FUZZY $r$-REGULAR OPEN SETS AND FUZZY ALMOST $r$-CONTINUOUS MAPS

SEOK JONG LEE AND EUN PYO LEE

ABSTRACT. We introduce the concepts of fuzzy $r$-regular open sets and fuzzy almost $r$-continuous maps in the fuzzy topology of Chattopadhyay. Also we investigate the equivalent conditions of the fuzzy almost $r$-continuity.

1. Introduction

Chang [2] introduced fuzzy topological spaces and other authors continued the investigation of such spaces. Azad [1] introduced the concepts of fuzzy regular open set and fuzzy almost continuous maps in Chang’s fuzzy topology. Chattopadhyay et al. [4] introduced another definition of fuzzy topology as a generalization of Chang’s fuzzy topology. By generalizing the definitions of Azad, we introduce the concepts of fuzzy $r$-regular open sets and fuzzy almost $r$-continuous maps in the fuzzy topology of Chattopadhyay. Then the concepts introduced by Azad become special cases of our definition. Also we investigate the equivalent conditions of the fuzzy almost $r$-continuity.

2. Preliminaries

In this paper, we denote by $I$ the unit interval $[0, 1]$ of the real line and $I_0 = (0, 1]$. A member $\mu$ of $I^X$ is called a fuzzy set in $X$. For any $\mu \in I^X$, $\mu^c$ denotes the complement $1 - \mu$. By $\tilde{0}$ and $\tilde{1}$ we denote constant maps on $X$ with value 0 and 1, respectively. All other notations are standard notations of fuzzy set theory.

Received July 24, 2001.
2000 Mathematics Subject Classification: 54A40.
Key words and phrases: fuzzy $r$-regular open, fuzzy almost $r$-continuous.
This work was supported partly by the Basic Science Research Institute (BSRI-01-4) of Chungbuk National University.
A Chang’s fuzzy topology on $X$ is a family $T$ of fuzzy sets in $X$ which satisfies the following three properties:

1. $\emptyset, \overline{1} \in T$.
2. If $\mu_1, \mu_2 \in T$ then $\mu_1 \land \mu_2 \in T$.
3. If $\mu_i \in T$ for each $i$, then $\bigvee \mu_i \in T$.

The pair $(X, T)$ is called a Chang’s fuzzy topological space.

A fuzzy topology on $X$ is a map $T : I^X \to I$ which satisfies the following properties:

1. $T(\overline{0}) = T(\overline{1}) = 1$,
2. $T(\mu_1 \land \mu_2) \geq T(\mu_1) \land T(\mu_2)$,
3. $T(\bigvee \mu_i) \geq \bigwedge T(\mu_i)$.

The pair $(X, T)$ is called a fuzzy topological space.

For each $\alpha \in (0, 1]$, a fuzzy point $x_\alpha$ in $X$ is a fuzzy set in $X$ defined by

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

In this case, $x$ and $\alpha$ are called the support and the value of $x_\alpha$, respectively. A fuzzy point $x_\alpha$ is said to belong to a fuzzy set $\mu$ in $X$, denoted by $x_\alpha \in \mu$, if $\alpha \leq \mu(x)$. A fuzzy point $x_\alpha$ in $X$ is said to be quasi-coincident with $\mu$, denoted by $x_\alpha \mu$, if $\alpha + \mu(x) > 1$. A fuzzy set $\rho$ in $X$ is said to be quasi-coincident with a fuzzy set $\mu$ in $X$, denoted by $\rho \mu$, if there is an $x \in X$ such that $\rho(x) + \mu(x) > 1$.

**Definition 2.1.** ([5]) Let $\mu$ be a fuzzy set in a fuzzy topological space $(X, T)$ and $r \in I_0$. Then $\mu$ is called

1. a fuzzy $r$-open set in $X$ if $T(\mu) \geq r$,
2. a fuzzy $r$-closed set in $X$ if $T(\mu^c) \geq r$.

**Definition 2.2.** ([3]) Let $(X, T)$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the fuzzy $r$-closure is defined by

$$\text{cl}(\mu, r) = \bigwedge \{\rho \in I^X : \mu \leq \rho, T(\rho^c) \geq r\}.$$ 

**Definition 2.3.** ([5]) Let $(X, T)$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the fuzzy $r$-interior is defined by

$$\text{int}(\mu, r) = \bigvee \{\rho \in I^X : \mu \geq \rho, T(\rho) \geq r\}.$$
THEOREM 2.4. ([5]) For a fuzzy set $\mu$ in a fuzzy topological space $(X, T)$ and $r \in I_0$, we have:

1. $\text{int}(\mu, r)^c = \text{cl}(\mu^c, r)$.
2. $\text{cl}(\mu, r)^c = \text{int}(\mu^c, r)$.

