{n+1} za_n - a_n za_{n+1}) yc
$$
for all \( y, z \in R \), which gives

\[
0 = (a_{n-1}za_n - a_nza_{n-1})ya, \quad y, z \in R, \tag{2.8}
\]

since \( a_n, a_{n+1} \) are \( C \)-dependent (c.f. Remark 2). Now it follows from

(2.8) that we have either that \( a_{n-1}, a_n \) are \( C \)-dependent or that \( a = 0 \) by primeness of \( R \). If \( a_{n-1}, a_n \) are \( C \)-dependent, then from (2.4)

\[
0 = a_{n-2}ya + a_{n-1}yb + a_nyc, \quad y \in R. \tag{2.9}
\]

Replacing \( za_{n-1}y \) for \( y \) in (2.9), we obtain

\[
0 = a_{n-2}za_{n-1}ya + a_{n-1}za_{n-1}yb + a_nza_{n-1}yc, \quad y, z \in R.
\]

On the other hand, we see from (2.9) that \( a_{n-1} za_{n-1} yb = -a_{n-1} za_{n-2} ya - a_{n-1} za_{n} yc \). Comparing the last two relations, we arrive at

\[
0 = (a_{n-2}za_{n-1} - a_{n-1}za_{n-2})ya + (a_nza_{n-1} - a_{n-1}za_{n})yc, \quad y, z \in R.
\]

which gives

\[
0 = (a_{n-2}za_{n-1} - a_{n-1}za_{n-2})ya, \quad y, z \in R, \tag{2.10}
\]

since \( a_{n-1}, a_n \) are \( C \)-dependent. Now it follows from (2.10) that we have either that \( a_{n-2}, a_{n-1} \) are \( C \)-dependent or that \( a = 0 \) by primeness of \( R \). Continuing the process, we have either that \( a_1, a_2 \) are \( C \)-dependent or that \( a = 0 \) by primeness of \( R \). If \( a_1, a_2 \) are \( C \)-dependent, then from (2.3)

\[
0 = ya + a_1yb + a_2yc, \quad y \in R. \tag{2.11}
\]

Replacing \( za_1y \) for \( y \) in (2.9), we obtain that \( 0 = za_1ya + a_1za_1yb + a_2za_1yc \), for all \( y, z \in R \). But on the other hand we see from (2.11) that \( a_1za_1yb = -a_1za_1ya - a_1za_2yc \). Comparing the last two relations, we arrive at

\[
0 = (za_1 - a_1z)ya + (a_2za_1 - a_1za_2)yc, \quad y, z \in R,
\]

which gives

\[
0 = (za_1 - a_1z)ya, \quad y, z \in R, \tag{2.12}
\]

since \( a_1, a_2 \) are \( C \)-dependent. From (2.12), we have either \( a_1 \in Z \) or \( a = 0 \) by primeness of \( R \). If \( a = 0 \), since \( [D(x), x]x^{n+1} = x[D(x), x]x^n \) by assumption, we obtain

\[
0 = [3xD(x)x^{n+1} - 2x^2D(x)x^n, x]
= 3xD(x), x)x^{n+1} - 2x[D(x), x]x^{n+1}
= 3[D(x), x]x^{n+2} - 2[D(x), x]x^{n+1}
= [D(x), x]x^{n+2},
\]
which is also true when \( a_1 \in Z \). We have thus proved that \( [D(x), x]x^{n+2} = 0 \) in case that \( a_n, a_{n+1} \) are \( C \)-dependent.

