PROJECTIVE LIMIT OF A SEQUENCE
OF BANACH FUNCTION ALGEBRAS
AS A FRÉCHET FUNCTION ALGEBRA

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Abstract. Let $X$ be a hemicompact space with $(K_n)$ as an ad-
missible exhaustion, and for each $n \in \mathbb{N}$, $A_n$ a Banach function
algebra on $K_n$ with respect to $\| \cdot \|_{K_n}$ such that $A_{n+1} | K_n \subseteq A_n$ and
$\| f |_{K_n} \| \leq \| f \|_{n+1}$ for all $f \in A_{n+1}$. We consider the subalgebra
$A = \{ f \in C(X) : f |_{K_n} \in A_n, \forall n \in \mathbb{N} \}$ of $C(X)$ as a Fréchet func-
tion algebra and give a result related to its spectrum when each
$A_n$ is natural. We also show that if $X$ is moreover noncompact,
then any closed subalgebra of $A$ cannot be topologized as a regu-
lar Fréchet $Q$-algebra. As an application, the Lipschitz algebra of
infinitely differentiable functions is considered.

1. Introduction

Let $X$ be a compact Hausdorff space. We denote the algebra of all
continuous functions on $X$ by $C(X)$ and the uniform norm of $f \in C(X)$
by $\| f \|_X$. Under a norm, a Banach subalgebra of $C(X)$, which contains
the constants and separates the points of $X$, is called a Banach function
algebra on $X$. The uniform norm of an element in a Banach function
algebra does not exceed from its norm. A Banach function algebra $B$ on
$X$ is called natural if each complex homomorphism on $B$ is an evaluation
homomorphism at some point of $X$.

By a Fréchet algebra $(A, (p_n))$ we mean a topological algebra $A$ whose
topology can be defined by a sequence $(p_n)$ of separating and submul-
tiplicative seminorms, $p_n(fg) \leq p_n(f)p_n(g), f, g \in A$, and which is
complete with respect to this topology. Without loss of generality we
can assume that $p_n \leq p_{n+1}$ and that $p_n(1) = 1$ if $A$ has unit $1$ (see

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chitz algebra, Fréchet algebra.
A Fréchet algebra $A$ is called a $Q$-algebra if the set of quasi-regular elements of $A$ is open in $A$. This is equivalent to say that the set of quasi-regular elements of $A$ has an interior (see [1]).

In this paper, we assume that all algebras are unital.

The spectrum of a commutative Fréchet algebra $(A, (p_n))$, which is denoted by $M_A$, is the set of all non-zero continuous complex homomorphisms on $A$, and for each $f \in A$, $\hat{f} : M_A \to \mathbb{C}$ is the Gelfand transform of $f$. We always endow $M_A$ with the Gelfand topology. The Fréchet algebra $A$ is called functionally continuous if each complex homomorphism on $A$ is continuous. It is unanswered for about 50 years whether or not each Fréchet algebra is functionally continuous (Michael’s problem).

**Definition 1.1.** A Hausdorff space $X$ is called hemicompact if there exists a sequence $(K_n)$ of increasing compact subsets of $X$ such that each compact subset of $X$ is contained in some $K_n$. The sequence $(K_n)$ with this property is called an admissible exhaustion of $X$.

Let $(A, (p_n))$ be a Fréchet algebra. For each $n$, let $A_n$ be the completion of $A/\ker p_n$ with respect to the norm $p_n'(f + \ker p_n) = p_n(f)$. Then $A_n$ is a Banach algebra, $A = \lim A_n$, projective limit of $(A_n)$, and $M_A = \bigcup M_{A_n}$ as sets. Moreover, $M_A$ is a hemicompact space with $(M_{A_n})$ as an admissible exhaustion and $M_{A_n} = \{ \phi \in M_A : |\phi(f)| \leq p_n(f), \forall f \in A \}, n \in \mathbb{N}$ (see [4]).

**Definition 1.2.** Let $X$ be a hemicompact space and $A$ a subalgebra of $C(X)$ which contains the constants and separates the points of $X$. We call $A$ a Fréchet function algebra or $Ff$-algebra on $X$ if it is a Fréchet algebra with respect to some topology such that the evaluation homomorphism $\varphi_x$ at each $x \in X$ is continuous, that is, $\varphi_x \in M_A$.

We can consider each commutative unital semisimple Fréchet algebra as an $Ff$-algebra on its spectrum. So indeed the class of $Ff$-algebras and the class of commutative unital semisimple Fréchet algebras are the same.

