STRONGLY $\Pi$-REGULAR MORITA CONTEXTS

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Abstract. In this paper, we show that if the ring of a Morita context $(A, B, M, N, \psi, \phi)$ with zero pairings is a strongly $\pi$-regular ring of bounded index if and only if so are $A$ and $B$. Furthermore, we extend this result to the ring of a Morita context over quasi-duo strongly $\pi$-regular rings.

Let $R$ be an associative ring with identity. We say that $R$ is strongly $\pi$-regular if for each $x \in R$ there exists a positive integer $m = m(a)$, depending on $a$, such that $a^nR = a^{n+1}R$. This concept is left-right symmetric and is equivalent to the condition that every cyclic left or right $R$-module is co-hopfian. It is well known that every strongly $\pi$-regular ring has stable range one and every element in a strongly $\pi$-regular ring is either a two-sided zero divisor or a unit. Many authors have studied such rings such as [1], [3]-[6] and [9]-[12].

Recall that a Morita context denoted by $(A, B, M, N, \psi, \phi)$ consists of two rings $A, B$, two bimodules $A_{B, B} M_A$ and a pair of bimodule homomorphisms (called pairings) $\psi : N \otimes_B M \to A$ and $\phi : M \otimes_A N \to B$ which satisfy the following associativity:

$\psi(n \otimes m)n' = n\phi(m \otimes n)n'$, $\phi(m \otimes n)m' = m\psi(n \otimes m').$

These conditions insure that the set $T$ of generalized matrices

$T = \left\{ \begin{pmatrix} a & n \\ m & b \end{pmatrix} \mid a \in A, b \in B, m \in M, n \in N \right\}$

forms a ring, called the ring of the context $(A, B, M, N, \psi, \phi)$. In [8], A. Haghany and K. Varadarajan studied Morita contexts with all $N = 0$ (i.e., formal triangular rings). In [6], A. Haghany investigated hopficity and co-hopficity for Morita contexts with zero pairings. He showed that if $T$ is the ring of a Morita context $(A, B, M, N, \psi, \phi)$ with zero pairings then $T$ is strongly $\pi$-regular provided that $A$ and $B$ are strongly $\pi$-regular, and that zero divisors in $A$ and $B$ annihilate $M$ and $N$.

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Following a new route, we now investigate the conditions under which the ring of a Morita context \((A, B, M, N, \psi, \phi)\) with zero pairings is strongly \(\pi\)-regular. We prove that \(T\) is a strongly \(\pi\)-regular ring of bounded index if and only if so are \(A\) and \(B\). Furthermore, we extend this result to right (left) quasi-duo strongly \(\pi\)-regular rings.

Throughout, rings are associative with identity. \(U(R)\) denotes the set of units of \(R\) and \(J(R)\) denotes the Jacobson radical of \(R\). We always use \(T\) to denote the ring of a Morita context \((A, B, M, N, \psi, \phi)\).

**Lemma 1.** Let \(T\) be the ring of a Morita context \((A, B, M, N, \psi, \phi)\) with zero pairings. Then \(T/J(T) \cong A/J(A) \oplus B/J(B)\).

**Proof.** One easily checks that \(J(T) = \begin{pmatrix} J(A) & N \\ M & J(B) \end{pmatrix}\). We construct a map \(\theta : T \to \begin{pmatrix} A/J(A) & 0 \\ 0 & B/J(B) \end{pmatrix}\) given by \(\begin{pmatrix} a & n \\ m & b \end{pmatrix}\) for any \(\begin{pmatrix} a & n \\ m & b \end{pmatrix} \in T\). Because of zero pairings, we claim that \(\theta\) is a ring epimorphism. Therefore \(T/J(T) \cong T/\text{Ker}(\theta) \cong A/J(A) \oplus B/J(B)\), as asserted. \(\square\)

Recall that a ring \(R\) is of bounded index provided that there exists some positive integer \(n\) such that \(a^n = 0\) for all nilpotent \(a \in R\). It is well known that every regular ring (or weakly \(P\)-exchange ring) of bounded index is strongly \(\pi\)-regular ring. For the Morita contexts over strongly \(\pi\)-regular rings of bounded index, we derive the following.

**Theorem 2.** Let \(T\) be the ring of a Morita context \((A, B, M, N, \psi, \phi)\) with zero pairings. Then \(T\) is strongly \(\pi\)-regular of bounded index if and only if so are \(A\) and \(B\).