Now let us consider the case that \( a_n, a_{n+1} \) are \( C \)-independent. Since \( a_n, a_{n+1}, a_{n+2} \) are \( C \)-dependent from (2.6), we have \( a_{n+2} = \lambda a_n + \mu a_{n+1} \) for some \( \lambda, \mu \in C \). From (2.4) we know that

\[
0 = a_n ya + a_{n+1} yb + a_{n+2} yc = a_n ya + a_{n+1} yb + (\lambda a_n + \mu a_{n+1}) yc
\]

\[
= a_n y(a + \lambda c) + a_{n+1} y(b + \mu c) \quad \text{for all } y \in R.
\]

Since \( a_n \) and \( a_{n+1} \) are \( C \)-independent, \( a + \lambda c = 0 = b + \mu c \) by Remark 2. So we have

\[
(2.13) \quad b + \mu c = a_1 c - 3ca_1 + \mu c = 0.
\]

On the other hand, we get from (2.6)

\[
0 = (3a_1ca_1 - 2ca_2)y a_n + (ca_1 - 3a_1c)ya_{n+1} + cy(\lambda a_n + \mu a_{n+1})
\]

\[
= (3a_1ca_1 - 2ca_2 + \lambda c)ya_n + (ca_1 - 3a_1c + \mu c)ya_{n+1}, \quad y \in R.
\]

Since \( a_n \) and \( a_{n+1} \) are \( C \)-independent, we obtain by Remark 2

\[
(2.14) \quad 0 = ca_1 - 3a_1c + \mu c.
\]

Subtracting (2.13) from (2.14), we arrive at \( 0 = [c, a_1] \), which yields \( 0 = [D(x), x]x^n \). We have thus proved that \( [D(x), x]x^n = 0 \) in case that \( a_n, a_{n+1} \) are \( C \)-independent. As a result, if \( D(x)x^n \neq 0 \), one obtains \( 0 = [D(x), x]x^{n+2} \) in any case. Obviously, \( D(x)x^n = 0 \) implies that \( [D(x), x]x^{n+2} = 0 \). Hence \( [D(x), x]x^{n+2} = 0 \) for all \( x \in R \). By Lemma 2.2, \( D = 0 \). The proof of the lemma is complete.

\[ \square \]

3. Main results

Using the previous results for the ring theory, we obtain the main theorems for the Banach algebra theory.

**Theorem 3.1.** Let \( D \) be a continuous linear Jordan derivation on a Banach algebra \( A \) such that either \( [[D(x), x], x]x^n \in \text{rad}(A) \) or \( x^n[[D(x), x], x] \in \text{rad}(A) \) for a positive integer \( n \) and for all \( x \in A \). Then \( D(A) \subseteq \text{rad}(A) \).
Proof. Let $P$ be a primitive ideal of $A$. Since $D$ is continuous, by [11, Lemma 3.2] we have $D(P) \subseteq P$. Thus we can define a Jordan derivation $D_P$ on $A/P$ by $D_P(\hat{x}) = D(x) + P$, $\hat{x} = x + P$ for all $x \in A$. The factor algebra $A/P$ is prime and semisimple since $P$ is a primitive ideal. Thus $D_P$ is a derivation by Brešar’s result [1]. Johnson [6] has proved that every derivation on a semisimple Banach algebra is continuous. Combining this result with Singer-Wermer theorem, we obtain that there are no nonzero derivations on a commutative semisimple Banach algebra. Hence in case $A/P$ is commutative, we have $D_P = 0$. It remains to show that $D_P = 0$ in case $A/P$ is noncommutative. The assumption of the theorem gives either $[[D_P(\hat{x}), \hat{x}], \hat{x}] \hat{x} = 0$ or $\hat{x}^n [[D_P(\hat{x}), \hat{x}], \hat{x}] = 0$, $\hat{x} \in A/P$. Then the assumption of Lemma 2.3 is fulfilled and thus we have $D_P = 0$. In any case $D_P = 0$. Hence we see that $D(A) \subseteq P$. Since $P$ is any primitive ideal, the result follows. This completes the proof.