Now let $(A, (p_n))$ be an $Ff$-algebra on $X$. Since $J : X \to M_A, x \mapsto \varphi_x$, is a continuous injective map, $\{ \varphi_x : x \in K_n \}$ is a compact subset of $M_A$ for each $n \in \mathbb{N}$. So for each $n$ there exists an integer $m$ such that $\{ \varphi_x : x \in K_n \} \subset M_{A_m}$. Therefore,

\[
(1) \quad \|f\|_{K_n} = \sup_{x \in K_n} |\varphi_x(f)| \leq \sup_{\varphi \in M_{A_m}} |\varphi(f)| = \|\hat{f}\|_{M_{A_m}} \leq p_m(f)
\]

for all $f \in A$. 
For each \( n \in \mathbb{N} \), let \( i(n) \geq n \) be the smallest integer that \( \|f\|_{K_n} \leq p_{i(n)}(f) \) holds for all \( f \in A \) and define \( p_n' \) on \( A|_{K_n} \) by
\[
p_n'(f|_{K_n}) = \inf \{ p_{i(n)}(g) : g|_{K_n} = f|_{K_n}, \ g \in A \}
\]
for each \( f \in A \). Then \( p_n' \) is an algebra norm on \( A|_{K_n} \). Let \( A_{K_n} \) be the completion of \( A|_{K_n} \) with respect to the norm \( p_n' \). Then we have the following result:

**Theorem 1.3** ([6]). Let \( (A,(p_n)) \) be an \( Ff \)-algebra on \( X, (K_n) \) an admissible exhaustion of \( X \) and \( (A_{K_n}) \) as defined above. Then \( (A_{K_n}) \) is a sequence of Banach algebras and \( A \) is dense in \( \varprojlim A_{K_n} \). Moreover, if \( \ker q_n \subset \ker p_{i(n)} \) for each positive integer \( n \), then \( A \) is algebraically and topologically a projective limit \( \varprojlim A_{K_n} \), where \( q_n \) is defined by \( q_n(f) = \|f\|_{K_n} \).

**Theorem 1.4** ([6]). Let \( (A,(p_n)) \) and \( (B,(q_n)) \) be \( Ff \)-algebras on hemicompact spaces \( X \) and \( Y \), respectively, and let \( T : (A,p_n) \rightarrow (B,q_n) \) be a continuous monomorphism with a dense range. Then the injective adjoint spectral map \( T^* : M_B \rightarrow M_A, \psi \mapsto \psi \circ T \), is surjective and proper, that is, the inverse image of each compact set is compact, if and only if for each \( m \in \mathbb{N} \), there exists an integer \( n \) such that
\[
\|\tilde{f}\|_{M_{A_m}} \leq q_n(T(f))
\]
for all \( f \in A \).

**2. Main results**

Let \( X \) be a hemicompact space and \( (K_n) \) an admissible exhaustion of \( X \). In this section, we assume that \( (A_n) \) is a sequence of Banach function algebras such that for each \( n \in \mathbb{N} \), \( A_n \) is a Banach function algebra on \( K_n \) with respect to \( \| \cdot \|_n \), \( A_{n+1} |_{K_n} \subseteq A_n \) and \( \|f|_{K_n}\|_n \leq \|f\|_{n+1} \) for all \( f \in A_{n+1} \). Consider
\[
A = \{ f \in C(X) : f|_{K_n} \in A_n, \ n \in \mathbb{N} \}.
\]
Clearly, \( A \) contains the constants and for each \( n \in \mathbb{N} \), \( p_n(f) = \|f|_{K_n}\|_n \), \( f \in A \), defines a submultiplicative seminorm on \( A \). It is easy to check that \( A \) is a Fréchet algebra with respect to the topology defined by the sequence \( (p_n) \) of seminorms. Moreover, the evaluation map \( \varphi_x \) at each
$x \in X$ is continuous. So if $A$ separates the points of $X$, then $A$ is an $Ff$-algebra on $X$.

