STRONG LAWS FOR WEIGHTED SUMS
OF I.I.D. RANDOM VARIABLES (II)

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ABSTRACT. Let \( \{X, X_n, n \geq 1\} \) be a sequence of i.i.d. random variables and \( \{a_{ni}, 1 \leq i \leq n, n \geq 1\} \) an array of constants. Let \( \phi(x) \) be a positive increasing function on \((0, \infty)\) satisfying \( \phi(x) \uparrow \infty \) and \( \phi(Cx) = O(\phi(x)) \) for any \( C > 0 \). When \( EX = 0 \) and \( E[\phi(|X|)] < \infty \), some conditions on \( \phi \) and \( \{a_{ni}\} \) are given under which \( \sum_{i=1}^{n} a_{ni}X_i \to 0 \) a.s.

1. Introduction

Let \( \{X, X_n, n \geq 1\} \) be a sequence of i.i.d. random variables and \( \{a_{ni}, 1 \leq i \leq n, n \geq 1\} \) an array of constants. Throughout this paper, we assume that \( \phi(x) \) is a positive increasing function on \((0, \infty)\) satisfying

\[
\phi(x) \uparrow \infty \quad \text{and} \quad \phi(Cx) = O(\phi(x)), \quad \forall C > 0.
\]

(1)

We also assume that \( EX = 0 \) and \( E[\phi(|X|)] < \infty \).

When \( \phi(x) = x^p (p \geq 1) \), \( \phi \) satisfies (1). In this case, the a.s. (almost sure) limiting behavior for the weighted sums \( \sum_{i=1}^{n} a_{ni}X_i \) was studied by many authors (see, Bai and Cheng [1], Choi and Sung [2], Cuzick [3], and Li et al. [4]). We recommend the paper of Rosalsky and Sreehari [5] for more information.

However, when \( \phi(x) = e^{h|x|^\gamma} (h > 0, \gamma > 0) \), \( \phi \) does not satisfy (1). In this case, the a.s. limiting behavior for the weighted sums was studied by Bai and Cheng [1], Sung [7], and Wu [8].
The purpose of this work is to present various conditions on $\phi$ and 
$\{a_n\}$ under which $\sum_{i=1}^{n} a_n X_i \to 0$ a.s. Our result extends that of Bai 
and Cheng [1], Cuzick [3], and Li et al. [4].

Throughout this paper, $C$ denotes a positive constant which may be 
different in various places.

2. Main results

Let $\psi(x)$ be the inverse function of $\phi(x)$. Since $\phi(x) \uparrow \infty$, it follows 
that $\psi(x) \uparrow \infty$. For easy notation, we let $\phi(0) = 0$ and $\psi(0) = 0$.

To prove our main results, we will need the following lemma.

**Lemma 1.** Assume that the inverse function $\psi(x)$ of $\phi(x)$ satisfies

$$
(2) \quad \psi(n) \sum_{i=1}^{n} \frac{1}{\psi(i)} = O(n), \quad \psi^2(n) \sum_{i=n}^{\infty} \frac{1}{\psi^2(i)} = O(n).
$$

If $E[\phi(|X|)] < \infty$, then the followings hold.

(i) $\sum_{n=1}^{\infty} \frac{1}{n \psi(n)} E[X|I(|X| > \psi(n)) < \infty,$

(ii) $\sum_{n=1}^{\infty} \frac{1}{n \psi^2(n)} E[X^2|I(|X| \leq \psi(n)) < \infty.$

**Proof.** Since $\psi(x)$ is increasing function, we have that

$$
\sum_{n=1}^{\infty} \frac{1}{n \psi(n)} E[X|I(|X| > \psi(n))
$$

$$
= \sum_{n=1}^{\infty} \frac{1}{n \psi(n)} \sum_{i=n}^{\infty} E[X|I(\psi(i) < |X| \leq \psi(i+1))]
$$

$$
= \sum_{i=1}^{\infty} E[X|I(\psi(i) < |X| \leq \psi(i+1)) \sum_{n=1}^{i} \frac{1}{\psi(n)}
$$

$$
\leq \sum_{i=1}^{\infty} P(\psi(i) < |X| \leq \psi(i+1)) \sum_{n=1}^{i} \frac{1}{\psi(n)}
$$

$$
\leq C \sum_{i=1}^{\infty} P(\psi(i) < |X| \leq \psi(i+1)) i
$$

So (i) is proved. The proof of (ii) is similar to that of (i) and is omitted. □

Now, we state and prove one of our main results.
Theorem 1. Let \{X, X_n, n \geq 1\} be a sequence of i.i.d. random variables with \( EX = 0 \) and \( E[\phi(|X|)] < \infty \). Assume that the inverse function \( \psi(x) \) of \( \phi(x) \) satisfies (2). Let \{a_{ni}, 1 \leq i \leq n, n \geq 1\} be an array of constants such that

