A REMARK OF EISENSTEIN SERIES AND THETA SERIES

Daeeyeoul Kim and Ja Kyung Koo

ABSTRACT. As a by-product of [5], we produce algebraic integers of certain values of quotients of Eisenstein series. And we consider the relation of $\theta_3(0, \tau)$ and $\theta_3(0, \tau^n)$. That is, we show that $|\theta_3(0, \tau^n)| = |\theta_3(0, \tau)|$, $\Delta(0, \tau) = \Delta(0, \tau^n)$ and $J(\tau) = J(\tau^n)$ for some $\tau \in \mathbb{H}$.

1. Introduction

Let $k$ be an imaginary quadratic field, $\mathbb{H}$ the complex upper half plane, $\tau \in \mathbb{H} \cap k$, $p = e^{\pi i \tau}$ and $\Delta(\tau)$ the modular discriminant. The Weierstrass $\wp$-function (relative to a lattice $\Lambda_\tau$) is defined by the series

$$\wp(z) = \wp(z; \Lambda_\tau) = \frac{1}{z^2} + \sum_{\substack{\omega \in \Lambda_\tau \setminus \mathbb{Z} \setminus \{0\}}} \left\{ \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right\} \quad ([10]).$$

In [5], [6], [7] and [8], we dealt with certain algebraic integers as values of elliptic functions constructed from Weierstrass $\wp$-function by using infinite products.

Kim [9] has obtained the leading finite-size corrections to the spectra of the asymmetric XXZ chain and the related six-vertex model in statistical physics. He established this at zero vertical field, and at zero horizontal field. The results are related by a $90^\circ$ rotation. Kim’s results are related to theta series. Baxter [1] had a direct proof of Kim’s identities.

Received September 14, 2001.
2000 Mathematics Subject Classification: 11R04, 11F11.
Key words and phrases: infinite product, Eisenstein series, theta series.
The first author was supported by the Korea Research Foundation Grant (KRF-2000-D00002), and the second named author by KOSEF Research Grant 98-0701-01-01-3.
In [4], we dealt with certain algebraic integers as values of theta functions and derived analogues of Burndt-Chan-Zhang’s results ([2]), which would be a generalization for the case of $m$ even.

In Section 2, we shall consider the following problem:

**Problem.** Find $\tau \in \mathfrak{h}$ which satisfies $|\theta_3(0, z^n)| = |\theta_3(0, x)|$.

We also consider the same problem for the modular discriminant $\Delta(z)$ and $J(z)$-invariant. Using these relations, we find the algebraic integers derived from infinite products.

Throughout the article we adopt the following notations:

- $\Lambda_r = \mathbb{Z} + r\mathbb{Z}$
- $\theta_3(0, \tau) := \prod_{n=1}^{\infty} (1 - p^{2n})(1 + p^{2n-1})^2$
- $G_k(\Lambda_r) := G_k(\tau) = \sum_{\omega \in \Lambda_r \cdot \{0\}} \frac{1}{\omega^k}$ Eisenstein series with weight $k$
- $g_2(\tau) = 60G_4(\tau)$
- $g_3(\tau) = 140G_6(\tau)$
- $\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2 = (2\pi)^{12}\eta(\tau)^{24}$
- $j(\tau) = 1728 \frac{g_2(\tau)^3}{\Delta(\tau)}$

2. Some relations of modular functions

We know from [3] and [4] that

$$\varphi \left( \frac{\tau}{2} \right) - \varphi \left( \frac{1}{2} \right) = -\pi^2 \prod_{n=1}^{\infty} (1 - p^{2n})(1 + p^{2n-1})^6,$$

$$\varphi \left( \frac{\tau + 1}{2} \right) - \varphi \left( \frac{1}{2} \right) = -\pi^2 \prod_{n=1}^{\infty} (1 - p^{2n})(1 - p^{2n-1})^6,$$

$$\varphi \left( \frac{\tau + 1}{2} \right) - \varphi \left( \frac{\tau}{2} \right) = 16\pi^2 p \prod_{n=1}^{\infty} (1 - p^{2n})(1 + p^{2n})^8.$$