**Proof.** Suppose that \(T\) is a strongly \(\pi\)-regular ring of bounded index. Set \(e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\). It is easy to check that \(A \cong eTe\) is also a strongly \(\pi\)-regular ring of bounded index. Likewise, \(B\) is a strongly \(\pi\)-regular ring of bounded index, as required.

Conversely, assume now that \(A\) and \(B\) are both strongly \(\pi\)-regular rings of bounded index. Then \(A/J(A)\) and \(B/J(B)\) are also strongly \(\pi\)-regular. It follows by Lemma 1 that \(T/J(T)\) is strongly \(\pi\)-regular.

Using Lemma 1 again, we have \(J(T) \cong \begin{pmatrix} J(A) & N \\ M & J(B) \end{pmatrix}\). Assume that the bounded indices of \(A\) and \(B\) are \(s\) and \(t\) respectively. Since \(A\) and \(B\) are strongly \(\pi\)-regular, \(J(A)\) and \(J(B)\) are nil. Given any
\[
\begin{pmatrix} a & n \\ m & b \end{pmatrix} \in J(T), \text{ then } a^{s+t} = 0 \text{ and } b^{s+t} = 0. \text{ So there exist } m_1, m_2 \in M \text{ and } n_1, n_2 \in N \text{ such that }
\]
\[
\begin{pmatrix} a & n \\ m & b \end{pmatrix}^{2(s+t)} = \begin{pmatrix} a & n \\ m & b \end{pmatrix}^{s+t} \begin{pmatrix} a & n \\ m & b \end{pmatrix}^{s+t} = \begin{pmatrix} a^{s+t} & n_1 \\ m_1 & b^{s+t} \end{pmatrix} \begin{pmatrix} a^{s+t} & n_2 \\ m_2 & b^{s+t} \end{pmatrix} = \begin{pmatrix} 0 & n_1 \\ m_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & n_2 \\ m_2 & 0 \end{pmatrix} = 0.
\]
Therefore \(J(T)\) is a nil ideal of bounded index.

Assume that the bounded index of \(J(T)\) is \(k\). Given any \(x \in T\), we have a positive integer \(l\) such that
\[
(x + J(T))^l \in (T/J(T))
\]
\[
= (x + J(T))^{l+1} \in (T/J(T))
\]
\[
= (x + J(T))^{kl+1} \in (T/J(T)).
\]
So we have \(y + J(T) \in T/J(T)\) such that \((x + J(T))^l = (x + J(T))^{kl+1} (y + J(T))\). Hence \(x^l - x^{kl+1} y \in J(T)\). Therefore \((x^l - x^{kl+1} y)^k = 0\). Thus we can find some \(z \in T\) such that \(x^{kl} = x^{kl+1} z\). That is, \(T\) is a strongly \(\pi\)-regular ring.

Suppose that \(\begin{pmatrix} a & n \\ m & b \end{pmatrix}^p = 0\) for some \(p \geq 1\). One easily checks that \(\begin{pmatrix} a & n \\ m & b \end{pmatrix}^p = \begin{pmatrix} a^p & n_3 \\ m_3 & b^p \end{pmatrix}\) for some \(m_3 \in M, n_3 \in N\). So \(a^p = 0\) in \(A\) and \(b^p = 0\) in \(B\). Hence \(a^s = 0\) and \(b^t = 0\). Analogously to the consideration above, we claim that \(\begin{pmatrix} a & n \\ m & b \end{pmatrix}^{2(s+t)} = 0\). Therefore \(T\) is of bounded index, as asserted. \(\square\)

Let \(A = B = k[x]/(x^2) = \{a + bt \mid a, b \in k, t^2 = 0\}\), where \(k\) is a field of characteristic 2. Take \(M = N = k\) made into an \(A\)-module by \(\alpha * (a + bt) = \alpha a\) with \(\alpha, a, b \in k\). By [6, p.488], we know that \(A\) and \(B\) are both strongly \(\pi\)-regular rings. Assume that \((a + bt)^n = 0\) in \(A\). Then \((a + bt)^{2n} = 0\), hence \(a^{2n} = ((a + bt)^2)^n = 0\). So \(a = 0\).
Therefore \((a + bt)^2 = a^2 = 0\). That is, \(A = B\) is a strongly \(\pi\)-regular ring of bounded index \(2\). Then with the zero pairings, all the conditions in Theorem 2 hold.

