(WEAK) IMPLICATIVE HYPER $K$-IDEALS

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ABSTRACT. In this note first we define the notions of weak implicative and implicative hyper $K$-ideals of a hyper $K$-algebra $H$. Then we state and prove some theorems which determine the relationship between these notions and (weak) hyper $K$-ideals. Also we give some relations between these notions and all types of positive implicative hyper $K$-ideals. Finally we classify the implicative hyper $K$-ideals of a hyper $K$-algebra of order 3.

1. Introduction

The hyperalgebraic structure theory was introduced by F. Marty [9] in 1934. Imai and Iseki [5] in 1966 introduced the notion of a BCK-algebra. Recently [2, 3, 12] Borzooei, Jun and Zahedi et.al. applied the hyperstructure to BCK-algebras and introduced the concept of hyper $K$-algebra which is a generalization of BCK-algebra. Now, in this note we define the notions of (weak) implicative hyper $K$-ideals, then we obtain some related results which have been mentioned in the abstract.

2. Preliminaries

DEFINITION 2.1. [2] Let $H$ be a nonempty set and “$\circ$” be a hyperoperation on $H$, that is “$\circ$” is a function from $H \times H$ to $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$. Then $H$ is called a hyper $K$-algebra if it contains a constant “0” and satisfies the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) < x \circ y$
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$
(HK3) $x < x$
(HK4) $x < y, y < x \Rightarrow x = y$

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(HK5) \(0 < x\)
for all \(x, y, z \in H\), where \(x < y\) is defined by \(0 \in x \circ y\) and for every \(A, B \subseteq H\), \(A < B\) is defined by \(\exists a \in A, \exists b \in B\) such that \(a < b\).

Note that if \(A, B \subseteq H\), then by \(A \circ B\) we mean the subset \(\bigcup_{a \in A, b \in B} a \circ b\) of \(H\).

**Example 2.2.** [2] Define the hyperoperation \(\circ\) on \(H = [0, +\infty)\) as follows:

\[
x \circ y = \begin{cases} 
[0, x] & \text{if } x \leq y \\
(0, y] & \text{if } x > y \\n\{x\} & \text{if } y = 0
\end{cases}
\]

for all \(x, y \in H\). Then \((H, \circ, 0)\) is a hyper \(K\)-algebra.

**Theorem 2.3.** [2] Let \((H, \circ, 0)\) be a hyper \(K\)-algebra. Then for all \(x, y, z \in H\) and for all nonempty subsets \(A, B\) and \(C\) of \(H\) the following hold:

(i) \(x \circ y < z \iff x \circ z < y\),

(ii) \((x \circ z) \circ (x \circ y) < y \circ z\),

(iii) \(x \circ (x \circ y) < y\),

(iv) \(x \circ y < x\),

(v) \(A \subseteq B\) implies \(A < B\),

(vi) \(x \in x \circ 0\),

(vii) \((A \circ C) \circ (A \circ B) < B \circ C\),

(viii) \((A \circ C) \circ (B \circ C) < A \circ B\),

(ix) \(A \circ B < C \iff A \circ C < B\).

**Definition 2.4.** [2] Let \(I\) be a nonempty subset of a hyper \(K\)-algebra \((H, \circ, 0)\) and \(0 \in I\). Then,

(i) \(I\) is called a **weak hyper \(K\)-ideal** of \(H\) if \(x \circ y \subseteq I\) and \(y \in I\) imply that \(x \in I\) for all \(x, y \in H\).

(ii) \(I\) is called a **hyper \(K\)-ideal** of \(H\) if \(x \circ y < I\) and \(y \in I\) imply that \(x \in I\) for all \(x, y \in H\).

**Theorem 2.5.** [2] Any hyper \(K\)-ideal of a hyper \(K\)-algebra \(H\), is a weak hyper \(K\)-ideal.

**Definition 2.6.** [1] Let \(I\) be a nonempty subset of a hyper \(K\)-algebra \((H, \circ, 0)\) such that \(0 \in I\). Then \(I\) is called a **positive implicative hyper \(K\)-ideal** of

(i) **type 1**, if for all \(x, y, z \in H\), \((x \circ y) \circ z \subseteq I\) and \(y \circ z \subseteq I\) imply that
(Weak) implicative hyper $K$-ideals

\[ x \circ z \subseteq I, \]

(ii) \textit{type 2}, if for all \(x, y, z \in H\), \((x \circ y) \circ z < I\) and \(y \circ z \subseteq I\) imply that \(x \circ z \subseteq I\),

(iii) \textit{type 3}, if for all \(x, y, z \in H\), \((x \circ y) \circ z < I\) and \(y \circ z < I\) imply that \(x \circ z \subseteq I\),

(iv) \textit{type 4}, if for all \(x, y, z \in H\), \((x \circ y) \circ z \subseteq I\) and \(y \circ z < I\) imply that \(x \circ z \subseteq I\),

(v) \textit{type 5}, if for all \(x, y, z \in H\), \((x \circ y) \circ z \subseteq I\) and \(y \circ z \subseteq I\) imply that \(x \circ z < I\),

(vi) \textit{type 6}, if for all \(x, y, z \in H\), \((x \circ y) \circ z < I\) and \(y \circ z < I\) imply that \(x \circ z < I\),

(vii) \textit{type 7}, if for all \(x, y, z \in H\), \((x \circ y) \circ z \subseteq I\) and \(y \circ z < I\) imply that \(x \circ z < I\),

(viii) \textit{type 8}, if for all \(x, y, z \in H\), \((x \circ y) \circ z < I\) and \(y \circ z \subseteq I\) imply that \(x \circ z < I\).