Note that if $X$ is compact and if each $A_n$ is inverse closed, that is, $\frac{1}{n} \in A_n$ if $f \in A_n$ and $f(x) \neq 0$ for all $x \in K_n$, then $A$ is a $Q$-algebra. This is because $A$ is also inverse closed and there is an integer $N$ such that $K_n = X$ for all $n \geq N$. Let $G = \{f \in A : 1 + f \in A^{-1}\}$, where $A^{-1}$ is the set of all invertible elements of $A$. If $f \in A$ and $p_N(f) < \frac{1}{2}$, then $\|f\|_X \leq \|f|_{K_n}\|_N = p_N(f) < \frac{1}{2}$, since the norm of a Banach function algebra is greater than the uniform norm. Thus $(1 + f)(x) \neq 0$ for all $x \in X$. Since $A$ is inverse closed, $1 + f \in A$, that is, $f \in G$. Hence the open neighborhood $V = \{f \in A : p_N(f) < \frac{1}{2}\}$ of the origin is contained in $G$. So $G$ has an interior point.

**Theorem 2.1.** Let $X$ be a hemicompact space and let $(A_n, \| \cdot \|_n)$ and $(A, (p_n))$ be as defined above. Suppose that $A$ separates the points of $X$ and that for each $n$, $A_n$ is natural. If $(B, (q_n))$ is an $Ff$-algebra on $X$ which contains $A$ as a dense subalgebra and the identity map $I : (A, (p_n)) \rightarrow (B, (q_n))$ is continuous, then $M_A = M_B$ as sets.

**Proof.** Let $i(n)$, $p'_n$ and $A_{K_n}$ be as defined in Theorem 1.3. Here we notice that $i(n) = n$ and if $f, g \in A$ and $f|_{K_n} = g|_{K_n}$, then $\|f - g\|_{K_n} = p_n(f - g) = 0$ so that $p_n(f) = p_n(g)$. This shows that for each $f \in A$, $p'_n(f|_{K_n}) = p_n(f) = \|f|_{K_n}\|_n$, and so $A_{K_n}$ is indeed the closure of $A|_{K_n}$ in the Banach function algebra $(A_n, \| \cdot \|_n)$. Therefore, in this case, each $A_{K_n}$ is a Banach function algebra on $K_n$ and $A = \lim A_{K_n}$ by Theorem 1.3.

Since $I$ is a continuous monomorphism with a dense range, $I^* : M_B \rightarrow M_A$, defined by $I^*(\varphi) = \varphi|_{A}$, is an injective continuous map. For each $m \in N$ and each $f \in A$,

$$\|\hat{f}\|_{M_{A_{K_m}}} = r_{A_{K_m}}(f|_{K_m}) = r_{A_{m}}(f|_{K_m}) = \|f\|_{K_m},$$

where $r_{A_{m}}(f|_{K_m})$ is the spectral radius of $f|_{K_m}$ in $A_m$ and the last equality is a consequence of the naturality of $A_m$. On the other hand, since $(B, (q_n))$ is an $Ff$-algebra on $X$, for each $m \in N$, there exists an integer $n \in N$ such that

$$\|f\|_{K_m} \leq \|\hat{f}\|_{M_{B_n}} \leq q_n(f), \quad f \in B,$$

where $B_n$ is the completion of $B/\ker q_n$ with respect to the norm $q'_n(f + \ker q_n) = q_n(f), \quad f \in B$ (see the inequality (1)). So by Theorem 1.4, $I^*$ is surjective and proper. Thus $M_A = M_B$ as sets. $\square$
REMARK 1.

(a) In Theorem 2.1, if \( M_A \) is a \( k \)-space, then the restriction of \( I^{*-1} \) to each compact subset of \( M_A \) is continuous, since \( I^* \) is a proper map. So \( I^{*-1} \) is continuous on \( M_A \). Hence \( M_A \) is homeomorphic to \( M_B \).

(b) The naturality of each \( A_n \) cannot be omitted in Theorem 2.1. For example, let \( X = [0,1], \ K_n = X, \ A_n = A(D)\{[-1,1]\}, \) where \( D \) is the closed unit disk in \( C \) and \( A(D) \) is the uniform Banach algebra of continuous functions on \( D \) which are analytic on \( D \). For each \( f \in A_n \), there is a unique \( g \in A(D) \) such that \( g|_{[-1,1]} = f \). Define \( ||f||_n = ||g||_{17} \). Then \( A = \{ f \in C(X) : f|_{K_n} \in A_n \} = A(D)\{[-1,1]\}, \ M_A = D, \) and \( A \) is dense in \( C([-1,1]) \). But \( M_{C([-1,1])} = [-1,1] \).