Thus we get

$$(K - 1) \varphi \left( \frac{\tau}{2} \right) - \varphi \left( \frac{1}{2} \right) = -\pi^2 \theta_3(0, \tau)^4.$$

Let $f(X) = X^2 + UX + V \in \mathbb{Z}[X]$. If $\tau \in \mathfrak{h}$ and $\tau^2 + \tau U + V = 0$, then $
\tau = \frac{-u + \sqrt{u^2 - 4v}}{2} = r \{ \cos \alpha + i \sin \alpha \}$ where $r = \sqrt{v}$, $\cos \alpha = \frac{-u}{2\sqrt{v}}$, and $\sin \alpha = \frac{\sqrt{4v-u^2}}{2\sqrt{v}}$. 

Theorem 2.1. If $0 < \alpha < \frac{\pi}{n}$ and $u$ is odd, then the followings are satisfied:

1. If $v$ is even then

$$\theta_3(0, \tau)^4 = \theta_3(0, \tau^k)^4,$$

where $k = 1, 2, \ldots, n$.

In particular, if $|\tau| = 1$ then there does not exist $u, v \in \mathbb{Z}$ satisfying $\theta_3(0, \tau)^4 = \theta_3(0, \tau^2)^4$.

2. If $v$ is odd then

$$\begin{cases} 
\theta_3(0, \tau^2)^4 = \theta_3(0, \tau + 1)^4 & \text{if } n \geq 2, \\
\theta_3(0, \tau^4)^4 = \theta_3(0, \tau)^4 & \text{if } n \geq 3.
\end{cases}$$

In particular, if $|\tau| = 1$ then $\tau = \frac{1 + \sqrt{-3}}{2}$, $u = -1$, and $v = 1$.

Proof. Since the Weierstrass $\wp$-function is an even elliptic function, $\wp(\tau) = \wp(-\tau)$ and $\wp(\tau + 1) = \wp(\tau)$. By $(K - 1)$,

$$-\pi^2 \theta_3(0, \tau^2)^4 = \wp\left(\frac{\tau^2}{2}\right) - \wp\left(\frac{1}{2}\right)$$

$$= \wp\left(\frac{-u\tau - v}{2}\right) - \wp\left(\frac{1}{2}\right)$$

$$= \wp\left(\frac{\tau + \epsilon(v)}{2}\right) - \wp\left(\frac{1}{2}\right)$$

$$= -\pi^2 \theta_3(0, \tau + \epsilon(v))^4,$$

where

$$\epsilon(v) = \begin{cases} 
0 & \text{if } v \text{ is even} \\
1 & \text{if } v \text{ is odd}
\end{cases}.$$

Thus we consider the following two cases.

1. Since $\tau^2 = -u\tau - v = u_2\tau + v_2$, $\tau^3 = \tau \tau^2 = \tau(u_2\tau + v_2) = u_2\tau^2 + v_2\tau = u_2(u_2\tau + v_2) + v_2\tau = (u_2^2 + v_2)\tau + u_2v_2 = u_3\tau + v_3$, $\tau^4 = \tau \tau^3 = \tau(-u_3\tau - v_3) = u_4\tau + v_4$, $\cdots$, and $\tau^n = \tau \tau^{n-1} = \tau(-u_{n-1}\tau - v_{n-1}) = u_n\tau + v_n$ where $u_i$ is odd and $v_i$ even with $i = 2, \cdots, n$, we obtain $\theta_3(0, \tau)^4 = \theta_3(0, \tau^2)^4 = \theta_3(0, \tau^3)^4 = \cdots = \theta_3(0, \tau^n)^4$.

2. Since $\tau^2 = -u\tau - v$ and $\tau^4 = (-u\tau - v)^2 = u'\tau + v'$ where $u, v, u'$ are odd and $v'$ even, we derive $\theta_3(0, \tau + 1)^4 = \theta_3(0, \tau^2)^4$ and $\theta_3(0, \tau^4)^4 = \theta_3(0, \tau)^4$. 
Furthermore, if $\tau^2 + u\tau + v = 0$ and $\tau = \frac{-u \pm \sqrt{u^2 - 4v}}{2}$ then $1 = |\tau| = \frac{u^2 + 4v - u^2}{4} = v$. Thus case (2) is possible, while case (1) is impossible. Since $u^2 - 4v < 0$ and $\tau^2 \in \mathfrak{h}$, we get $u = -1$. Consequently, $\tau = \frac{1 + \sqrt{-3}}{2}$. We have the theorem. 