\textsc{Definition 2.7.} Let \(I\) be a nonempty subset of \(H\). Then we say that \(I\) satisfies the \textit{additive condition} if for all \(x, y \in H\), \(x < y\) and \(y \in I\) imply that \(x \in I\).

\textsc{Definition 2.8.} Let \(H\) be a hyper \(K\)-algebra. An element \(a \in H\) is called a \textit{left (resp. right) scalar} if \(|a \circ x| = 1\) (resp. \(|x \circ a| = 1\)) for all \(x \in H\). If \(a \in H\) is both left and right scalar, we say that \(a\) is an \textit{scalar element}.

\textsc{Definition 2.9.} We say that the hyper \(K\)-algebra \(H\) satisfies the \textit{transitive condition} if for all \(x, y, z \in H\), \(x < y\) and \(y < z\) imply that \(x < z\).

3. Some results on hyper \(K\)-ideals

From now on \(H\) is a hyper \(K\)-algebra, unless otherwise is stated.

\textsc{Proposition 3.1.} Let \(I\) be a hyper \(K\)-ideal of \(H\), and \(A, B \subseteq H\). If \(A \circ B < I\) and \(B \subseteq I\), then \(A < I\).
Proof. We have \( A \circ B = \bigcup_{a \in A, b \in B} a \circ b \) and \( A \circ B < I \). Thus there exist \( t \in a \circ b \) for some \( a \in A \), \( b \in B \) and \( s \in I \) such that \( t < s \). Hence \( a \circ b < I \). Since \( I \) is a hyper K-ideal and \( b \in I \) we conclude that \( a \in I \), thus \( A < I \).

Remark 3.2. (i) In the above proposition it is not necessary that \( A \subseteq I \). To show this, let \( H = \{0, 1, 2\} \). Then the following table shows a hyper K-algebra structure on \( H \).

<table>
<thead>
<tr>
<th>( \circ )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0, 1, 2</td>
<td>0, 1, 2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0, 1, 2</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0, 1, 2</td>
</tr>
</tbody>
</table>

Now, \( I = \{0, 1\} \) is a hyper K-ideal of \( H \), \( \{1, 2\} \circ \{0, 1\} = \{0, 1, 2\} < I \) and \( \{0, 1\} \subseteq I \), but \( \{1, 2\} \not\subseteq I \).

(ii) If in Proposition 3.1, we use \( B < I \) instead of \( B \subseteq I \), then the result does not hold. Because consider \( H = \{0, 1, 2\} \), then the following table shows a hyper K-algebra structure on \( H \).

<table>
<thead>
<tr>
<th>( \circ )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0, 1, 2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0, 1</td>
</tr>
</tbody>
</table>

Let \( I = \{0\} \), clearly \( I \) is a hyper K-ideal. We have \( \{1\} \circ \{0, 1, 2\} < I \) and \( \{0, 1, 2\} < I \), but \( \{1\} \not\in I \).

Lemma 3.3. Let \( I \) be a weak hyper K-ideal of \( H \). If for all \( A, B \subseteq H \), \( A \circ B \subseteq I \) and \( B \subseteq I \), then \( A \subseteq I \).

Proof. For all \( a \in A \), \( b \in B \) we have \( a \circ b \subseteq A \circ B \subseteq I \) and \( b \in I \). Since \( I \) is a weak hyper K-ideal, we get that \( a \in I \), thus \( A \subseteq I \).

Remark 3.4. In the above lemma the condition \( B \subseteq I \) can not be replaced by \( B < I \). Because let \( H = \{0, 1, 2\} \). Then the following table
(Weak) implicative hyper $K$-ideals

shows a hyper $K$-algebra structure on $H$.

\[
\begin{array}{c|ccc}
\circ & 0 & 1 & 2 \\
\hline
0 & \{0\} & \{0\} & \{0\} \\
1 & \{1\} & \{0, 1, 2\} & \{2\} \\
2 & \{2\} & \{0, 1, 2\} & \{0, 1\} \\
\end{array}
\]

Now, $I = \{0, 1\}$ is a weak hyper $K$-ideal of $H$, $2 \circ (1 \circ 2) \subseteq I$ and $1 \circ 2 < I$, while $\{2\} \not\subseteq I$.

**Definition 3.5.** We say that $H$ satisfies the **strong transitive condition** if for all $A, B, C \subseteq H$, $A < B$ and $B < C$ imply that $A < C$.

**Corollary 3.6.** Let $H$ satisfies the **strong transitive condition**. Then it satisfies the transitive condition.

**Proof.** It is easy.

The following example shows that the converse of the above corollary is not true in general. To show this let $H = \{0, 1, 2\}$. Then the following table shows a hyper $K$-algebra structure on $H$.

\[
\begin{array}{c|ccc}
\circ & 0 & 1 & 2 \\
\hline
0 & \{0\} & \{0\} & \{0\} \\
1 & \{1\} & \{0\} & \{1\} \\
2 & \{2\} & \{2\} & \{0, 1\} \\
\end{array}
\]

It is easy to check that $H$ satisfies the transitive condition, while it does not satisfy the strong transitive condition. Because $\{2\} < \{1, 2\}$ and $\{1, 2\} < \{1\}$, but $\{2\} \not< \{1\}$.