THEOREM 2.2. Let \( X \) be a hemicompact noncompact space with \( (K_n) \) as an admissible exhaustion. Let \( (A_n, || \cdot ||_n) \) and \( (A, (p_n)) \) be as defined in the beginning of this section such that \( A \) separates the points of \( X \). Then any closed subalgebra \( B \) of the \( Ff \)-algebra \( (A, (p_n)) \) cannot be normable as a regular Banach algebra.

Proof. Let \( || \cdot || \) be a norm on \( B \) such that \( (B, || \cdot ||) \) is a regular Banach algebra on \( M_B \). Since \( B \) is closed in \( A \), \( (B, (p_n)) \) is a commutative semisimple Fréchet algebra. By the Carpenter's theorem, i.e., each commutative semisimple Fréchet algebra has a unique topology as a Fréchet algebra, the identity map \( I : (B, || \cdot ||) \to (B, (p_n)) \) is a homeomorphism. So there exist an \( n_0 \in \mathbb{N} \) and an \( M > 0 \) such that

\[
(2) \quad ||f|| \leq M \cdot p_{n_0}(f)
\]

holds for all \( f \in B \).

Since \( X \) is noncompact, one can choose an \( x \in X \setminus K_{n_0} \). By the compactness of \( K_{n_0} \) in \( X \) and hence in \( M_B \) and by the regularity of \( B \) on \( M_B \), there exists an \( f \in B \) with \( \hat{f}(\varphi_x) = 1 \) and \( \hat{f}(\varphi_y) = 0 \) for all \( y \in K_{n_0} \). That is, \( f(x) = 1 \) and \( f|_{K_{n_0}} = 0 \). Thus \( p_{n_0}(f) = 0 \). Now the inequality (2) implies that \( ||f|| = 0 \) and hence \( f = 0 \) as an element of \( B \), which is a contradiction. \( \square \)

REMARK 2. By the same method as the proof of Theorem 2.2, one can show that the closed subalgebra \( B \) of \( A \) cannot be topologized as a regular Fréchet \( Q \)-algebra.
Example 2.3. Let \((X, d)\) be a metric space and \(0 < \alpha \leq 1\). The collection of all complex bounded Lipschitz functions of order \(\alpha\) on \(X\) is denoted by \(\text{Lip}(X, \alpha)\). It is well-known (see \([7]\)) that \(\text{Lip}(X, \alpha)\) with respect to pointwise multiplication is a Banach algebra under the norm \(\| \cdot \|_\alpha\), defined by

\[
\| f \|_\alpha = \| f \|_X + p_\alpha(f), \quad f \in \text{Lip}(X, \alpha),
\]

where \(p_\alpha(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d^\alpha(x, y)}\) and \(\| f \|_X = \sup_{x \in X} |f(x)|\).

Now let \(X\) be a hemi-compact metric space, \((K_n)\) an admissible exhaustion of \(X\), and \(0 < \alpha \leq 1\). Let \(A_n = \text{Lip}(K_n, \alpha)\) and

\[
\| f \|_n = \| f \|_{K_n} + \sup_{x, y \in K_n \atop x \neq y} \frac{|f(x) - f(y)|}{d^\alpha(x, y)}, \quad f \in A_n.
\]

Clearly, \(A_{n+1} |_{K_n} \subset A_n\) and \(\| f \|_{K_n} \leq \| f \|_{n+1}, \; f \in A_{n+1}\). So by the above argument, \(\text{FLip}(X, \alpha) = \{ f \in C(X) : f|_{K_n} \in \text{Lip}(K_n, \alpha), \; n \in \mathbb{N} \}\) is an \(F\)-algebra on \(X\) with respect to the topology defined by the sequence \((p_n)\) of seminorms, where \(p_n(f) = \| f \|_{K_n} \| n \) for all \(f \in \text{FLip}(X, \alpha)\) and all \(n \in \mathbb{N}\). Using \([7, \text{Proposition 1.4}]\), one can show that \(\text{FLip}(X, \alpha)\) is dense in \(C(X)\) in the compact-open topology. So by Theorem 2.1, \(M_{\text{FLip}(X, \alpha)} = M_{C(X)} = X\). Indeed, one can show that the Gelfand topology on \(X\) inherited from \(M_{\text{FLip}(X, \alpha)}\) coincides on the metric topology and so \(M_{\text{FLip}(X, \alpha)} \cong X\).