DEFINITION 2.5. ([5]) Let $\mu$ be a fuzzy set in a fuzzy topological space $(X, T)$ and $r \in I_0$. Then $\mu$ is said to be

1. fuzzy $r$-semiopen if there is a fuzzy $r$-open set $\rho$ in $X$ such that $\rho \leq \mu \leq \text{cl}(\rho, r)$,
2. fuzzy $r$-semiclosed if there is a fuzzy $r$-closed set $\rho$ in $X$ such that $\text{int}(\rho, r) \leq \mu \leq \rho$.

DEFINITION 2.6. ([5]) Let $x_\alpha$ be a fuzzy point in a fuzzy topological space $(X, T)$ and $r \in I_0$. Then a fuzzy set $\mu$ in $X$ is called

1. a fuzzy $r$-neighborhood of $x_\alpha$ if there is a fuzzy $r$-open set $\rho$ in $X$ such that $x_\alpha \in \rho \leq \mu$,
2. a fuzzy $r$-quasi-neighborhood of $x_\alpha$ if there is a fuzzy $r$-open set $\rho$ in $X$ such that $x_\alpha \rho \leq \mu$.

DEFINITION 2.7. ([5]) Let $f : (X, T) \rightarrow (Y, U)$ be a map from a fuzzy topological space $X$ to another fuzzy topological space $Y$ and $r \in I_0$. Then $f$ is called

1. a fuzzy $r$-continuous map if $f^{-1}(\mu)$ is a fuzzy $r$-open set of $X$ for each fuzzy $r$-open set $\mu$ in $Y$,
2. a fuzzy $r$-semiopen map if $f^{-1}(\mu)$ is a fuzzy $r$-semiopen set of $X$ for each fuzzy $r$-open set $\mu$ in $Y$,
3. a fuzzy $r$- irresolute map if $f^{-1}(\mu)$ is a fuzzy $r$-semiopen set of $X$ for each fuzzy $r$-semiopen set $\mu$ in $Y$.

All the other nonstandard definitions and notations can be found in [5] and [6].

3. Fuzzy $r$-regular open sets

We define the notions of fuzzy $r$-regular open sets and fuzzy $r$-regular closed sets, and investigate some of their properties.

DEFINITION 3.1. Let $\mu$ be a fuzzy set in a fuzzy topological space $(X, T)$ and $r \in I_0$. Then $\mu$ is said to be

1. fuzzy $r$-regular open if $\text{int}(\text{cl}(\mu, r), r) = \mu$,
2. fuzzy $r$-regular closed if $\text{cl}(\text{int}(\mu, r), r) = \mu$. 
**Theorem 3.2.** Let $\mu$ be a fuzzy set in a fuzzy topological space $(X, T)$ and $r \in I_0$. Then $\mu$ is fuzzy $r$-regular open if and only if $\mu^c$ is fuzzy $r$-regular closed.

**Proof.** It follows from Theorem 2.4. \qed

**Remark 3.3.** Clearly, every fuzzy $r$-regular open ($r$-regular closed) set is fuzzy $r$-open ($r$-closed). That the converse need not be true is shown by the following example. The example also shows that the union (intersection) of any two fuzzy $r$-regular open ($r$-regular closed) sets need not be fuzzy $r$-regular open ($r$-regular closed).

**Example 3.4.** Let $X = I$ and $\mu_1, \mu_2$ and $\mu_3$ be fuzzy sets in $X$ defined by

$$
\mu_1(x) = \begin{cases} 
0 & \text{if } 0 \leq x \leq \frac{1}{2}, \\
2x - 1 & \text{if } \frac{1}{2} \leq x \leq 1;
\end{cases}
$$

$$
\mu_2(x) = \begin{cases} 
1 & \text{if } 0 \leq x \leq \frac{1}{4}, \\
-4x + 2 & \text{if } \frac{1}{4} \leq x \leq \frac{1}{2}, \\
0 & \text{if } \frac{1}{2} \leq x \leq 1;
\end{cases}
$$

and

$$
\mu_3(x) = \begin{cases} 
0 & \text{if } 0 \leq x \leq \frac{1}{4}, \\
\frac{1}{3}(4x - 1) & \text{if } \frac{1}{4} \leq x \leq 1.
\end{cases}
$$

Define $T : I^X \to I$ by

$$
T(\mu) = \begin{cases} 
1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\
\frac{1}{2} & \text{if } \mu = \mu_1, \mu_2, \mu_1 \vee \mu_2, \\
0 & \text{otherwise}.
\end{cases}
$$

Then clearly $T$ is a fuzzy topology on $X$.

1. Clearly, $\mu_1 \vee \mu_2$ is fuzzy $\frac{1}{4}$-open. Since $\text{int}(\text{cl}(\mu_1 \vee \mu_2, \frac{1}{4}), \frac{1}{4}) = \tilde{1} \neq \mu_1 \vee \mu_2, \mu_1 \vee \mu_2$ is not a fuzzy $\frac{1}{2}$-regular open set.