**Proposition 3.7.** Let $H$ satisfies the strong transitive condition. If $I$ is a hyper $K$-ideal of $H$ and $A, B \subseteq H$, $A \circ B < I$ and $B < I$, then $A < I$.

**Proof.** Let $A \circ B < I$. Then by Theorem 2.3 (ix) we have $A \circ I < B$, and $B < I$. Since $H$ satisfies the strong transitive condition we get that $A \circ I < I$. Now by Proposition 3.1 we have $A < I$. \qed
4. Implicative hyper $K$-ideal

Definition 4.1. A nonempty subset $I$ of $H$ is called a weak implicative hyper $K$-ideal if it satisfies:

(i) $0 \in I$

(ii) $(x \circ z) \circ (y \circ x) \subseteq I$ and $z \in I$ imply $x \in I$, for all $x, y, z \in H$.

Example 4.2. Let $H = \{0, 1, 2\}$. Then the following table shows a hyper $K$-algebra structure on $H$.

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0}</td>
<td>{0}</td>
<td>{0}</td>
</tr>
<tr>
<td>1</td>
<td>{1}</td>
<td>{0, 1}</td>
<td>{1}</td>
</tr>
<tr>
<td>2</td>
<td>{1, 2}</td>
<td>{0, 1}</td>
<td>{0, 1}</td>
</tr>
</tbody>
</table>

Then $I = \{0, 2\}$ is a weak implicative hyper $K$-ideal of $H$.

Definition 4.3. A nonempty subset $I$ of $H$ is called an implicative hyper $K$-ideal if it satisfies:

(i) $0 \in I$

(ii) $(x \circ z) \circ (y \circ x) < I$ and $z \in I$ imply $x \in I$, for all $x, y, z \in H$.

Example 4.4. Let $H = \{0, 1, 2\}$. The following table shows a hyper $K$-algebra structure on $H$.

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0}</td>
<td>{0}</td>
<td>{0}</td>
</tr>
<tr>
<td>1</td>
<td>{1}</td>
<td>{0, 2}</td>
<td>{1}</td>
</tr>
<tr>
<td>2</td>
<td>{2}</td>
<td>{0, 2}</td>
<td>{0, 2}</td>
</tr>
</tbody>
</table>

Then $I = \{0, 2\}$ is an implicative hyper $K$-ideal, while $I = \{0, 1\}$ is not an implicative hyper $K$-ideal, because $(2 \circ 0) \circ (1 \circ 2) < I$, and $0 \in I$ but $2 \notin I$.

Proposition 4.5. Each implicative hyper $K$-ideal of $H$ is a weak implicative.

Proof. Let $I$ be an implicative hyper $K$-ideal and $(x \circ z) \circ (y \circ x) \subseteq I$, $z \in I$. Then by Theorem 2.3 (v) we have $(x \circ z) \circ (y \circ x) < I$, thus $x \in I$. So $I$ is a weak implicative hyper $K$-ideal. $\square$

The following example shows that the converse of the above proposition is not correct in general. Consider $H = \{0, 1, 2\}$. The following
(Weak) implicative hyper $K$-ideals

The table shows a hyper $K$-algebra structure on $H$.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
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<tbody>
<tr>
<td>0</td>
<td>{0}</td>
<td>{0}</td>
<td>{0}</td>
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<tr>
<td>1</td>
<td>{1}</td>
<td>{0}</td>
<td>{0}</td>
</tr>
<tr>
<td>2</td>
<td>{2}</td>
<td>{1,2}</td>
<td>{0,1,2}</td>
</tr>
</tbody>
</table>

Then $I = \{0,1\}$ is a weak implicative hyper $K$-ideal, while it is not an implicative hyper $K$-ideal, because $(2 \circ 0) \circ (1 \circ 2) < I$, $0 \in I$ but $2 \not\in I$.

**Theorem 4.6.** Every implicative hyper $K$-ideal of $H$ is a hyper $K$-ideal.

**Proof.** Let $I$ be an implicative hyper $K$-ideal of $H$, $x \circ y < I$ and $y \in I$. Then there exist $t \in x \circ y$ and $z \in I$ such that $t < z$. We have $t \in t \circ 0 \subseteq (x \circ y) \circ (0 \circ x)$. Thus $(x \circ y) \circ (0 \circ x) < I$ and $y \in I$, therefore $x \in I$. □

The following example shows that the converse of the above theorem is not correct in general. Let $H = \{0,1,2\}$. Then the following table shows a hyper $K$-algebra structure on $H$.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
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<tbody>
<tr>
<td>0</td>
<td>{0}</td>
<td>{0}</td>
<td>{0}</td>
</tr>
<tr>
<td>1</td>
<td>{1}</td>
<td>{0,2}</td>
<td>{1}</td>
</tr>
<tr>
<td>2</td>
<td>{2}</td>
<td>{0,1,2}</td>
<td>{0,2}</td>
</tr>
</tbody>
</table>

Now, we can see that $I = \{0,2\}$ is a hyper $K$-ideal, while it is not an implicative hyper $K$-ideal, since $(1 \circ 0) \circ (2 \circ 1) = \{0,1,2\} < I$ and $0 \in I$, but $1 \not\in I$.