One can easily show the following relations by induction for a positive integer $n$.

$$D(x^n) = D(x)x^{n-1} + xD(x)x^{n-2} + \cdots + x^{n-1}D(x),$$

(3.1) 

$$[D(x), x^n] = [D(x), x]x^{n-1} + x[D(x), x]x^{n-2} + \cdots + x^{n-1}[D(x), x].$$

Combining the above two relations, we have the following two equations.

$$[D(x^n), x^n] = [D(x), x]x^{2n-2} + 2x[D(x), x]x^{2n-3} + \cdots + nx^{n-1}[D(x), x]x^{n-1} + (n - 1)x^n[D(x), x]x^{n-2} + \cdots + 2x^{2n-3}[D(x), x]x + x^{2n-2}[D(x), x],$$

(3.2)

$$[[D(x), x], x^n] = [[[D(x), x], x]x^{n-1} + x[[D(x), x], x]x^{n-2} + \cdots + x^{n-1}[[D(x), x], x].$$

(3.3)

Using the relations (3.2) and (3.3), we obtain the following equation,
which is needed to prove the next theorem.

\[
\begin{align*}
[[D(x^n), x^n], x^n] &= [[D(x), x], x^n]x^{2n-2} + 2x[[D(x), x], x^n]x^{2n-3} \\
&\quad + \cdots + nx^{n-1}[[D(x), x], x^n]x^{n-1} \\
&\quad + (n-1)x^n[[D(x), x], x^n]x^{n-2} \\
&\quad + \cdots + 2x^{2n-3}[[D(x), x], x^n]x \\
&\quad + x^{2n-2}[[D(x), x], x^n].
\end{align*}
\]

(3.4)

**Theorem 3.2.** Let \( D \) be a continuous linear Jordan derivation on a Banach algebra \( A \) such that \([[[D(x^n), x^n], x^n]] \in \text{rad}(A)\) for a positive integer \( n \) and for all \( x \in A \). Then \( D \) maps \( A \) into its radical.

**Proof.** Let \( P \) be a primitive ideal of \( A \). Since \( D \) is continuous, by [11, Lemma 3.2] we have \( D(P) \subseteq P \). Thus we can define a Jordan derivation \( D_P \) on \( A/P \) by \( D_P(\hat{x}) = D(x) + P \), \( \hat{x} = x + P \) for all \( x \in A \). As in the proof of Theorem 3.1, we can see that in case \( A/P \) is commutative, we have \( D_P = 0 \). We claim that \( D_P = 0 \) in case \( A/P \) is noncommutative. Then we will see that \( D(A) \subseteq P \) in any case and so the result of the theorem follows since \( P \) is any primitive ideal. Now the assumption of the theorem gives \([[[D_P(\hat{x}^n), \hat{x}^n], \hat{x}^n]] = 0 \), \( \hat{x} \in A/P \). Thus without any loss of generality we may assume that \( A \) is noncommutative primitive Banach algebra and the condition \([[[D(x^n), x^n], x^n]] = 0 \) holds for all \( x \in A \). We use the notations \( f, F \) in Lemma 2.3. The assumption of the theorem can now be written in the form by virtue of (3.3) and (3.4)

\[
0 = \sum_{k=1}^{n} \frac{k(k+1)}{2} x^{k-1} [f(x), x] x^{3n-k-2} + \sum_{k=0}^{n-3} \{(k+2) + \cdots \\
+ n + \cdots + (n-k-1)\} x^{n+k} [f(x), x] x^{2n-k-3} \\
+ \sum_{k=n}^{\frac{1}{2}} \frac{k(k+1)}{2} x^{2n-k-2} [f(x), x] x^{k-1}
\]

(3.5)

for all \( x \in A \), where we write \( x^0 = 1 \) conveniently. The above relation contains the sum of these \( 3n-2 \) terms. Replacing \( x \) by \( x + \lambda y \) in (3.5) and expanding the resulting equation, we obtain

\[
0 = \lambda a_1(x, y) + \lambda^2 a_2(x, y) + \cdots + \lambda^{3n-1} a_{3n-1}(x, y),
\]
$\lambda \in \mathbb{Z}$, $x, y \in A$, where $a_i(x, y)$ denotes the sum of these terms in which $y$ appears as a term in the product $i$ times. Applying Lemma 2.1, we have $0 = a_1(x, y)$ for all $x, y \in A$. Arranging the resulting relation, we obtain