2. Since $\text{int}(\text{cl}(\mu_1, \frac{1}{2}), \frac{1}{2}) = \text{int}(\mu_2, \frac{1}{2})$ and $\text{int}(\text{cl}(\mu_2, \frac{1}{2}), \frac{1}{2}) = \text{int}(\mu_1, \frac{1}{2}) = \mu_2, \mu_1$ and $\mu_2$ are fuzzy $\frac{1}{2}$-regular open sets. But $\mu_1 \vee \mu_2$ is not a fuzzy $\frac{1}{2}$-regular open set.

3. In view of Theorem 3.2, $\mu_1^c$ and $\mu_2^c$ are fuzzy $\frac{1}{2}$-regular closed sets but $\mu_1^c \wedge \mu_2^c = (\mu_1 \vee \mu_2)^c$ is not a fuzzy $\frac{1}{2}$-regular closed set.
THEOREM 3.5. (1) The intersection of two fuzzy \( r \)-regular open sets is fuzzy \( r \)-regular open.

(2) The union of two fuzzy \( r \)-regular closed sets is fuzzy \( r \)-regular closed.

Proof. (1) Let \( \mu \) and \( \rho \) be any two fuzzy \( r \)-regular open sets in a fuzzy topological space \( X \). Then \( \mu \) and \( \rho \) are fuzzy \( r \)-open sets and hence \( T(\mu) \cap \rho \geq T(\mu) \cap T(\rho) \geq r \). Thus \( \mu \cap \rho \) is a fuzzy \( r \)-open set. Since \( \mu \cap \rho \leq \operatorname{cl}(\mu \cap \rho, r) \),
\[
\text{int}(\operatorname{cl}(\mu \cap \rho, r), r) \geq \text{int}(\mu \cap \rho, r) = \mu \cap \rho.
\]

Now, \( \mu \cap \rho \leq \mu \) and \( \mu \cap \rho \leq \rho \) implies
\[
\text{int}(\operatorname{cl}(\mu \cap \rho, r), r) \leq \text{int}(\operatorname{cl}(\mu, r), r) = \mu
\]
and
\[
\text{int}(\operatorname{cl}(\mu \cap \rho, r), r) \leq \text{int}(\operatorname{cl}(\rho, r), r) = \rho.
\]
Hence \( \text{int}(\operatorname{cl}(\mu \cap \rho, r), r) \leq \mu \cap \rho \). Therefore \( \mu \cap \rho \) is fuzzy \( r \)-regular open.

(2) It follows from (1) and Theorem 3.2.

THEOREM 3.6. (1) The fuzzy \( r \)-closure of a fuzzy \( r \)-open set is fuzzy \( r \)-regular closed.

(2) The fuzzy \( r \)-interior of a fuzzy \( r \)-closed set is fuzzy \( r \)-regular open.

Proof. (1) Let \( \mu \) be a fuzzy \( r \)-open set in a fuzzy topological space \( X \). Then clearly \( \text{int}(\operatorname{cl}(\mu, r), r) \leq \operatorname{cl}(\mu, r) \) implies that
\[
\operatorname{cl}(\text{int}(\operatorname{cl}(\mu, r), r), r) \leq \operatorname{cl}(\operatorname{cl}(\mu, r), r) = \operatorname{cl}(\mu, r).
\]
Since \( \mu \) is fuzzy \( r \)-open, \( \mu = \text{int}(\mu, r) \). Also since \( \mu \leq \operatorname{cl}(\mu, r) \), \( \mu = \text{int}(\mu, r) \leq \text{int}(\operatorname{cl}(\mu, r), r) \). Thus \( \operatorname{cl}(\mu, r) \leq \text{cl}(\text{int}(\operatorname{cl}(\mu, r), r), r) \). Hence \( \operatorname{cl}(\mu, r) \) is a fuzzy \( r \)-regular closed set.

(2) Similar to (1).

Let \( (X, T) \) be a fuzzy topological space. For an \( r \)-cut \( T_r = \{ \mu \in I^X \mid T(\mu) \geq r \} \), it is obvious that \( (X, T_r) \) is a Chang’s fuzzy topological space for all \( r \in I_0 \).

Let \( (X, T) \) be a Chang’s fuzzy topological space and \( r \in I_0 \). Recall [4] that a fuzzy topology \( T^r : I^X \to I \) is defined by
\[
T^r(\mu) = \begin{cases} 
1 & \text{if } \mu = \hat{0}, \hat{1}, \\
r & \text{if } \mu \in T - \{\hat{0}, \hat{1}\}, \\
0 & \text{otherwise.}
\end{cases}
\]
Theorem 3.7. Let $\mu$ be a fuzzy set in a fuzzy topological space $(X, T)$ and $r \in I_0$. Then $\mu$ is fuzzy $r$-regular open (regular closed) in $(X, T)$ if and only if $\mu$ is fuzzy regular open (regular closed) in $(X, T_r)$.