For an example of Theorem 2.1, we let

$$S_v = \{ \tau \in \mathfrak{h} : \theta_2(0, \tau^2)^4 = \theta_2(0, \tau)^4, \ |\tau|^2 = v, \tau^2 + u\tau + v = 0 \}$$

with $u$ odd, $v$ even,

$$S'_v = \{ \tau \in \mathfrak{h} : \theta_3(0, \tau^2)^4 = \theta_3(0, \tau + 1)^4, \ |\tau|^2 = v, \tau^2 + u\tau + v = 0 \}$$

with $u, v$ odd.

By Theorem 2.1, we have:

$$S_1 = S_{2v+1} = \emptyset \text{ for all } v \in \mathbb{Z}^+,$$

$$S_2 = \left\{ \frac{1 + \sqrt{-7}}{2} \right\},$$

$$S_4 = \left\{ \frac{1 + \sqrt{-15} + 3 + \sqrt{-7}}{2}, \frac{3 + \sqrt{-7}}{2} \right\},$$

$$S_6 = \left\{ \frac{1 + \sqrt{-23} + 3 + \sqrt{-19}}{2}, \frac{3 + \sqrt{-19}}{2} \right\},$$

$$S_8 = \left\{ \frac{1 + \sqrt{-31} + 3 + \sqrt{-23} + 5 + \sqrt{-7}}{2}, \frac{3 + \sqrt{-23} + 5 + \sqrt{-7}}{2} \right\},$$

$$S_{10} = \left\{ \frac{1 + \sqrt{-39} + 3 + \sqrt{-31} + 5 + \sqrt{-15}}{2}, \frac{3 + \sqrt{-31} + 5 + \sqrt{-15}}{2} \right\},$$

$$S_{12} = \left\{ \frac{1 + \sqrt{-47} + 3 + \sqrt{-39} + 5 + \sqrt{-23}}{2}, \frac{3 + \sqrt{-39} + 5 + \sqrt{-23}}{2} \right\}, \cdots,$$

$$S_{2v} = \left\{ \frac{1 + \sqrt{-215} + 3 + \sqrt{-207} + 5 + \sqrt{-191} + 7 + \sqrt{-167}}{2}, \frac{3 + \sqrt{-207} + 5 + \sqrt{-191} + 7 + \sqrt{-167}}{2}, \frac{9 + \sqrt{-135} + 11 + \sqrt{-95} + 13 + \sqrt{-47}}{2}, \frac{11 + \sqrt{-95} + 13 + \sqrt{-47}}{2} \right\}, \cdots,$$
\[ S'_1 = \left\{ \frac{1 + \sqrt{-3}}{2} \right\}, \]
\[ S'_2 = \emptyset \text{ for all } v \in \mathbb{Z}^+, \]
\[ S'_3 = \left\{ \frac{1 + \sqrt{-11}}{2}, \frac{3 + \sqrt{-3}}{2} \right\}, \]
\[ S'_5 = \left\{ \frac{1 + \sqrt{-19}}{2}, \frac{3 + \sqrt{-11}}{2} \right\}, \]
\[ S'_7 = \left\{ \frac{1 + 3\sqrt{-3}}{2}, \frac{3 + \sqrt{-19}}{2}, \frac{5 + \sqrt{-3}}{2} \right\}, \]
\[ S'_9 = \left\{ \frac{1 + \sqrt{-35}}{2}, \frac{3 + 5i}{2}, \frac{5 + \sqrt{-11}}{2} \right\}, \]
\[ S'_{11} = \left\{ \frac{1 + \sqrt{-43}}{2}, \frac{3 + \sqrt{-35}}{2}, \frac{5 + \sqrt{-19}}{2} \right\}, \ldots. \]

By Theorem 2.1.(1), we get the pair \((u, v)\) of the coefficients of the polynomial \(f_1(X) = X^2 + uX + v\) with \(f_1(\tau) = 0\) and \(\tau, u, v\) as in Theorem 2.1.(1). If \(k\) is a positive integer less than or equal to \(n\) then \(\theta_3(0, \tau^k)^4 = \theta_3(0, \tau)^4\).