**Remark 4.7.** (i) In general, a weak implicative hyper $K$-ideal does not need to be a weak hyper $K$-ideal. To show this, consider $H = \{0,1,2\}$, then the following table shows a hyper $K$-algebra structure on $H$.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0,1,2}</td>
<td>{0,1,2}</td>
<td>{0,1,2}</td>
</tr>
<tr>
<td>1</td>
<td>{1}</td>
<td>{0,1,2}</td>
<td>{1,2}</td>
</tr>
<tr>
<td>2</td>
<td>{1,2}</td>
<td>{0,1}</td>
<td>{0,1,2}</td>
</tr>
</tbody>
</table>

We can check that $I = \{0,1\}$ is a weak implicative hyper $K$-ideal, while it is not a weak hyper $K$-ideal, because $2 \circ 1 \subseteq I$ and $1 \in I$, but $2 \not\in I$.

(ii) In general, a weak hyper $K$-ideal does not need to be a weak implicative hyper $K$-ideal. For this consider the hyper $K$-algebra $H$ of Remark 3.4. Then $I = \{0,1\}$ is a weak hyper $K$-ideal, while it is not a
weak implicative hyper $K$-ideal, since $(2 \circ 0) \circ (1 \circ 2) \subseteq I$, and $0 \in I$, but $2 \not\in I$.

**Theorem 4.8.** Let $I$ be a weak hyper $K$-ideal of $H$. Then the following statements hold:

(i) If for all $x, y, z \in H$, $x \circ (y \circ x) \subseteq I$ implies $x \in I$, then $I$ is a weak implicative hyper $K$-ideal.

(ii) Let $0 \in H$ be a right scalar element and $I$ be a weak implicative hyper $K$-ideal. Then for all $x, y \in H$, $x \circ (y \circ x) \subseteq I$, implies that $x \in I$.

**Proof.** (i) Let $I$ be a weak hyper $K$-ideal, $(x \circ z) \circ (y \circ x) \subseteq I$ and $z \in I$. Then $(x \circ (y \circ x)) \circ z \subseteq I$. By Lemma 3.3, we have $x \circ (y \circ x) \subseteq I$. Now by hypothesis $x \in I$. So $I$ is a weak implicative hyper $K$-ideal.

(ii) Let $I$ be a weak implicative hyper $K$-ideal, $x \circ (y \circ x) \subseteq I$ and $0 \in H$ is a right scalar element. We have $(x \circ 0) \circ (y \circ x) = x \circ (y \circ x) \subseteq I$ and $0 \in I$, thus $x \in I$. □

The following theorem shows that if we restrict to a hyper $K$-algebra of order 3, then we can omit the condition “$0 \in H$ be a right scalar element”, in the above theorem.

**Theorem 4.9.** Let $H = \{0, 1, 2\}$ be a hyper $K$-algebra of order 3, and $I$ be a proper hyper $K$-ideal of $H$. Then $I$ is a weak implicative hyper $K$-ideal if and only if for all $x, y \in H$, $x \circ (y \circ x) \subseteq I$ implies $x \in I$.

**Proof.** Let $I = \{0, 1\}$ be a proper weak hyper $K$-ideal and also a weak implicative hyper $K$-ideal of $H$. If $x \circ (y \circ x) \subseteq I$, for arbitrary elements $x, y \in H$, then we show that $x \in I$. If $x = 0$ or 1, then it is done. So let $x = 2$, therefore

$$(1) \quad 2 \circ (y \circ 2) \subseteq I.$$

We know that $0 \notin 2 \circ 0$ and $2 \in 2 \circ 0$. Thus $2 \circ 0 = \{2\}$ or $2 \circ 0 = \{1, 2\}$. If $2 \circ 0 = \{2\}$, then $(2 \circ 0) \circ (y \circ 2) = 2 \circ (y \circ 2) \subseteq I$, by (1). Since $0 \in I$ and $I$ is a weak implicative hyper $K$-ideal, we get that $2 \in I$, which is a contradiction.

If $2 \circ 0 = \{1, 2\}$, then we consider the following different cases.

(i) If $y = 0$, then $2 \in 2 \circ 0 \subseteq 2 \circ (0 \circ 2) \subseteq I$, by (1), which is a contradiction.

(ii) If $y = 1$ and $1 \leq 2$, then $0 \in 1 \circ 2$. Thus $2 \in 2 \circ 0 \subseteq 2 \circ (1 \circ 2) \subseteq I$, by (1). Which is a contradiction.
If \( y = 1 \) and \( 1 \not\preceq 2 \), then \( 1 \circ 2 = \{1\} \) or \( \{1, 2\} \) or \( \{2\} \). So we must discuss on the above different cases:

(a) If \( 1 \circ 2 = \{1\} \), then \( 2 \circ 1 = 2 \circ (1 \circ 2) \subseteq I \), by (1). Since \( 1 \in I \) and \( I \) is a weak hyper \( K \)-ideal, we conclude that \( 2 \in I \), which is a contradiction.

(b) If \( 1 \circ 2 = \{1, 2\} \), then \( (2 \circ 1) \cup (2 \circ 2) = 2 \circ \{1, 2\} = 2 \circ (1 \circ 2) \subseteq I \), by (1). Hence \( 2 \circ 1 \subseteq I \). Therefore \( 2 \in I \), which is a contradiction.