$$
0 = [f(x), x](x^{3n-4} + yx^{3n-5} + \cdots + yx^{3n-4}) \\
+ ((f(x), y) + 2[F(x, y), x])x^{3n-3} \\
+ \sum_{k=2}^{n} \frac{k(k+1)}{2} \left\{ x^{k-1}[f(x), x](x^{3n-k-3} + yx^{3n-k-4}) \\
+ \cdots + yx^{3n-k-3} \\
+ x^{k-1}((f(x), y) + 2[F(x, y), x])x^{3n-k-2} \\
+ (x^{n-k-1}y + x^{n-k-2}yx + \cdots + yx^{k-1}) \\
\cdot [f(x), x)x^{3n-k-2} \right\} \\
+ \sum_{k=0}^{n-3} \{(k + 2) + \cdots + n + \cdots + (n - k - 1) \} \\
\cdot \left\{ x^{n+k}[f(x), x](x^{2n-k-4} + yx^{2n-k-5} + \cdots + yx^{2n-k-4}) \\
+ x^{n+k}((f(x), y) + 2[F(x, y), x])x^{2n-k-3} \\
+ (x^{n+k-1}y + x^{n+k-2}yx + \cdots + yx^{n+k-1}) \\
\cdot [f(x), x)x^{2n-k-3} \right\} \\
+ \sum_{k=n}^{2} \frac{k(k+1)}{2} \left\{ x^{3n-k-2}[f(x), x] \\
\cdot (x^{k-2}y + x^{k-3}yx + \cdots + yx^{k-2}) \\
+ x^{3n-k-2}((f(x), y) + 2[F(x, y), x])x^{k-1} \\
+ (x^{3n-k-3} + yx^{3n-k-4}yx + \cdots + yx^{3n-k-4}) \\
\cdot [f(x), x)x^{k-1} \right\} \\
+ x^{3n-3}((f(x), y) + 2[F(x, y), x]) \\
+ (x^{3n-4} + yx^{3n-5} + \cdots + yx^{3n-4})[f(x), x]
$$

for all $x, y \in A$. Let us put $x^2$ instead of $y$ in (3.6). Putting in order the resulting long relation with the use of identity $[f(x), x^2] + 2[F(x, x^2), x] =$
3[f(x), x]x + 3x[f(x), x], we get
\[0 = 3n \sum_{k=1}^{n} \frac{k(k+1)}{2} x^{k-1}[f(x), x]x^{3n-k-1}\]
\[+ 3n \sum_{k=0}^{n-2} ((k+1) + \cdots + n \]
\[+ \cdots + (n-k-1)) x^{n+k}[f(x), x]x^{2n-k-2}\]
\[+ 3n \sum_{k=n}^{1} \frac{k(k+1)}{2} x^{3n-k-1}[f(x), x]x^{k-1}\]
for all \( x \in A \), which can be written in the form
\[0 = 3n \left\{ \sum_{k=1}^{n} \frac{k(k+1)}{2} x^{k-1}[f(x), x]x^{3n-k-2}\right.\]
\[\left. + \sum_{k=0}^{n-3} ((k+2) + \cdots + n \]
\[+ \cdots + (n-k-1)) x^{n+k}[f(x), x]x^{2n-k-3}\right.\]
\[+ \sum_{k=n}^{1} \frac{k(k+1)}{2} x^{3n-k-2}[f(x), x]x^{k-1}\right\} x\]
\[+ 3n \left\{ x^{n}[f(x), x]x^{2n-2} + 2x^{n+1}[f(x), x]x^{2n-3} + \cdots \right.\]
\[+ nx^{2n-1}[f(x), x]x^{n-1} + (n-1)x^{2n}[f(x), x]x^{n-2} + \cdots \right.\]
\[+ 2x^{3n-3}[f(x), x]x + x^{3n-2}[f(x), x]\}.\]

Applying (3.5) to the last relation, we have
\[0 = x^{n}[f(x), x]x^{2n-2} + 2x^{n+1}[f(x), x]x^{2n-3} + \cdots \]
\[+ nx^{2n-1}[f(x), x]x^{n-1} + (n-1)x^{2n}[f(x), x]x^{n-2} \]
\[+ \cdots + 2x^{3n-3}[f(x), x]x + x^{3n-2}[f(x), x]\].\]