In this way, we put \(\tau^{2k} = u_k\tau + v_k\). Then, \((u_k, v_k)\) is the pair of the coefficients of the polynomial \(f_k(X) = X^k - u_kX - v_k\) with \(f_k(\tau) = 0\) and \(\tau\) as in Theorem 2.1.(1). If we choose \((u_2, v_2)\) with \(\tau \in S_n\) and \(0 < \alpha < \frac{\pi}{n}\), then we can find \((u_k, v_k)\) by the following relations
\[
(u_2, v_2) = (-u, -v),
\]
\[
(u_k, v_k) = (u_{k-1} + v_{k-1}, u_{k-1}v_2) \text{ for } 3 \leq k \leq n.
\]

**Example 2.2.** If \(u_1 = -5\), \(v_1 = 8\), \(\tau = \frac{5 + \sqrt{-7}}{2}\), then we have
\[
(u_1, v_1) = (-5, 8),
\]
\[
(u_2, v_2) = (5, -8),
\]
\[
(u_3, v_3) = (17, -40),
\]
\[
(u_4, v_4) = (45, -136),
\]
\[
(u_5, v_5) = (361, -360),
\]
\[
(u_6, v_6) = (1445, -2888),
\]
\[
(u_7, v_7) = (10125, -34696),
\]
\[
(u_8, v_8) = (15929, -81000).
\]

From Theorem 2.1, we get the following corollary.
COROLLARY 2.3. If $0 < \alpha < \frac{\pi}{2}$ and $u\tau^2 + v\tau + 1 = 0$ where $u, v$ integer, then the followings hold.

1. If $u$ is odd and $v$ even, then $\theta_3(0, \frac{1}{\tau})^4 = \theta_3(0, \tau)^4$.
2. If $u$ is odd and $v$ odd, then $\theta_3(0, \frac{1}{\tau})^4 = \theta_3(0, \tau + 1)^4$.

Proof. Note that $\theta_3(0, \frac{1}{\tau})^4 = \theta_3(0, \frac{u\tau^2 + v\tau}{\tau})^4 = \theta_3(0, u\tau + v)^4$. By Theorem 2.1, the conclusion is immediate.

THEOREM 2.4. Let $u$ be an odd integer and $v$ any integer. If $0 < \alpha < \frac{\pi}{n}$ and $\tau^n + u\tau + v = 0$ where $n$ is an integer $(\neq 0, 1)$, then

$$\Delta(\tau^n) = \Delta(\tau).$$

In particular, if $n = 2$, then there does not exist $u, v \in \mathbb{Z}$ such that $|\tau| = 1$.

Proof. It follows from [3, p. 69] that

$$(K - 2)$$

$$\Delta(\tau) = 16\pi^{12} \theta_3(0, \tau)^6 \theta_3(0, \tau + 1)^6 (\theta_3(0, \tau)^4 - \theta_3(0, \tau + 1)^4)^2.$$

Let $v$ be even. By $(K - 2)$ and Theorem 2.1, we obtain that

$$\Delta(\tau^n) = 16\pi^{12} \theta_3(0, \tau^n)^6 \theta_3(0, \tau^n + 1)^6 (\theta_3(0, \tau^n)^4 - \theta_3(0, \tau^n + 1)^4)^2$$

$$= 16\pi^{12} \theta_3(0, \tau)^6 \theta_3(0, \tau + 1)^6 (\theta_3(0, \tau)^4 - \theta_3(0, \tau + 1)^4)^2$$

$$= \Delta(\tau).$$

When $v$ is odd,

$$\Delta(\tau^n) = 16\pi^{12} \theta_3(0, \tau^n)^8 \theta_3(0, \tau^n + 1)^8 (\theta_3(0, \tau^n)^4 - \theta_3(0, \tau^n + 1)^4)^2$$

$$= 16\pi^{12} \theta_3(0, \tau)^8 \theta_3(0, \tau)^8 (\theta_3(0, \tau)^4 - \theta_3(0, \tau)^4)^2$$

$$= 16\pi^{12} \theta_3(0, \tau + 1)^8 \theta_3(0, \tau)^8 (\theta_3(0, \tau + 1)^4 - \theta_3(0, \tau)^4)^2$$

$$= \Delta(\tau).$$

Finally, if $n = 2$ and $|\tau| = 1$, then we check the following two cases.