Proof. Straightforward.

Theorem 3.8. Let $\mu$ be a fuzzy set of a Chang's fuzzy topological space $(X, T)$ and $r \in I_0$. Then $\mu$ is a fuzzy regular open (regular closed) in $(X, T)$ if and only if $\mu$ is fuzzy $r$-regular open (regular closed) in $(X, T^r)$.

Proof. Straightforward.

4. Fuzzy almost $r$-continuous maps

We are going to introduce the notions of fuzzy almost $r$-continuous maps and investigate some of their properties. Also, we describe the relations among fuzzy almost $r$-continuous maps, fuzzy $r$-continuous maps and fuzzy $r$-semicontinuous maps.

Definition 4.1. Let $f : (X, T) \to (Y, U)$ be a map from a fuzzy topological space $X$ to another fuzzy topological space $Y$ and $r \in I_0$. Then $f$ is called

1. a fuzzy almost $r$-continuous map if $f^{-1}(\mu)$ is a fuzzy $r$-open set of $X$ for each fuzzy $r$-regular open set $\mu$ in $Y$, or equivalently, $f^{-1}(\mu)$ is a fuzzy $r$-closed set in $X$ for each fuzzy $r$-regular closed set $\mu$ in $Y$,

2. a fuzzy almost $r$-open map if $f(\rho)$ is a fuzzy $r$-open set in $Y$ for each fuzzy $r$-regular open set $\rho$ in $X$,

3. a fuzzy almost $r$-closed map if $f(\rho)$ is a fuzzy $r$-closed set in $Y$ for each fuzzy $r$-regular closed set $\rho$ in $X$.

Theorem 4.2. Let $f : (X, T) \to (Y, U)$ be a map and $r \in I_0$. Then the following statements are equivalent:

1. $f$ is a fuzzy almost $r$-continuous map.
2. $f^{-1}(\mu) \subseteq \text{int}(f^{-1}(\text{cl}(\mu, r)), r)$ for each fuzzy $r$-open set $\mu$ in $Y$.
3. $\text{cl}(f^{-1}(\text{cl}(\mu, r)), r) \subseteq f^{-1}(\mu)$ for each fuzzy $r$-closed set $\mu$ in $Y$.
Proof. (1) $\Rightarrow$ (2) Let $f$ be fuzzy almost $r$-continuous and $\mu$ any fuzzy $r$-open set in $Y$. Then

$$\mu = \text{int}(\mu, r) \leq \text{int}(\text{cl}(\mu, r), r).$$

By Theorem 3.6(2), $\text{int}(\text{cl}(\mu, r), r)$ is a fuzzy $r$-regular open set in $Y$. Since $f$ is fuzzy almost $r$-continuous, $f^{-1}(\text{int}(\text{cl}(\mu, r), r))$ is a fuzzy $r$-open set in $X$. Hence

$$f^{-1}(\mu) \leq f^{-1}(\text{int}(\text{cl}(\mu, r), r)) = \text{int}(f^{-1}(\text{int}(\text{cl}(\mu, r), r)), r).$$

(2) $\Rightarrow$ (3) Let $\mu$ be a fuzzy $r$-closed set of $Y$. Then $\mu^c$ is a fuzzy $r$-open set in $Y$. By (2),

$$f^{-1}(\mu^c) \leq \text{int}(f^{-1}(\text{cl}(\mu^c, r), r)), r).$$

Hence

$$f^{-1}(\mu) = f^{-1}(\mu^c)^c \geq \text{int}(f^{-1}(\text{cl}(\mu^c, r), r))^c = \text{cl}(f^{-1}(\text{cl}(\mu, r), r), r).$$

(3) $\Rightarrow$ (1) Let $\mu$ be a fuzzy $r$-regular closed set in $Y$. Then $\mu$ is a fuzzy $r$-closed set in $Y$ and hence

$$f^{-1}(\mu) \leq \text{int}(f^{-1}(\text{cl}(\mu, r), r)), r) = \text{int}(f^{-1}(\mu), r).$$

Thus $f^{-1}(\mu) = \text{cl}(f^{-1}(\mu), r)$ and hence $f^{-1}(\mu)$ is a fuzzy $r$-closed set in $X$. Therefore, $f$ is a fuzzy almost $r$-continuous map. 

**Theorem 4.3.** Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map and $r \in I_0$. Then $f$ is fuzzy almost $r$-open if and only if $f(\text{int}(\rho, r)) \leq \text{int}(f(\rho), r)$ for each fuzzy $r$-semiclosed set $\rho$ in $X$.