Thus the last relation contains the sum of these \(2n-1\) terms, which the number of \(n-1\) terms decreased by virtue of our process starting from (3.5). But the power of \(x\) in (3.7) has increased by 1 in comparison to
(3.5). We repeat similarly the process from the relation (3.5) to (3.7) as follows. Substituting \( x + \lambda y \) for \( x \) in (3.7) and expanding the resulting equation, we obtain

\[
0 = \lambda b_1(x, y) + \lambda^2 b_2(x, y) + \cdots + \lambda^{3n} b_{3n}(x, y),
\]

\( \lambda \in \mathbb{Z}, x, y \in A \), where \( b_i(x, y) \) denotes the sum of these terms in which \( y \) appears as a term in the product \( i \) times. By Lemma 2.1 we have \( 0 = b_1(x, y) \) for all \( x, y \in A \). Thus

\[
(3.8)
\]

\[
0 = \sum_{k=1}^{n} k \left\{ x^{n-k+1} [f(x), x] (x^{2n-k-2} + x^{2n-k-3} yx + \cdots + y x^{2n-k-2}) \\
+ x^{n-k+1} (\{f(x), y\} + 2[F(x, y), x]) x^{2n-k} \\
+ (x^{n-k+2} y + x^{n+k-3} yx + \cdots + y x^{n+k-2}) [f(x), x] x^{2n-k-1} \right\} \\
+ \sum_{k=n-1}^{2} k \left\{ x^{3n-k-1} [f(x), x] (x^{k-2} y + x^{k-3} yx + \cdots + y x^{k-2}) \\
+ x^{3n-k-1} (\{f(x), y\} + 2[F(x, y), x]) x^{k-1} \\
+ (x^{3n-k-2} y + x^{3n-k-3} yx + \cdots + y x^{3n-k-2}) [f(x), x] x^{k-1} \right\} \\
+ x^{3n-2} (\{f(x), y\} + 2[F(x, y), x]) \\
+ (x^{3n-3} y + x^{3n-4} yx + \cdots + y x^{3n-3}) [f(x), x]
\]

for all \( x, y \in A \). Putting \( x^2 \) instead of \( y \) in (3.8) and using the identity \( [f(x), x^2] + 2[F(x, x^2), x] = 3[f(x), x] x + 3x[f(x), x] \), we obtain the resulting long relation

\[
0 = \sum_{k=0}^{n-1} \{(3k+2)n + (2k+1)\} x^{n+k} [f(x), x] x^{2n-k-1} \\
+ \sum_{k=n-1}^{0} \{(k+1)3n + (2k+1)\} x^{3n-k-1} [f(x), x] x^{k} \\
= \sum_{k=0}^{n-1} \{(k+1)(2n+1) + (n+1)k\} x^{n+k} [f(x), x] x^{2n-k-1} \\
+ \sum_{k=n-1}^{0} \{k(2n+1) + (k+1)(n+1) + 2n\} x^{3n-k-1} [f(x), x] x^{k}
\]
for all $x \in A$, which can be rearranged in the form

\[(3.9)\quad 0 = (2n + 1)\left\{ x^n[f(x), x]x^{2n-2} + 2x^{n+1}[f(x), x]\right.\]
\[\quad + \cdots + nx^{2n-1}[f(x), x]x^{n-1}\]
\[\quad + (n - 1)x^{2n}[f(x), x]x^{n-2}\]
\[\quad + \cdots + 2x^{3n-3}[f(x), x]x + x^{3n-2}[f(x), x]\}
\[\quad + (n + 1)x\left\{ x^n[f(x), x]x^{2n-2} + 2x^{n+1}[f(x), x]x^{2n-3}\right.\]
\[\quad + \cdots + nx^{2n-1}[f(x), x]x^{n-1}\]
\[\quad + (n - 1)x^{2n}[f(x), x]x^{n-2}\]
\[\quad + \cdots + 2x^{3n-3}[f(x), x]x + x^{3n-2}[f(x), x]\}
\[\quad + 2n\left\{ x^{2n}[f(x), x]x^{n-1} + nx^{2n+1}[f(x), x]x^{n-2}\right.\]
\[\quad + \cdots + x^{3n-1}[f(x), x]\}\]