(c) If \( 1 \circ 2 = \{2\} \), then we claim that \( 1 \circ 0 = \{1\} \). Suppose \( 1 \circ 0 \neq \{1\} \).

Since \( 1 \in 1 \circ 0 \) and \( 0 \not\in 1 \circ 0 \), we must have \( 1 \circ 0 = \{1, 2\} \). Then \( 0 \in 2 \circ 2 \subseteq \{2\} \cup 2 \circ 2 = 1 \circ 2 \cup 2 \circ 2 = \{1, 2\} \circ 2 = (1 \circ 0) \circ 2 \), so

\[
0 \in (1 \circ 0) \circ 2.
\]

On the other hand \( (1 \circ 0) \circ 2 = (1 \circ 2) \circ 0 = 2 \circ 0 \). Since \( 0 \not\in 2 \circ 0 \), we get that \( 0 \not\in (1 \circ 0) \circ 2 \), which is a contradiction by (2). Thus we must have \( 1 \circ 0 = \{1\} \). Therefore

\[
(1 \circ 2) \circ 0 = 2 \circ 0 = \{1, 2\}
\]

and

\[
(1 \circ 0) \circ 2 = 1 \circ 2 = \{2\}.
\]

Since \( (1 \circ 2) \circ 0 = (1 \circ 0) \circ 2 \). So (3), (4) given a contradiction. Thus \( y = 1 \) does not happen.

(iii) Let \( y = 2 \). Then \( 2 \in 2 \circ 0 \subseteq 2 \circ (2 \circ 2) \subseteq I \), by (1). Which is a contradiction. Therefore the above argument shows that \( x \neq 2 \), i.e., \( x \in I \). Finally by considering Theorem 4.8, the proof of the converse is obvious. \( \square \)

**Definition 4.10.** [11] Let \( H = \{0, 1, 2\} \) be a hyper \( K \)-algebra of order 3. We say that \( H \) satisfies the simple condition if \( 1 \not\preceq 2 \) and \( 2 \not\preceq 1 \).

**Theorem 4.11.** Let \( H = \{0, 1, 2\} \) be a hyper \( K \)-algebra of order 3, that satisfies the simple condition, and let \( \{0\} \not\subseteq I \subseteq H \). Then \( I \) is a weak hyper \( K \)-ideal of \( H \) if and only if \( I \) is a weak implicative \( K \)-ideal of \( H \).

**Proof.** Let \( I \) be a weak hyper \( K \)-ideal of \( H \). By hypothesis we have \( I = \{0, 1\} \) or \( \{0, 2\} \). Let \( I = \{0, 1\} \). By Theorem 4.9 it is enough to show that if \( x \circ (y \circ x) \subseteq I \), for any two arbitrary elements \( x, y \) of \( H \), then \( x \in I \). So let \( x \circ (y \circ x) \subseteq I \). If \( x = 0 \) or \( 1 \), then it is done. Thus let \( x = 2 \). Consider the following different cases:

**Case (1).** If \( y = 0 \), then \( 2 \in 2 \circ 0 \subseteq 2 \circ (2 \circ 0) \subseteq I \) and hence \( 2 \in I \), which is a contradiction.
Case (2). If \( y = 1 \), since \( H \) satisfies the simple condition then \( 1 \not\subset 2 \) and \( 0 \not\subset 1 \). Hence \( 1 \circ 2 = \{1\}, \{2\} \) or \( \{1,2\} \).

(i) If \( 1 \circ 2 = \{1\} \), then \( 2 \circ 1 = 2 \circ (1 \circ 2) \subseteq I \). Since \( I \) is a weak hyper \( K \)-ideal and \( 1 \in I \) then we get that \( 2 \in I \), which is a contradiction.

(ii) The case \( 1 \circ 2 = \{2\} \) does not happen, by Theorem 3.17 of [11].

(iii) If \( 1 \circ 2 = \{1,2\} \), then \( (2 \circ 1) \cup (2 \circ 2) = 2 \circ \{1,2\} = 2 \circ (1 \circ 2) \subseteq I \).

Thus \( 2 \circ 1 \subseteq I \). Now \( 1 \in I \) implies that \( 2 \in I \), which is a contradiction.  

Case (3). If \( y = 2 \), then \( 2 \in 2 \circ 0 \subseteq 2 \circ (2 \circ 2) \subseteq I \). Hence \( 2 \in I \), which is a contradiction.

Thus \( x \neq 2 \). Hence \( x \) is in \( I \). Note that the proof of the case \( I = \{0,2\} \) is similar as above.

Conversely, let \( I \) be a weak implicative hyper \( K \)-ideal of \( H \). Without loss of generality we assume that \( I = \{0,1\} \). Let \( x \circ y \subseteq I \) and \( y \in I \). If \( x = 0 \) or \( 1 \), then \( x \in I \). So let \( x = 2 \). We consider the following cases:

Case (1). The case \( y = 0 \) does not happen, because \( 2 \not\subseteq I \).

Case (2). If \( y = 1 \), since \( 2 \not\subset 1 \), then \( 0 \not\subset 2 \circ 1 \). Hence \( 2 \circ 1 = \{1\}, \{2\} \) or \( \{1,2\} \). Since \( H \) satisfies the simple condition, then by Theorem 3.17 of [11] \( 2 \circ 1 \neq \{1\} \). So the cases \( 2 \circ 1 = \{2\} \) or \( \{1,2\} \) do not happen, since \( 2 \circ 1 \not\subseteq I \).