**Proof.** Let $f$ be fuzzy almost $r$-open and $\rho$ a fuzzy $r$-semiclosed set in $X$. Then $\text{int}(\rho, r) \leq \text{int}(\text{cl}(\rho, r), r) \leq \rho$. Note that $\text{cl}(\rho, r)$ is a fuzzy $r$-closed set of $X$. By Theorem 3.6(2), $\text{int}(\text{cl}(\rho, r), r)$ is a fuzzy $r$-regular open set in $X$. Since $f$ is fuzzy almost $r$-open, $f(\text{int}(\text{cl}(\rho, r), r))$ is a fuzzy $r$-open set in $X$. Thus we have

$$f(\text{int}(\rho, r)) \leq f(\text{int}(\text{cl}(\rho, r), r)) = \text{int}(f(\text{int}(\text{cl}(\rho, r), r)), r) \leq \text{int}(f(\rho), r).$$
Conversely, let $\rho$ be a fuzzy $r$-regular open set of $X$. Then $\rho$ is fuzzy $r$-open and hence $\text{int}(\rho, r) = \rho$. Since $\text{int}(\text{cl}(\rho, r), r) = \rho$, $\rho$ is fuzzy $r$-semiclosed. So

$$f(\rho) = f(\text{int}(\rho, r)) \leq \text{int}(f(\rho), r) \leq f(\rho).$$

Thus $f(\rho) = \text{int}(f(\rho), r)$ and hence $f(\rho)$ is a fuzzy $r$-open set in $Y$. □

The global property of fuzzy almost $r$-continuity can be rephrased to the local property in terms of neighborhood and quasi-neighborhood, respectively, in the following two theorems.

**Theorem 4.4.** Let $f : (X, T) \to (Y, \mathcal{U})$ be a map and $r \in I_0$. Then $f$ is fuzzy almost $r$-continuous if and only if for every fuzzy point $x_\alpha$ in $X$ and every fuzzy $r$-neighborhood $\mu$ of $f(x_\alpha)$, there is a fuzzy $r$-neighborhood $\rho$ of $x_\alpha$ such that $x_\alpha \in \rho$ and $f(\rho) \subseteq \text{int}(\text{cl}(\mu, r), r)$.

**Proof.** Let $x_\alpha$ be a fuzzy point in $X$ and $\mu$ a fuzzy $r$-neighborhood of $f(x_\alpha)$. Then there is a fuzzy $r$-open set $\lambda$ in $Y$ such that $f(x_\alpha) \subseteq \lambda \subseteq \mu$. So $x_\alpha \in f^{-1}(\lambda) \subseteq f^{-1}(\mu)$. Since $f$ is fuzzy almost $r$-continuous,

$$f^{-1}(\lambda) \subseteq \text{int}(f^{-1}(\text{cl}(\lambda, r)), r) \subseteq \text{int}(f^{-1}(\text{cl}(\mu, r)), r).$$

Put $\rho = f^{-1}(\text{int}(\text{cl}(\mu, r), r))$. Then $x_\alpha \in f^{-1}(\lambda) \subseteq \text{int}(\rho, r) \subseteq \rho$. By Theorem 3.6(2), $\text{int}(\text{cl}(\mu, r), r)$ is fuzzy $r$-regular open. Since $f$ is fuzzy almost $r$-continuous, $\rho = f^{-1}(\text{int}(\text{cl}(\mu, r), r))$ is fuzzy $r$-open. Thus $\rho$ is a fuzzy $r$-neighborhood of $x_\alpha$ and

$$f(\rho) = ff^{-1}(\text{int}(\text{cl}(\mu, r), r)) \subseteq \text{int}(\text{cl}(\mu, r), r).$$

Conversely, let $\mu$ be a fuzzy $r$-regular open set in $Y$ and $x_\alpha \in f^{-1}(\mu)$. Then $\mu$ is fuzzy $r$-open and hence $\mu$ is a fuzzy $r$-neighborhood of $f(x_\alpha)$. By hypothesis, there is a fuzzy $r$-neighborhood $\rho_{x_\alpha}$ of $x_\alpha$ such that $x_\alpha \in \rho_{x_\alpha}$ and $f(\rho_{x_\alpha}) \subseteq \text{int}(\text{cl}(\mu, r), r) = \mu$. Since $\rho_{x_\alpha}$ is a fuzzy $r$-neighborhood of $x_\alpha$, there is a fuzzy $r$-open set $\lambda_{x_\alpha}$ in $X$ such that

$$x_\alpha \in \lambda_{x_\alpha} \subseteq \rho_{x_\alpha} \subseteq f^{-1}f(\rho_{x_\alpha}) \subseteq f^{-1}(\mu).$$