Example 2.4. Let \(0 < \alpha \leq 1\) and \(X\) a perfect compact plane set which is a finite union of regular sets. The algebra of all functions \(f\) on \(X\) which are \(n\) times differentiable and for each \(k\), \(0 \leq k \leq n\), \(f^{(k)} \in C(X)\) (resp. \(f^{(k)} \in \text{Lip}(X, \alpha)\)) is denoted by \(D^n(X)\) (resp. \(\text{Lip}^n(X, \alpha)\)) and the algebra of all functions \(f\) with derivatives of all orders (resp. \(f^{(k)} \in \text{Lip}(X, \alpha) \; \forall k \in \mathbb{N}\)) is denoted by \(D^\infty(X)\) (resp. \(\text{Lip}^\infty(X, \alpha)\)).

It is well-known (see \([3, 5]\)) that for each \(n \in \mathbb{N}\), \(D^n(X)\) and \(\text{Lip}^n(X, \alpha)\) are natural Banach function algebras on \(X\) under the norms, defined by

\[
\| f \|_n = \sum_{k=0}^{n} \frac{\| f^{(k)} \|_X}{k!}
\]

and

\[
\| f \|_n = \sum_{k=0}^{n} \frac{\| f^{(k)} \|_X + p_\alpha(f^{(k)})}{k!},
\]
respectively.

Now for each \( n \in \mathbb{N} \), set \( K_n = X \) and \( A_n = D^n(X) \) (resp. \( \text{Lip}^n(X, \alpha) \)). Then \( A = \{ f \in C(X) : f|_{K_n} \in A_n, \ n \in \mathbb{N} \} = D^\infty(X) \) (resp. \( A = \text{Lip}^\infty(X, \alpha) = \cap A_n \) and \( (A, (\| \cdot \|_n)) \) is an \( Ff \)-algebra on \( X \). Moreover, we have the following inclusions:

\[
R_0(X) \subseteq \text{Lip}^\infty(X, \alpha) \subseteq \text{Lip}^n(X, \alpha) \subseteq D^n(X) \subseteq D^1(X)
\]

and \( D^1(X) \subseteq R(X) \), where \( R_0(X) \) is the algebra of all rational functions with poles off \( X \) and \( R(X) \) is the uniform closure of \( R_0(X) \) (see [3]). Thus \( A \) is dense in \( R(X) \), and since each \( A_n \) is natural, we have \( M_A = M_{R(X)} = X \) by Theorem 2.1. Indeed, by the compactness of \( X \), \( M_A \) is homeomorphic to \( X \).

**Remark 3.**

(a) Notice that the algebra \( \text{FLip}(X, \alpha) \), defined in Example 2.3, is not in general a Banach algebra. Indeed, it is a Banach algebra if and only if \( X \) is compact.

(b) In Example 2.4, the algebras \( \text{Lip}^\infty(X, \alpha) \) and \( D^\infty(X) \) are \( Q \)-algebras, since each \( A_n \) is inverse closed. Moreover, there is no topology which makes these algebras Banach algebras, since \( f \mapsto f' \) defines a nontrivial derivation.

Now let \( (A_n) \) and \( (A, (p_n)) \) be as defined before such that \( A \) is an \( Ff \)-algebra on \( X \). Set \( b(A) = \{ f \in A : \sup p_n(f) < \infty \} \) and \( \| f \|_\infty = \sup p_n(f) \) for each \( f \in b(A) \). Then it is not difficult to check that \( (b(A), \| \cdot \|_\infty) \) is a Banach algebra. For instance, if \( A = \text{FLip}(X, \alpha) \), then \( b(A) = \text{Lip}(X, \alpha) \) and \( \| f \|_\infty \) is the Banach algebra norm on \( \text{Lip}(X, \alpha) \), which was defined earlier, and if \( A = \text{Lip}^\infty(X, \alpha) \) then \( b(A) = \text{Lip}(X, M, \alpha) = \{ f \in \text{Lip}^\infty(X, \alpha) : \sum_{k=0}^\infty \| f^{(k)} \|_X + p_n(f^{(k)}) < \infty \} \), and \( \| \cdot \|_\infty \) is the summation applied in the definition of \( \text{Lip}(X, M, \alpha) \) which makes \( \text{Lip}(X, M, \alpha) \) a Banach algebra (see [5]), where \( M = (k!) \).

Assume that \( b(A) = A \). Since \( A \) is semisimple and the identity map \( I : (b(A), \| \cdot \|_\infty) \mapsto (A, (p_n)) \) is continuous, the identity map \( I \) is a homeomorphism. So \( (A, (p_n)) \) is a Banach algebra.