Case 1. If $v$ is odd then $\theta_3(0, \tau^2) = \theta_3(0, \tau)$. By Theorem 2.1, it cannot happen.

Case 2. If $v$ is even then $\theta_3(0, \tau^2 + 1) = \theta_3(0, \tau)$. We see by Theorem 2.1 that it is impossible.

Consequently, if $\tau^2 + u\tau + v = 0$ and $|\tau| = 1$, then $\Delta(\tau^2) \neq \Delta(\tau)$. □
COROLLARY 2.5. Let $a, u$ be odd integers and $b, v$ be any integers. If $0 < \alpha < \frac{\pi}{\max\{n', m'\}}$ and $\tau^{n'} + u\tau + v = \tau^{m'} + a\tau + b = 0$, where $n, m \neq 0, 1$ integers then $\Delta(\tau^{n'}) = \Delta(\tau^{m'}) = \Delta(\tau)$. That is,

$$e^{2\pi i \tau^{n'}} \prod_{k=1}^{\infty} (1 - e^{2\pi i k\tau^{n'}})^{24} = e^{2\pi i \tau} \prod_{k=1}^{\infty} (1 - e^{2\pi i k\tau})^{24}.$$ 

Proof. It is clear from Theorem 2.4. \hfill \Box

COROLLARY 2.6. Let $u$ be odd integer and $v$ be any integer. If $0 < \alpha < \frac{\pi}{n}$ and $\tau^n + u\tau + v = 0$ where $(n \neq 0, 1)$ is an integer, then

$$J(\tau^n) = J(\tau).$$

In particular, if $n = 2$, then there does not exist $u, v \in \mathbb{Z}$ such that $|\tau| = 1$.

Proof. We see from [3] that

$$J(\tau) = \frac{1}{54} \left( \frac{\theta_3(0, \tau)^8 + \theta_3(0, \tau + 1)^8 + (\theta_3(0, \tau)^4 - \theta_3(0, \tau + 1)^4)^2}{\theta_3(0, \tau)^8 \theta_3(0, \tau + 1)^8 (\theta_3(0, \tau)^4 - \theta_3(0, \tau + 1)^4)^2} \right)^3.$$ 

Then, in a similar way as in Theorem 2.4, we have the conclusion. \hfill \Box

In like manner, we also obtain analogous formulae for Eisenstein series.

COROLLARY 2.7. Let $u$ be odd integer and $v$ be any integer. If $0 < \alpha < \frac{\pi}{n}$ and $\tau^n + u\tau + v = 0$ where $(n \neq 0, 1)$ is an integer, then

$$g_2(\tau^n) = g_2(\tau) \text{ and } g_3(\tau^n) = g_3(\tau).$$

3. Eisenstein series

We adopt the following notation

$$M_{2k} = \{ \text{modular forms of weight } 2k \text{ for } \Gamma(1) \},$$

which is a $\mathbb{C}$-vector space([11, 12]). As is well known [11], the set \{ $G_b^cG_0^c$ : $b, c \in \mathbb{Z}, 2b + 3c = k$ \} constitutes a basis for $M_{2k}$. 
LEMMA 3.1. For \( k \geq 2 \), \( G_{2k} = \sum_{i=1}^{t} a_i G_{4i}^b G_6^c \) where \( t = \text{dim } M_{2k} \), \( a_i \in \mathbb{Q} \) and \( 2b_i + 3c_i = k \).