So we have

$$f^{-1}(\mu) = \bigvee \{x_\alpha : x_\alpha \in f^{-1}(\mu)\}$$

$$\leq \bigvee \{\lambda_{x_\alpha} : x_\alpha \in f^{-1}(\mu)\}$$

$$\leq f^{-1}(\mu).$$

Thus $f^{-1}(\mu) = \bigvee \{\lambda_{x_\alpha} : x_\alpha \in f^{-1}(\mu)\}$ is fuzzy $r$-open in $X$ and hence $f$ is almost $r$-continuous. □
Fuzzy r-regular open sets and fuzzy almost r-continuous maps

THEOREM 4.5. Let $f : (X,T) \rightarrow (Y,U)$ be a map and $r \in I_0$. Then $f$ is a fuzzy almost $r$-continuous map if and only if for every fuzzy point $x_\alpha$ in $X$ and every fuzzy $r$-quasi-neighborhood $\mu$ of $f(x_\alpha)$, there is a fuzzy $r$-quasi-neighborhood $\rho$ of $x_\alpha$ such that $x_\alpha q \rho$ and $f(\rho) \leq \text{int}(\text{cl}(\mu), r)$.

Proof. Let $x_\alpha$ be a fuzzy point in $X$ and $\mu$ a fuzzy $r$-quasi-neighborhood of $f(x_\alpha)$. Then there is a fuzzy $r$-open set $\lambda$ in $Y$ such that $f(x_\alpha) q \lambda \leq \mu$. So $x_\alpha q f^{-1}(\lambda)$. Since $f$ is fuzzy almost $r$-continuous,

$$f^{-1}(\lambda) \leq \text{int}(f^{-1}(\text{int}(\text{cl}(\lambda), r)), r) \leq \text{int}(f^{-1}(\text{int}(\text{cl}(\mu), r)), r).$$

Put $\rho = f^{-1}(\text{int}(\text{cl}(\mu), r))$. Then $x_\alpha q f^{-1}(\lambda) \leq \text{int}(\rho, r) \leq \rho$. So $x_\alpha q \rho$. Since $\text{int}(\text{cl}(\mu), r)$ is fuzzy $r$-regular open and $f$ is fuzzy almost $r$-continuous, $\rho = f^{-1}(\text{int}(\text{cl}(\mu), r))$ is fuzzy $r$-open. Thus $\rho$ is a fuzzy $r$-quasi-neighborhood of $x_\alpha$ and

$$f(\rho) = ff^{-1}(\text{int}(\text{cl}(\mu), r)) \leq \text{int}(\text{cl}(\mu), r).$$

Conversely, let $\mu$ be a fuzzy $r$-regular open set in $Y$. If $f^{-1}(\mu) = \emptyset$, then it is obvious. Suppose $x_\alpha$ is a fuzzy point in $f^{-1}(\mu)$ such that $\alpha < f^{-1}(\mu)(x)$. Then $\alpha < \mu(f(x))$ and hence $f(x)_{1-\alpha} q \mu$. So $\mu$ is a fuzzy $r$-quasi-neighborhood of $f(x)_{1-\alpha} = f(x_{1-\alpha})$. By hypothesis, there is a fuzzy $r$-quasi-neighborhood $\rho_{x_{1-\alpha}}$ of $x_{1-\alpha}$ such that $x_{1-\alpha} q \rho_{x_{1-\alpha}}$ and $f(\rho_{x_{1-\alpha}}) \leq \text{int}(\text{cl}(\mu), r) = \mu$. Since $\rho_{x_{1-\alpha}}$ is a fuzzy $r$-quasi-neighborhood of $x_{1-\alpha}$, there is a fuzzy $r$-open set $\lambda_{x_{1-\alpha}}$ in $X$ such that

$$x_{1-\alpha} q \lambda_{x_{1-\alpha}} \leq \rho_{x_{1-\alpha}} \leq f^{-1}(\mu).$$

Then $\alpha < \lambda_{x_{\alpha}}(x)$ and hence $x_\alpha \in \lambda_{x_{\alpha}}$. So

$$f^{-1}(\mu) = \bigvee \{x_\alpha : x_\alpha \text{ is a fuzzy point in } f^{-1}(\mu) \text{ such that } \alpha < f^{-1}(\mu)(x)\}$$

$$\leq \bigvee \{\lambda_{x_\alpha} : x_\alpha \text{ is a fuzzy point in } f^{-1}(\mu) \text{ such that } \alpha < f^{-1}(\mu)(x)\}$$

$$\leq f^{-1}(\mu),$$

and hence

$$f^{-1}(\mu) = \bigvee \{\lambda_{x_\alpha} : x_\alpha \text{ is a fuzzy point in } f^{-1}(\mu) \text{ such that } \alpha < f^{-1}(\mu)(x)\}.$$

Thus $f^{-1}(\mu)$ is fuzzy $r$-open in $X$. Therefore $f$ is fuzzy almost $r$-continuous. \qed
Theorem 4.6. Let \( f : (X, T) \to (Y, U) \) be fuzzy \( r \)-semitopological and fuzzy almost \( r \)-open. Then \( f \) is fuzzy \( r \)- irresolute.