(i) \( \max_{1 \leq i \leq n} a_{ni} = O\left(\frac{1}{\psi(n)}\right) \),

(ii) \( \max_{1 \leq j \leq n} \frac{\psi(j)}{j} \sum_{i=j}^{n} a_{ni}^2 = O\left(\frac{1}{n^\alpha}\right) \) for some \( \alpha > 0 \).

Then \( \sum_{i=1}^{n} a_{ni}X_i \to 0 \) a.s.

Proof. First we prove that

\[
\sum_{i=1}^{n} a_{ni}X_i \to 0 \text{ in probability.}
\]

Define \( Y'_n = X_n I(|X_n| \leq \psi(n)) \) and \( Y''_n = X_n - Y'_n \) for \( n \geq 1 \). Then for any \( \epsilon > 0 \)

\[
P\left(\left| \sum_{i=1}^{n} a_{ni}X_i \right| > \epsilon \right)
\leq P\left(\left| \sum_{i=1}^{n} a_{ni}(Y'_i - EY'_i) \right| > \frac{\epsilon}{2} \right) + P\left(\left| \sum_{i=1}^{n} a_{ni}(Y''_i - EY''_i) \right| > \frac{\epsilon}{2} \right)
\leq \frac{4}{\epsilon^2} E\left| \sum_{i=1}^{n} a_{ni}(Y'_i - EY'_i) \right|^2 + \frac{2}{\epsilon} E\left| \sum_{i=1}^{n} a_{ni}(Y''_i - EY''_i) \right|
\leq \frac{C}{\psi^2(n)} \sum_{i=1}^{n} EY'_i^2 + \frac{C}{\psi(n)} \sum_{i=1}^{n} E|Y''_i|
\]

by (i). From Lemma 1 and the Kronecker lemma, the two terms on the last expression converge to 0. Thus (3) is proved.

From (3) it follows at once that \( \mu(\sum_{i=1}^{n} a_{ni}X_i) \to 0 \), where \( \mu(Y) \) is a median of \( Y \). Hence, by Theorem 3.2.1 in Stout [6], it suffices to prove that

\[
\sum_{i=1}^{n} a_{ni}X_i^* \to 0 \text{ a.s.,}
\]

where \( \{X_i^*\} \) is a symmetrized version of \( \{X_n\} \). So we need only to prove the result for \( \{X_n\} \) symmetric. Since \( E[\phi(|X|)] < \infty \) and \( \phi(Cx) = O(\phi(x)) \) for any \( C > 0 \), \( \sum_{n=1}^{\infty} P(|X_n| > \epsilon\psi(n)) < \infty \) for any \( \epsilon > 0 \).
Thus it is possible to construct a sequence \( \{b_n\} \) of real numbers such that \( 0 < b_n \leq 1, b_n \downarrow 0 \), and

\[
(5) \quad \sum_{n=1}^{\infty} P(|X_n| > b_n \psi(n)) < \infty.
\]

Define \( X'_n = X_n I(|X_n| \leq b_n \psi(n)) \) and \( X''_n = X_n - X'_n \) for \( n \geq 1 \). Then we have by (5) and the Borel-Cantelli lemma that

\[
|\sum_{i=1}^{n} a_{ni} X''_i| \leq C \sum_{i=1}^{n} \frac{|X''_i|}{\psi(n)} \rightarrow 0 \text{ a.s.}
\]

Thus it is enough to show that

\[
(6) \quad \sum_{i=1}^{n} a_{ni} X'_i \rightarrow \text{ a.s.}
\]