Case (3). The case of \( y = 2 \) does not happen, because \( 2 \not\in I \).

Consequently \( x \neq 2 \), hence \( x \circ y \subseteq I \) and \( y \in I \) imply that \( x \in I \), for all \( x, y \in H \). This shows that \( I \) is a weak implicative hyper \( K \)-ideal. Note that the proof of the case \( I = \{0,2\} \) is similar as above. \( \square \)

**Theorem 4.12.** Let \( I \) be a hyper \( K \)-ideal of \( H \). Then \( I \) is an implicative hyper \( K \)-ideal if and only if

\[
(5) \quad x \circ (y \circ x) < I \text{ implies that } x \in I, \text{ for any } x, y \in H.
\]

**Proof.** Let \( I \) satisfies in (5) and \( (x \circ x) \circ (y \circ x) < I, z \in I \). Then by Proposition 3.1 we have \( x \circ (y \circ x) < I \). So by (5) we get that \( x \in I \). Therefore \( I \) is an implicative hyper \( K \)-ideal.

Conversely, let \( I \) be an implicative hyper \( K \)-ideal, and \( x \circ (y \circ x) < I \). Since \( x \circ (y \circ x) \subseteq (x \circ 0) \circ (y \circ x) \), we conclude that \( (x \circ 0) \circ (y \circ x) < I \). Thus \( 0 \in I \) implies that \( x \in I \). \( \square \)

**Theorem 4.13.** Let \( H \) satisfies the strong transitive condition. If \( I \) is an implicative hyper \( K \)-ideal of \( H \), then \( I \) is a positive implicative hyper \( K \)-ideal of type 1-8.

**Proof.** By considering Theorem 3.5 of [1], it is enough to show that \( I \) is a positive implicative hyper \( K \)-ideal of type 3. Let \((x \circ y) \circ z < I\),
and \( y \circ z < I \), we must show that \( x \circ z \subseteq I \). Let \( t \in x \circ z \). Then by (HK1) we have
\[
(t \circ z) \circ (y \circ z) < t \circ y \subseteq (x \circ z) \circ y = (x \circ y) \circ z < I.
\]
Since \( H \) satisfies the strong transitive condition, then \( (t \circ z) \circ (y \circ z) < I \).
Since \( y \circ z < I \) by Proposition 3.7, we conclude that \( t \circ z < I \). Now, by Theorem 2.3 (ii) we have \( (x \circ z) \circ (x \circ t) < t \circ z \), thus by hypothesis we get that \( (x \circ z) \circ (x \circ t) < I \). Since \( (x \circ z) \circ (x \circ t) \subseteq (x \circ z) \circ (x \circ (x \circ z)) \), we conclude that \( (x \circ z) \circ (x \circ (x \circ z)) < I \). But for all \( t \in x \circ z \) we have \( t \circ (x \circ t) \subseteq (x \circ z) \circ (x \circ (x \circ z)) \), so by hypothesis \( t \circ (x \circ t) < I \). Thus by Theorem 4.12, \( t \in I \), and hence \( x \circ z \subseteq I \).

\[\square\]

Remark 4.14. In Theorem 4.13 the condition strong transitivity of \( H \) is essential. Because, let \( H = \{0, 1, 2\} \). Then the following table shows a hyper \( K \)-algebra structure on \( H \).

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0}</td>
<td>{0}</td>
<td>{0}</td>
</tr>
<tr>
<td>1</td>
<td>{1}</td>
<td>{0}</td>
<td>{1}</td>
</tr>
<tr>
<td>2</td>
<td>{1,2}</td>
<td>{0}</td>
<td>{0,1}</td>
</tr>
</tbody>
</table>

Now \( H \) does not satisfy the strong transitive condition, because \( \{1\} < \{1,2\} < \{2\} \) and \( \{1\} \not< \{2\} \). Clearly \( I = \{0, 2\} \) is an implicatve hyper \( K \)-ideal of \( H \), but it is not a positive implicatve hyper \( K \)-ideal of type 2 or 3. Because \( 2 \circ 0 \circ 0 < I \) and \( 0 \circ 0 \subseteq I \), but \( 2 \circ 0 \not\subseteq I \).

Theorem 4.15. Let \( H = \{0, 1, 2\} \) be a hyper \( K \)-algebra of order 3, that satisfies the simple condition, and \( \{0\} \neq I \subset H \). Then \( I \) is an implicatve hyper \( K \)-ideal if and only if \( I \) is a positive implicatve hyper \( K \)-ideal of type 3.