Proof. Let \( \mu \) be fuzzy \( r \)-semitopological in \( Y \). Then \( \text{int}(\text{cl}(\mu, r), r) \leq \mu \).

Since \( f \) is fuzzy \( r \)-semitopological,\n\[
\text{int}(\text{cl}(f^{-1}(\mu), r), r) \leq f^{-1}(\text{cl}(\mu, r)).
\]
Thus we have
\[
\text{int}(\text{cl}(f^{-1}(\mu), r), r) = \text{int}(\text{int}(\text{cl}(f^{-1}(\mu), r), r), r) \leq \text{int}(f^{-1}(\text{cl}(\mu, r)), r).
\]

Since \( f \) is fuzzy \( r \)-semitopological and \( \text{cl}(\mu, r) \) is fuzzy \( r \)-closed, \( f^{-1}(\text{cl}(\mu, r)) \) is a fuzzy \( r \)-semitopological set in \( X \). Since \( f \) is fuzzy almost \( r \)-open,
\[
f(\text{int}(f^{-1}(\text{cl}(\mu, r)), r) \leq \text{int}(f f^{-1}(\text{cl}(\mu, r)), r) \leq \text{int}(\text{cl}(\mu, r), r) \leq \mu.
\]
Hence we have
\[
\text{int}(\text{cl}(f^{-1}(\mu), r), r) \leq f^{-1} f(\text{int}(\text{cl}(f^{-1}(\mu), r), r)) \leq f^{-1} f(\text{int}(f^{-1}(\text{cl}(\mu, r)), r)) \leq f^{-1}(\mu).
\]
Thus \( f^{-1}(\mu) \) is fuzzy \( r \)-semitopological in \( X \) and hence \( f \) is fuzzy \( r \)- irresolute. \( \square \)

Remark 4.7. Clearly a fuzzy \( r \)-continuous map is a fuzzy almost \( r \)-continuous map. That the converse need not be true is shown by the following example. Also, the example shows that a fuzzy almost \( r \)-continuous map need not be a fuzzy \( r \)-semitopological map.

Example 4.8. Let \( X = I \) and \( \mu_1, \mu_2 \) and \( \mu_3 \) be fuzzy sets in \( X \) defined by\n\[
\mu_1(x) = x;
\]
\[
\mu_2(x) = 1 - x;
\]
and\n\[
\mu_3(x) = \begin{cases} 
x & \text{if } 0 \leq x \leq \frac{1}{2}, \\
0 & \text{if } \frac{1}{2} \leq x \leq 1.
\end{cases}
\]
Define $\mathcal{T}_1 : I^X \to I$ and $\mathcal{T}_2 : I^X \to I$ by

$$
\mathcal{T}_1(\mu) = \begin{cases} 
1 & \text{if } \mu = \bar{0}, \bar{1} \\
\frac{1}{2} & \text{if } \mu = \mu_1, \mu_2, \mu_1 \lor \mu_2, \mu_1 \land \mu_2 \\
0 & \text{otherwise},
\end{cases}
$$

and

$$
\mathcal{T}_2(\mu) = \begin{cases} 
1 & \text{if } \mu = \bar{0}, \bar{1} \\
\frac{1}{2} & \text{if } \mu = \mu_1, \mu_2, \mu_3, \mu_1 \lor \mu_2, \mu_1 \land \mu_2 \\
0 & \text{otherwise}.
\end{cases}
$$

Then clearly $\mathcal{T}_1, \mathcal{T}_2$ are fuzzy topologies on $X$. Consider the identity map $1_X : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$. It is clear that $\mu_1, \mu_2, \mu_1 \lor \mu_2$ and $\mu_1 \land \mu_2$ are fuzzy $\frac{1}{2}$-regular open in $(X, \mathcal{T}_2)$ while $\mu_3$ is not. Noting that $\mathcal{T}_1(\mu_3) = 0$, it is obvious that $1_X$ is a fuzzy $\frac{1}{2}$-almost continuous map which is not a fuzzy $\frac{1}{2}$-continuous map. Also, because $\bar{0}$ is the only fuzzy $\frac{1}{2}$-open set contained in $\mu_3$, $\mu_3 = 1_X^{-1}(\mu_3)$ is not a fuzzy $\frac{1}{2}$-semiopen set in $(X, \mathcal{T}_1)$ and hence $1_X$ is not a fuzzy $\frac{1}{2}$-semiopen continuous map.