**Proposition 2.5.** Let \( (A, (p_n)) \) be as in Theorem 2.2. If \( A \) is regular then it is a \( Q \)-algebra if and only if \( X \) is compact.

**Proof.** Assume that \( X \) is compact. Then it is also a compact subset of \( M_A \). If \( U \) is an open subset of \( M_A \) containing \( X \), then by the regularity of \( A \), there exists an \( f \in A \) with \( \widehat{f}(\varphi_x) = 0, \ x \in X \), and \( \widehat{f}|_{M_A \setminus U} = 1 \),
which is impossible if $U \neq M_A$. So $M_A$ is the only open subset which contains $X$. This shows that $A$ is dense in $M_A$, and so $M_A = X$. In particular, $M_A$ is compact and so $A$ is a $Q$-algebra (see [1, 6.3-2]).

The converse is a consequence of Remark 2. □

The following theorem is known for a regular Banach function algebra $A$ and a Banach algebra $B$ (see [2]). Applying [1, Proposition 5.6-1], we can obtain the same result when $A$ is a regular Fréchet function algebra and $B$ is a Fréchet algebra.

**Theorem 2.6.** Let $(A, (p_n))$ be a regular $Fr$-algebra on its spectrum $M_A$ which is locally compact. Let $(B, (q_n))$ be a commutative Fréchet algebra and $\theta : A \to B$ a continuous monomorphism with a dense range. Then $\theta^*(M_B) = M_A$.

**Proof.** Since $\theta(A)$ is dense in $B$, $\theta^*(\psi) = \psi \circ \theta \neq 0$ for each $\psi \in M_B$.

The continuity of $\theta$ shows that $\theta^*(M_B) \subseteq M_A$. Let $S = \overline{\theta^*(M_B)}$ and $\varphi \in M_A \setminus S$. Since $A$ is Gelfand normal (see [1]) and $M_A$ is locally compact, there exists an $f \in A$ with compact support such that $\widehat{f}(\varphi) = 1$ and $\text{supp} \widehat{f} \subset M_A \setminus S$. Let $I = \{f \in A : \widehat{f}|_S = 0\}$, $K = \text{supp} \widehat{f}$, and $J = k(K) = \{f \in A : \varphi(f) = 0, \varphi \in K\}$. Then $I$ and $J$ are closed ideals in $A$ with $h(I) \cap h(J) = \phi$, where $h(I)$ is the set of all closed maximal ideals containing $I$. Hence $I + J = A$ by [1, Proposition 5.6-1], and so there are $h \in J$ and $g \in I$ with $h + g = 1$. Since $h \in J$, $g|_K = 1$.

Consequently, $f = fg$. Since $g = 0$ on $S$, $\theta^*(\psi)(g) = 0$ for each $\psi \in M_B$ so that $\theta(g) \in \text{rad}(B)$. So we show that $\theta(f) = 0$ and hence $f = 0$, which is a contradiction. Suppose that $q_{n_0}(\theta(f)) \neq 0$ for some $n_0 \in \mathbb{N}$. The equality $\theta(f) = \theta(f)\theta(g)$ implies that $\theta(f) = \theta(f)\theta(g^n)$ for each $n \in \mathbb{N}$. So $q_{n_0}(\theta(f)) \leq q_{n_0}(\theta(f))q_{n_0}(\theta(g^n))$ and hence $q_{n_0}(\theta(g^n)) \geq 1$ for each $n \in \mathbb{N}$. But $\theta(g) \in \text{rad}(B)$ and so $\lim_{n \to \infty} \sqrt[n]{q_{n_0}(\theta(g^n))} = 0$. □

**Remark 4.**

(a) In Theorem 2.6, if $B$ is a $Q$-algebra, then $M_B$ is compact and so $\theta^*(M_B) = M_A$, that is, $M_B = M_A$ as sets.

(b) With the hypotheses of Theorem 2.6, $\theta^*$ is not surjective even if $A$ is a regular Banach function algebra. For example, let $X$ be a hemicompact noncompact metric space, $A = \text{Lip}(X, \alpha)$, $B = F\text{Lip}(X, \alpha)$ and $\theta$ the canonical inclusion map. Then $\theta^*(M_B) = \overline{X} = M_A$. 

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(c) The regularity of $A$ cannot be omitted in Theorem 2.6. For example, let $A$ be as given in Remark 1, $B = C([-1, 1])$, and $\theta$ the canonical inclusion map.

References


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