Proof. Let \( I \) be a positive implicatve hyper \( K \)-ideal of type 3. Without loss of generality assume that \( I = \{0, 1\} \). Let \( (x \circ z) \circ (y \circ z) < I \) and \( z \in I \), we show that \( x \in I \). By Theorems 17.3 and 19.3 of [11], we have \( 2 \circ 1 = \{2\}, 2 \circ 0 = \{2\}, 1 \circ 2 = \{1\}, 1 \circ 0 = \{1\}, x \circ y \neq \{0, 2\} \) and \( x \circ y \neq \{0, 1, 2\} \) for all \( x, y \in H \). Thus
\[(6) \quad x \circ y \subseteq \{0, 1\}, \text{ for all } x, y \in H.\]

Now, let \( x = 2 \). In the following we show that, this case is impossible. To this end consider three different cases:

(i) Let \( z = 0 \). We consider the following subcases:
(a) If \( y = 0 \), then by (6) we have \( 0 \circ 2 \subseteq \{0,1\} \). Hence \( (2 \circ 0) \circ (0 \circ 2) = 2 \circ (0 \circ 2) \subseteq 2 \circ \{0,1\} = (2 \circ 0) \cup (2 \circ 1) = \{2\} \cup \{2\} = \{2\} \). So by hypothesis \( (2 \circ 0) \circ (0 \circ 2) < \{0,1\} \), therefore \( \{2\} < \{0,1\} \), which implies that \( 2 < 1 \). Thus we obtain a contradiction, because \( H \) satisfies the simple condition.

(b) If \( y = 1 \), then \( (2 \circ 0) \circ (1 \circ 2) = \{2\} \circ \{1\} = \{2\} \). By hypothesis \( \{2\} < \{0,1\} \). Therefore \( 2 < 1 \), which is a contradiction.

(c) If \( y = 2 \), then by (6), \( 2 \circ 2 \subseteq \{0,1\} \). So \( (2 \circ 0) \circ (2 \circ 2) = 2 \circ (2 \circ 2) \subseteq 2 \circ \{0,1\} = (2 \circ 0) \cup (2 \circ 1) = \{2\} \cup \{2\} = \{2\} \). By hypothesis \( \{2\} < \{0,1\} \), hence \( 2 < 1 \), which is a contradiction.

(ii) Let \( z = 1 \). Then a similar argument as the case of (i), gives a contradiction.

Note that by hypothesis \( z \in I \) so \( z \neq 2 \). Hence \( x = 2 \) is impossible i.e., \( x \neq 2 \). Thus \( x \in I \), which implies that \( I \) is an implicitive hyper-ideal. Conversely, let \( I \) be an implicitive hyper-ideal. Without loss of generality assume that \( I = \{0,1\} \). Let \( (x \circ y) \circ z < I \) and \( y \circ z < I \) for \( x, y, z \in H \), we must show that \( x \circ z \subseteq I \). By Theorem 3.17 [11], we know that \( 1 \circ 0 = \{1\}, 2 \circ 0 = \{2\}, 1 \circ 2 \neq \{2\} \) and \( 2 \circ 1 \neq \{1\} \). Now we show that

(I) \( 1 \circ 2 = \{1\} \)

(II) \( 2 \circ 1 = \{2\} \)

(III) \( x \circ y \neq \{0,2\}, x \circ y \neq \{0,1,2\} \); for all \( x, y \in H \).

(I): Let \( 1 \circ 2 \neq \{1\} \). Then \( 1 \neq 2 \), since \( H \) is simple. Thus \( 0 \neq 1 \circ 2 \), therefore we must have \( 1 \circ 2 = \{1,2\} \). But

\[
0 \in 2 \circ 2 \subseteq (2 \circ 1) \cup (2 \circ 2) = 2 \circ \{1,2\} = 2 \circ (1 \circ 2) = (2 \circ 0) \circ (1 \circ 2).
\]

So \( (2 \circ 0) \circ (1 \circ 2) < I \). Since \( 0 \in I \), we conclude that \( 2 \in I \), which is a contradiction. Hence \( 1 \circ 2 = \{1\} \).

(II): Suppose \( 2 \circ 1 \neq \{2\} \). Since \( 2 \neq 1, 0 \neq 2 \circ 1 \) and since \( 2 \circ 1 \neq \{1\} \), thus we must have \( 2 \circ 1 = \{2,1\} \). Now \( \{1,2\} = 2 \circ 1 = (2 \circ 0) \circ (1 \circ 2) \), by (I), that is \( (2 \circ 0) \circ (1 \circ 2) < I \). Since \( 0 \in I \) and \( I \) is implicitative we get that \( 2 \in I \) which is a contradiction. Hence \( 2 \circ 1 = \{2\} \).

(III): By considering (I) and (II), it remains to show that none of \( 0 \circ 0, 0 \circ 1, 1 \circ 1 \) and \( 2 \circ 2 \) are equal to \( \{0,2\} \) or \( \{0,1,2\} \). Clearly all of them contain \( 0 \), so we show that none of them contain \( 2 \).

(a) \( 2 \neq 2 \circ 2 \): Let \( 2 \in 2 \circ 2 \). Then by (II) we have \( 0 \in 2 \circ 2 \subseteq (2 \circ 2) = (2 \circ 1) \circ (2 \circ 2) \), hence \( (2 \circ 1) \circ (2 \circ 2) < I \). Since \( 1 \in I \), then \( 2 \in I \),
which is a contradiction. Therefore $2 \not\in 2 \circ 2$.

(b) The proof of $2 \not\in 0 \circ 2$ is similar as (a).

(c) $2 \not\in 0 \circ 1$: Let $2 \in 0 \circ 1$. Then by (HK3) and (HK2) we have $2 \in 0 \circ 1 \subseteq (2 \circ 2) \circ 1 = (2 \circ 1) \circ 2$. By (I), $(2 \circ 1) \circ 2 = 2 \circ 2$, so $2 \in 2 \circ 2$, which is in contradiction with (a).