From an inequality \( e^x \leq 1 + x + \frac{1}{2} x^2 e^{|x|} \) for all \( x \in \mathbb{R} \), we have

\[
e^{\frac{t^2 a_{ni}}{2} X'_i} \leq 1 + \frac{1}{2} t^2 a_{ni}^2 E[X'_i^2 e^{t|X'_i|}] \\
\leq 1 + \frac{1}{2} t^2 a_{ni}^2 e^{t a_{ni} |b_i \psi(i)|} EX'_i^2 \\
\leq \exp\left(\frac{1}{2} t^2 a_{ni}^2 e^{t |b_i \psi(i)|} EX'_i^2\right)
\]

for any \( t > 0 \). Let \( u_n = \max_{1 \leq i \leq n} |a_{ni}| |b_i \psi(i)| \). Then it follows by (i), \( \psi(x) \uparrow \infty \), and \( b_n \downarrow 0 \) that \( u_n \rightarrow 0 \). From (ii), we obtain that

\[
\sum_{i=1}^{n} a_{ni}^2 EX'_i^2 \leq \sum_{i=1}^{n} a_{ni}^2 EX^2 I(|X| \leq \psi(i)) \quad (\because b_n \leq 1)
\]

\[
= \sum_{j=1}^{n} EX^2 I(\psi(j-1) < |X| \leq \psi(j)) \sum_{i=j}^{n} a_{ni}^2 \\
\leq \sum_{j=1}^{n} P(\psi(j-1) < |X| \leq \psi(j)) \psi^2(j) \sum_{i=j}^{n} a_{ni}^2 \\
\leq C \frac{1}{n^\alpha} \sum_{j=1}^{n} P(\psi(j-1) < |X| \leq \psi(j)) j \\
\leq C \frac{1}{n^\alpha} E[\phi(|X|)].
\]
Now, let $\epsilon > 0$ be given. By putting $t = 2 \log n/\epsilon$, we have that

$$P\left(\sum_{i=1}^{n} a_{ni}X'_i > \epsilon\right) \leq e^{-t\epsilon} E[e^{t\sum_{i=1}^{n} a_{ni}X'_i}]$$

$$\leq e^{-t\epsilon} \exp\left\{\frac{1}{2}t^2 \sum_{i=1}^{n} a_{ni}^2 E[X'_i^2]\right\}$$

$$\leq e^{-2 \log n} \exp\left\{\frac{2(\log n)^2}{\epsilon^2} \sum_{i=1}^{n} a_{ni}^2 E[X'_i^2]\right\}$$

$$\leq e^{-2 \log n} \exp\left\{C(\log n)^2 n^{-\alpha+2u_n/\epsilon}\right\}$$

$$\leq Cn^{-2}$$

for all sufficiently large $n$. Hence

$$\sum_{n=1}^{\infty} P\left(\sum_{i=1}^{n} a_{ni}X'_i > \epsilon\right) < \infty.$$ 

By the Borel-Cantelli lemma, we have

$$\lim_{n \to \infty} \sup_{n=1}^{\infty} \sum_{i=1}^{n} a_{ni}X'_i \leq 0 \text{ a.s.}$$

By replacing $X'_i$ by $-X'_i$ from the above statement, we obtain

$$\lim_{n \to \infty} \inf_{n=1}^{\infty} \sum_{i=1}^{n} a_{ni}X'_i \geq 0 \text{ a.s.}$$

Thus (6) is proved. \(\square\)

The following theorem shows that if the variance of $X$ exists, then the conditions of Theorem 1 can be replaced by more simple conditions. In particular, the additional condition (2) on $\psi$ (and $\phi$) is not necessary.

**Theorem 2.** Let $\{X_i, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $EX = 0, \text{Var}(X) < \infty$, and $E[\phi(|X|)] < \infty$. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants such that

(i) $\max_{1 \leq i \leq n} |a_{ni}| = O\left(\frac{1}{\psi(n)}\right)$,

(ii) $\sum_{i=1}^{n} a_{ni}^2 = O\left(\frac{1}{n^\alpha}\right)$ for some $\alpha > 0$.