**Example 4.9.** A fuzzy $r$-semicontinuous map need not be a fuzzy almost $r$-continuous map.

Let $(X, T)$ be a fuzzy topological space as described in Example 3.4 and let $f : (X, T) \to (X, T)$ be defined by $f(x) = \frac{x}{2}$. Simple computations give $f^{-1}(\bar{0}) = \bar{0}, f^{-1}(\bar{1}) = \frac{1}{2}$, $f^{-1}(\mu_1) = \bar{0}$ and $f^{-1}(\mu_2) = \mu_1^c = f^{-1}(\mu_1 \lor \mu_2)$. Since $\text{cl}(\mu_2, \frac{1}{2}) = \mu_1^c, \mu_1^c$ is a fuzzy $\frac{1}{2}$-semiopen set and hence $f$ is a fuzzy $\frac{1}{2}$-semiopen continuous map. But $f^{-1}(\mu_2) = \mu_1^c$ and

$$
\text{int}(f^{-1}(\text{int}(\text{cl}(\mu_2, \frac{1}{2})), \frac{1}{2})), \frac{1}{2}) = \text{int}(f^{-1}(\text{int}(\mu_1^c, \frac{1}{2})), \frac{1}{2})
$$

$$
= \text{int}(f^{-1}(\mu_2), \frac{1}{2})
$$

$$
= \text{int}(\mu_1^c, \frac{1}{2}) = \mu_2.
$$

Thus $f^{-1}(\mu_2) \not\subseteq \text{int}(f^{-1}(\text{int}(\text{cl}(\mu_2, \frac{1}{2})), \frac{1}{2})), \frac{1}{2})$ and hence $f$ is not a fuzzy almost $\frac{1}{2}$-continuous map.

From Example 4.8 and 4.9 we have the following result.

**Theorem 4.10.** Fuzzy $r$-semicontinuity and fuzzy almost $r$-continuity are independent notions.
Definition 4.11. Let \((X, T)\) be a fuzzy topological space and \(r \in I_0\). Then \((X, T)\) is called a fuzzy \(r\)-semiregular space if each fuzzy \(r\)-open set in \(X\) is a union of fuzzy \(r\)-regular open sets.

Theorem 4.12. Let \(r \in I_0\) and \(f : (X, T) \to (Y, \mathcal{U})\) be a map from a fuzzy topological space \(X\) to a fuzzy \(r\)-semiregular space \(Y\). Then \(f\) is fuzzy almost \(r\)-continuous if and only if \(f\) is fuzzy \(r\)-continuous.

Proof. Due to Remark 4.7, it suffices to show that if \(f\) is fuzzy almost \(r\)-continuous then it is fuzzy \(r\)-continuous. Let \(\mu\) be a fuzzy \(r\)-open set in \(Y\). Since \((Y, \mathcal{U})\) is a \(r\)-semiregular space, \(\mu = \bigvee \mu_i\), where \(\mu_i\)'s are fuzzy \(r\)-regular open sets in \(Y\). Then since \(f\) is a fuzzy almost \(r\)-continuous map, \(f^{-1}(\mu_i)\) is a fuzzy \(r\)-open set for each \(i\). So

\[ T(f^{-1}(\mu)) = T(f^{-1}(\bigvee \mu_i)) = T(\bigvee f^{-1}(\mu_i)) \geq \bigwedge T(f^{-1}(\mu_i)) \geq r. \]

Thus \(f^{-1}(\mu)\) is fuzzy \(r\)-open in \(X\) and hence \(f\) is a fuzzy \(r\)-continuous map.

Theorem 4.13. Let \(f : (X, T) \to (Y, \mathcal{U})\) be a map from a fuzzy topological space \(X\) to another fuzzy topological space \(Y\) and \(r \in I_0\). Then \(f\) is fuzzy almost \(r\)-continuous (\(r\)-open, \(r\)-closed) if and only if \(f : (X, T_r) \to (Y, \mathcal{U}_r)\) is fuzzy almost continuous (open, closed).

Proof. Straightforward.

Theorem 4.14. Let \(f : (X, T) \to (Y, U)\) be a map from a Chang's fuzzy topological space \(X\) to another Chang's fuzzy topological space \(Y\) and \(r \in I_0\). Then \(f\) is fuzzy almost continuous (open, closed) if and only if \(f : (X, T^r) \to (Y, U^r)\) is fuzzy almost \(r\)-continuous (\(r\)-open, \(r\)-closed).

Proof. Straightforward.

References

Fuzzy \( r \)-regular open sets and fuzzy almost \( r \)-continuous maps


Seok Jong Lee, Department of Mathematics, Chungbuk National University, Cheongju 361-763, Korea
E-mail: sjlee@cbnu.ac.kr

Eun Pyo Lee, Department of Mathematics, Seonam University, Namwon 590-711, Korea
E-mail: eplee@tiger.seonam.ac.kr