Proof. We see from [11, 12] that

\[
(K - 3) \quad G_{2k}(\tau) = 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{(2k - 1)!} \sum_{n \geq 1} \sigma_{2k-1}(n)p^n,
\]

where \( \zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} \) and \( \sigma_k(n) = \sum_{d|n} d^k \) are the Riemann \( \zeta \)-function and \( k \)-th-power divisor function, respectively.

We know from [11, 12] that \( \zeta(2k) \) is a rational multiple of \( \pi^{2k} \).

Since \( G_{2k} \in M_{2k} \), we have

\[
(K - 4) \quad G_{2k}(\tau) = \sum_{i=1}^{t} a_i G_{4i}^b G_6^c
\]

with \( t = \text{dim } M_{2k} \) and \( a_i \in \mathbb{C} \).

From \((K - 3)\) and \((K - 4)\), we see that

\[
(\pi)^{2k}(d_1 + d_2p + d_3p^2 + \cdots) = (\pi)^{2k}\{a_1(e_{11} + e_{12}p + e_{13}p^2 + \cdots) + a_2(e_{21} + e_{22}p + e_{23}p^2 + \cdots) + \cdots + a_t(e_{t1} + e_{t2}p + e_{t3}p^2 + \cdots)\},
\]

and hence

\[
\begin{pmatrix}
  e_{11} & e_{12} & \cdots & e_{1t} \\
  e_{21} & e_{22} & \cdots & e_{2t} \\
  \vdots & \vdots & \ddots & \vdots \\
  e_{t1} & e_{t2} & \cdots & e_{tt}
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2 \\
\vdots \\
a_t
\end{pmatrix}
=\begin{pmatrix}
d_1 \\
d_2 \\
\vdots \\
d_t
\end{pmatrix}
\]

with \( d_i, e_{i,j} \in \mathbb{Q} \). Since \( \{G_4^b G_6^c : a, b \in \mathbb{Z}, 2b + 3c = k\} \) is a basis for the space \( M_{2k} \), \( a_1, \cdots, a_t \) are uniquely determined in \( \mathbb{C} \). Therefore \( a_1, \cdots, a_t \in \mathbb{Q} \). \( \square \)

Using Corollary 2.7 and Lemma 3.1, we get the following.

COROLLARY 3.2. Let \( u \) be odd integer and \( v \) be any integer. If \( 0 < \alpha < \frac{\pi}{n} \) and \( \tau^n + ur + v = 0 \) where \( n \neq 0, 1 \) is an integer, then

\[
G_{2k}(\tau^n) = G_{2k}(\tau)
\]

with \( k \geq 2 \).
PROPOSITION 3.3. ([5]) If \( k \geq 2 \), \( \frac{G_{2k}(\tau)}{(\pi)^{2k} \eta(\tau)^{4k}} \) is an algebraic number.

And if \( k \geq 2 \) and \( G_{2k}(\tau) \neq 0 \), then \( \frac{G_{2k}(\tau)}{\eta(\tau)^{4k}} \) is a transcendental number.

Using Corollary 3.2 and Proposition 3.3, we derive

COROLLARY 3.4. Let \( u \) be an odd integer and \( v \) be any integer, and let \( 0 < \alpha < \frac{\pi}{n} \) and \( \tau^n + ut + v = 0 \) where \( n \neq 0, 1 \) is an integer. If \( k \geq 2 \),

\[ \frac{G_{2k}(\tau^n)}{(\pi)^{2k} \eta(\tau^n)^{4k}} \]

is an algebraic number. And if \( k \geq 2 \) and \( G_{2k}(\tau^n) \neq 0 \), then \( \frac{G_{2k}(\tau^n)}{\eta(\tau^n)^{4k}} \) is a transcendental number.

References

[8] ———, As values of transcendental and algebraic numbers of infinite products in imaginary quadratics(Korean), Submitted.

Daeeyeoul Kim, Department of Mathematics, Chonbuk National University, Chonju 561-756, Korea
E-mail: dykim@math.chonbuk.ac.kr

Ja Kyung Koo, Korea Advanced Institute of Science and Technology, Department of Mathematics, Taehon 305-701, Korea
E-mail: jkkoo@math.kaist.ac.kr