Then $\sum_{i=1}^{n} a_{ni}X_i \to 0 \text{ a.s.}$
Proof. We first observe that
\[
E\left(\sum_{i=1}^{n} a_{ni}X_i\right)^2 = EX^2 \sum_{i=1}^{n} a_{ni}^2 \to 0
\]
as \(n \to \infty\). It follows that \(\sum_{i=1}^{n} a_{ni}X_i \to 0\) in probability. The rest of the proof is similar to that of Theorem 1 except that (7) is replaced by
\[
\sum_{i=1}^{n} a_{ni}^2 EX_i^2 \leq EX^2 \sum_{i=1}^{n} a_{ni}^2 \leq C\frac{1}{n^\alpha}.
\]

\[
\square
\]

Corollary 1. Let \(\{X, X_n, n \geq 1\}\) be a sequence of i.i.d. random variables satisfying \(E X = 0\) and \(E|X|^p < \infty\) for some \(p \geq 1\). Let \(\{a_{ni}, 1 \leq i \leq n, n \geq 1\}\) be an array of constants such that
\[
(i) \max_{1 \leq i \leq n} |a_{ni}| = O(1/n^{1/p}),
\]
\[
(ii) \sum_{i=1}^{n} a_{ni}^2 = \begin{cases} O(1/n^{2(p-1)+\alpha}) & \text{for some } \alpha > 0, & \text{if } 1 < p < 2, \\ O(1/n^\alpha) & \text{for some } \alpha > 0, & \text{if } p \geq 2. \end{cases}
\]
Then \(\sum_{i=1}^{n} a_{ni}X_i \to 0\) a.s.

Proof. When \(p = 1\), the result is the content of Theorem 5 in Choi and Sung [2]. Let \(\phi(x) = x^p(p > 1)\). Then \(\phi^{-1}(x) = \psi(x) = x^{1/p}\). When \(1 < p < 2\), \(\psi\) satisfies (2), and
\[
\max_{1 \leq j \leq n} \sum_{i=j}^{n} a_{ni}^2 \leq \max_{1 \leq j \leq n} \sum_{i=j}^{n} \frac{\psi^2(j)}{j} \sum_{i=1}^{n} a_{ni}^2 \leq C\frac{1}{n^\alpha}.
\]
So the result follows by Theorem 1. When \(p \geq 2\), the result follows by Theorem 2. \(\square\)

Remark 1. Li et al. [4] proved Corollary 1 when \(p > 1\).

In some cases, it is not easy to check the condition (ii) of Corollary 1. To solve this problem, we need the following lemma.

Lemma 2. Let \(p > 0\) and \(0 < r < s\). Let \(\{a_{ni}, 1 \leq i \leq n, n \geq 1\}\) be an array of constants satisfying \(\max_{1 \leq i \leq n} |a_{ni}| = O(1/n^{1/p})\). Then the following statements are equivalent.
\[
(i) \sum_{i=1}^{n} |a_{ni}|^r = O(1/n^{r/p-1+\alpha}) \text{ for some } \alpha > 0,
\]
\[
(ii) \sum_{i=1}^{n} |a_{ni}|^s = O(1/n^{s/p-1+\beta}) \text{ for some } \beta > 0.
\]
Proof. The implication (i) \(\implies\) (ii) follows by

\[
\sum_{i=1}^{n} |a_{ni}|^s \leq \max_{1 \leq i \leq n} |a_{ni}|^{s-r} \sum_{i=1}^{n} |a_{ni}|^r = O\left(\frac{1}{n^{s/p-1+\alpha}}\right).
\]

To prove the converse, we take \(t > 0\) such that \(\beta - t(s - r) > 0\). Define \(A = \{1 \leq i \leq n : |a_{ni}| \leq 1/n^{l+1/p}\}\) and \(B = \{1, \ldots, n\} \setminus A\). Then we have by (ii) that

\[
\sum_{i=1}^{n} |a_{ni}|^r = \sum_{i \in A} |a_{ni}|^r + \sum_{i \in B} |a_{ni}|^r \leq n \left(\frac{1}{n^{l+1/p}}\right)^r + (n^{l+1/p})^{s-r} \sum_{i \in B} |a_{ni}|^s = O\left(\frac{1}{n^{r/p-1+\min\{t\beta-t(s-r)\}}}\right).
\]

Thus, the converse is proved. \(\Box\)

From Corollary 1 and Lemma 2, we can obtain the following theorem.

**Theorem 3.** Let \(\{X, X_n, n \geq 1\}\) be a sequence of i.i.d. random variables with \(EX = 0\) and \(E|X|^p < \infty\) for some \(1 < p < 2\). Let \(\{a_{ni}, 1 \leq i \leq n, n \geq 1\}\) be an array of constants such that

1. \(\max_{1 \leq i \leq n} |a_{ni}| = O\left(1/n^{1/p}\right)\),
2. \(\sum_{i=1}^{n} |a_{ni}|^r = O\left(1/n^{r/p-1+\alpha}\right)\) for some \(r > 0\) and \(\alpha > 0\).