for all $x \in A$. Comparing (3.7) with the above relation, we get

\[(3.10)\quad 0 = x^{2n}[f(x), x]x^{n-1} + x^{2n+1}[f(x), x]x^{n-2} + \cdots + x^{3n-1}[f(x), x]\]

for all $x \in A$. The last relation contains the sum of these $n$ terms, which the number of $n - 1$ terms has decreased by virtue of our process starting from (3.7). We repeat once more the process from the relation (3.7) to (3.10) as follows. Taking the place of $x$ by $x + \lambda y$ in (3.10), we get

\[0 = \lambda c_1(x, y) + \lambda^2 c_2(x, y) + \cdots + \lambda^{3n+1} c_{3n+1}(x, y),\]

$\lambda \in \mathbb{Z}, \ x, y \in A$, where $c_i(x, y)$ denotes the sum of these terms in which $y$ appears as a term in the product $i$ times. Applying Lemma 2.1 and expanding the resulting equation, we have $0 = c_1(x, y)$ for all $x, y \in A$. 


Therefore
\[(3.11)\]
\[
0 = \sum_{k=0}^{n-2} \left\{ x^{2n+k}[f(x), x](x^n y + x^{n-k-2}yx + \cdots + yx^{n-k-2}) \\
+ x^{2n+k}(f(x), y + 2F(x, y), x)x^{n-k-1} \\
+ (x^{2n+k-1}y + x^{2n+k-2}yx + \cdots + yx^{2n+k-1}) \\
\cdot [f(x), x]x^{n-k-1} \right\} \\
+ x^{3n-1}((f(x), x) + 2F(x, y), x) \\
+ (x^{3n-2}y + x^{3n-3}yx + \cdots + yx^{3n-1})[f(x), x]
\]
for all $x, y \in A$. Taking $x^2$ instead of $y$ in (3.11), we obtain
\[
0 = (n + 2) \left\{ x^{2n}[f(x), x]x^{n-1} + x^{2n+1}[f(x), x]x^{n-2} \\
+ \cdots + x^{3n-1}[f(x), x] \right\}x \\
+ (2n + 2)x \left\{ x^{2n}[f(x), x]x^{n-1} + x^{2n+1}[f(x), x]x^{n-2} + \cdots \\
+ x^{3n-1}[f(x), x] \right\} + nx^{3n}[f(x), x]
\]
for all $x \in A$. Thus, comparing the last relation with the relation (3.10), we have $x^{3n}[f(x), x] = 0$ for all $x \in A$. By Lemma 2.3, $D = 0$. In other words, $D_P = 0$ on $A/P$ and we see that $D(A) \subseteq P$ and so the result follows since $P$ is any primitive ideal. The proof of the theorem is complete.

The following corollaries are due to Theorem 3.2.

**Corollary 3.3.** Let $D$ be a continuous linear Jordan derivation on a Banach algebra $A$ such that $[[D(x), x], x] \in \text{rad}(A)$ for a positive integer $n$ and for all $x \in A$. Then $D(A) \subseteq \text{rad}(A)$.

**Corollary 3.4.** Let $D$ be a continuous linear Jordan derivation on a Banach algebra $A$ such that $[D(x^n), x^n] \in \text{rad}(A)$ for a positive integer $n$ and for all $x \in A$. Then $D(A) \subseteq \text{rad}(A)$.

**Corollary 3.5.** Let $D$ be a continuous linear Jordan derivation on a Banach algebra $A$ such that $[[D(x), x], x^n] \in \text{rad}(A)$ for a positive integer $n$ and for all $x \in A$. Then $D(A) \subseteq \text{rad}(A)$. 
COROLLARY 3.6. Let $D$ be a continuous linear Jordan derivation on a Banach algebra $A$ such that $[D(x), x^n] \in \text{rad}(A)$ for a positive integer $n$ and for all $x \in A$. Then $D(A) \subseteq \text{rad}(A)$.

References


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