(d) $2 \not\in 1 \circ 1$: Let $2 \in 1 \circ 1$. Then by (HK2) and (I) we have

\[(7) \quad 2 \in 1 \circ 1 = (1 \circ 2) \circ 1 = (1 \circ 1) \circ 2.
\]

Since $0 \in 1 \circ 1$ and $2 \in 1 \circ 1$, then $1 \circ 1$ contains $\{0, 2\}$. Thus $1 \circ 1 = \{0, 2\}$ or $\{0, 1, 2\}$. If $1 \circ 1 = \{0, 1, 2\}$, then by (7), (I) and (II) we have

\[2 \in (1 \circ 1) \circ 2 = \{0, 1, 2\} \circ 2 = (0 \circ 2) \cup (1 \circ 2) \cup (2 \circ 2) \subseteq \{0, 1\},\]

which is a contradiction. If $1 \circ 1 = \{0, 2\}$, then similarly we get a contradiction.

(e) $2 \not\in 0 \circ 0$: Let $2 \in 0 \circ 0$. Then by (HK2), (HK3) and (d) we have $2 \in 0 \circ 0 \subseteq (1 \circ 1) \circ 0 = (1 \circ 0) \circ 1 = 1 \circ 1 \subseteq \{0, 1\}$, which is a contradiction. Thus (III) is proved.

Now, (III) imposes that $(H, \circ, 0)$ must have the following hyper structure table:

<table>
<thead>
<tr>
<th>\circ</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0} or {0, 1}</td>
<td>{0} or {0, 1}</td>
<td>{0} or {0, 1}</td>
</tr>
<tr>
<td>1</td>
<td>{1}</td>
<td>{0} or {0, 1}</td>
<td>{1}</td>
</tr>
<tr>
<td>2</td>
<td>{2}</td>
<td>{2}</td>
<td>{0} or {0, 1}</td>
</tr>
</tbody>
</table>

As we see, in the above table except the cases $2 \circ 0 = \{2\}$ and $2 \circ 1 = \{2\}$, the other possible cases of $x \circ z$ are subsets of $I$. That is $x \circ z \subseteq I$. Now we prove that if $x = 2$, $z = 0$ or $x = 2$, $z = 1$, then $(x \circ y) \circ z \not\in I$, or $y \circ z \not\in I$. Therefore the proof will be completed.

First let $x = 2$ and $z = 0$. If $y = 0$, then we have

\[2 = 2 \circ 0 = (2 \circ 0) \circ 0 < I = \{0, 1\},\]

which is a contradiction. Similarly for $y = 1$ or $y = 2$ we obtain a contradiction.

Now, if $x = 2$ and $z = 1$, then by a similar argument as above we give a contradiction. Hence we proved that if $(x \circ y) \circ z < I$, and $y \circ z < I$, then $x \circ z \subseteq I$, for all $x, y, z \in H$. Thus $I$ is a positive implicative hyper $K$-ideal of type 3. \qed
COROLLARY 4.16. Let \( H = \{0, 1, 2\} \) be a hyper \( K \)-algebra of order 3, that satisfies the simple condition and \( I \) be an implicatice hyper \( K \)-ideal of \( H \). Then \( I \) is a positive hyper \( K \)-ideal of types 1-8.

Proof. The proof follows from Theorem 4.15 and Theorem 3.5 of [1].

Theorem 4.17. There are 12 non-isomorphic hyper \( K \)-algebras of order 3, with simple condition such that they have at least one proper implicatice hyper \( K \)-ideal.


Theorem 4.18. Let \( I \) be an implicatice hyper \( K \)-ideal of \( H \), that satisfies the strong transitive condition, \( A \) be a hyper \( K \)-ideal of \( H \) that contains \( I \). Then \( A \) is an implicatice hyper \( K \)-ideal of \( H \).

Proof. Let \( x \circ (y \circ x) < A \), we prove that \( x \in A \). By Theorem 2.3 (ix) we have \( x \circ A < y \circ x \). Since \( I \subseteq A \), we get that \( x \circ I < x \circ A \), hence \( x \circ I < y \circ x \). Thus \( x \circ (y \circ x) < I \), by Theorem 2.3 (ix). Since \( I \) is an implicatice hyper \( K \)-ideal we get that \( x \in I \), so \( x \in A \). Therefore by Theorem 4.12 \( A \) is an implicatice hyper \( K \)-ideal of \( H \).

Theorem 4.19. If \( \{I_i | i \in \Lambda\} \) is a family of (weak) implicatice hyper \( K \)-ideals, then \( \bigcap_{i \in \Lambda} I_i \) is also a (weak) implicatice hyper \( K \)-ideal.

Proof. The proof is straightforward.

Theorem 4.20. Let \( (H, \ast, 0) \) be a \( BCK \)-algebra and \( I \) be a nonempty subset of \( H \) which satisfies the additive condition. If we consider the hyperoperation \( x \circ y = \{x \ast y\} \) on \( H \), then \( I \) is a weak implicatice hyper \( K \)-ideal of \( H \) if and only if \( I \) is an implicatice hyper \( K \)-ideal of \( H \).

Proof. The proof is easy.

Open Problem. Under what suitable condition each weak implicatice hyper \( K \)-ideal is an implicatice hyper \( K \)-ideal?

References


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