Then \(\sum_{i=1}^{n} a_{ni}X_i \rightarrow 0\) a.s.

**Remark 2.** When \(1 < p < 2\), Corollary 1 follows by Theorem 3 with \(r = 2\).

The following corollary is due to Cuzick [3].

**Corollary 2.** Let \(\{X, X_n, n \geq 1\}\) be a sequence of i.i.d. random variables with \(EX = 0\) and \(E|X|^p < \infty\) for some \(p > 1\). Let \(\{a_{ni}, 1 \leq i \leq n, n \geq 1\}\) be an array of constants such that

\[
\sum_{i=1}^{n} |a_{ni}|^q = O\left(\frac{1}{n^{q/p}}\right),
\]

where \(\frac{1}{p} + \frac{1}{q} = 1\). Then \(\sum_{i=1}^{n} a_{ni}X_i \rightarrow 0\) a.s.
Proof. By (8), \( \max_{1 \leq i \leq n} |a_{ni}|^q = O(1/n^{q/n}) \), which implies that \( \max_{1 \leq i \leq n} |a_{ni}| = O(1/n^{1/p}) \). So when \( 1 < p < 2 \), the result follows by Theorem 3 with \( r = q \) and \( \alpha = 1 \).

Now let \( p \geq 2 \). Since \( q \leq 2 \), it follows that

\[
\sum_{i=1}^{n} a_{ni}^2 \leq \left( \sum_{i=1}^{n} |a_{ni}|^q \right)^{2/q} = O\left( \frac{1}{n^{2/p}} \right).
\]

So the result follows by Corollary 1. \( \square \)

The following corollary is due to Bai and Cheng [1]

**Corollary 3.** Let \( 1 < p < 2 \) and \( \frac{1}{p} = \frac{1}{\alpha} + \frac{1}{\beta} \) for \( 1 < \alpha, \beta < \infty \). Let \( \{X, X_n, n \geq 1\} \) be a sequence of i.i.d. random variables with \( E[X] = 0 \) and \( E|X|^{\beta} < \infty \). Let \( \{b_{ni}, 1 \leq i \leq n, n \geq 1\} \) be an array of constants such that

\[
\sum_{i=1}^{n} |b_{ni}|^{\alpha} = O(n).
\]

Then \( \sum_{i=1}^{n} b_{ni}X_i/n^{1/p} \to 0 \) a.s.

**Proof.** Define \( a_{ni} = b_{ni}/n^{1/p} \) for \( 1 \leq i \leq n \) and \( n \geq 1 \). By (9), \( \max_{1 \leq i \leq n} |b_{ni}|^{\alpha} = O(n) \), which implies that

\[
\max_{1 \leq i \leq n} |a_{ni}| = O\left( \frac{n^{1/\alpha}}{n^{1/p}} \right) = O\left( \frac{1}{n^{1/\beta}} \right).
\]

We also have that \( \sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(1/n^{\alpha/\beta}) \). Thus, when \( 1 < \beta < 2 \), the result follows by Theorem 3.

We now let \( \beta \geq 2 \). If \( \alpha < 2 \), we have that

\[
\sum_{i=1}^{n} a_{ni}^2 \leq \frac{1}{n^{2/p}} \left( \sum_{i=1}^{n} |b_{ni}|^{\alpha/\alpha} \right)^{2/\alpha} = O\left( \frac{n^{2/\alpha}}{n^{2/p}} \right) = O\left( \frac{1}{n^{2/\beta}} \right).
\]

If \( \alpha > 2 \), we obtain that

\[
\sum_{i=1}^{n} a_{ni}^2 = \frac{1}{n^{2/p}} \sum_{i=1}^{n} b_{ni}^2 \leq \frac{1}{n^{2/p}} \left( n + \sum_{i=1}^{n} |b_{ni}|^{\alpha} \right) = O\left( \frac{1}{n^{2/\beta-1}} \right).
\]

Thus, when \( \beta \geq 2 \), the result follows by Corollary 1. \( \square \)
Strong laws for weighted sums of i.i.d. random variables (II)

References


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