**Corollary 3.** Let \(T\) be the ring of a Morita context \((A, B, M, N, \psi, \phi)\) with zero pairings. If \(A\) and \(B\) are regular rings of bounded index, then \(T\) is a strongly \(\pi\)-regular ring.

**Proof.** Since \(A\) and \(B\) are regular rings of bounded index, they are strongly \(\pi\)-regular. Hence we get the result by Theorem 2.

**Corollary 4.** A ring \(R\) is a strongly \(\pi\)-regular ring of bounded index if and only if so is the ring of all \(n \times n\) lower triangular matrices over \(R\).

**Proof.** Suppose that the ring \(T\) of all \(n \times n\) lower triangular matrices over \(R\) is a strongly \(\pi\)-regular ring of bounded index. Then we have an idempotent \(e \in T\) such that \(R \cong eTe\). Thus we easily check that \(R\) is a strongly \(\pi\)-regular ring of bounded index as well.

Conversely, assume that \(R\) is a strongly \(\pi\)-regular ring of bounded index. Applying Theorem 2, the triangular matrix ring \(
\begin{pmatrix}
A & 0 \\
M & B
\end{pmatrix}
\) is a strongly \(\pi\)-regular rings of bounded index if and only if so are \(A\) and \(B\). By induction, we obtain the result.

Similarly, we deduce that a ring \(R\) is a strongly \(\pi\)-regular ring of bounded index if and only if so is the ring of all \(n \times n\) upper triangular matrices over \(R\).

Let \(A_1, A_2, A_3\) be rings with identities, and let \(M_{21}, M_{31}, M_{32}\) be \((A_2, A_1)-, (A_3, A_1)-, (A_3, A_2)\)-bimodules, respectively. Let

\[
\phi : M_{32} \otimes_{A_2} M_{21} \rightarrow M_{31}
\]

be an \((A_3, A_1)\)-homomorphism, and let \(A = \begin{pmatrix} A_1 & 0 & 0 \\ M_{21} & A_2 & 0 \\ M_{31} & M_{32} & A_3 \end{pmatrix}\) with usual matrix operations. Now we generalize Corollary 4 to formal triangular matrix rings.

**Theorem 5.** The following are equivalent:

1. \(A_1, A_2\) and \(A_3\) are strongly \(\pi\)-regular rings of bounded index.

2. The formal triangular matrix ring \(A = \begin{pmatrix} A_1 & 0 & 0 \\ M_{21} & A_2 & 0 \\ M_{31} & M_{32} & A_3 \end{pmatrix}\) is strongly \(\pi\)-regular rings of bounded index.
Proof. (1) \(\Rightarrow\) (2) Let \(B = \begin{pmatrix} A_2 & 0 \\ M_{32} & A_3 \end{pmatrix}\) and \(M = \begin{pmatrix} M_{21} \\ M_{31} \end{pmatrix}\). Because \(A_2\) and \(A_3\) are strongly \(\pi\)-regular rings of bounded index, so is the ring \(B\) by Theorem 2. In addition, \(A_1\) is a strongly \(\pi\)-regular rings of bounded index. By using Theorem 2 again, we have \(A = \begin{pmatrix} A_1 & 0 \\ M & B \end{pmatrix}\) is also a strongly \(\pi\)-regular rings of bounded index, as required.

(2) \(\Rightarrow\) (1) Suppose that the ring \(A\) is a strongly \(\pi\)-regular ring of bounded index. Then we have an idempotent \(e \in T\) such that \(R \cong e Ae\). Therefore we conclude that \(R\) is a strongly \(\pi\)-regular ring of bounded index.

COROLLARY 6. Let \(A_1, A_2\) and \(A_3\) be regular rings of bounded index. Then the formal triangular matrix ring \(A = \begin{pmatrix} A_1 & 0 & 0 \\ M_{21} & A_2 & 0 \\ M_{31} & M_{32} & A_3 \end{pmatrix}\) is strongly \(\pi\)-regular rings of bounded index.

Proof. Since every regular ring of bounded index is a strongly \(\pi\)-regular ring, by Theorem 5, the result follows.

Let \(I\) be an ideal of \(R\). If there exists a positive integer \(p\) such that \(I^p = 0\), then we call \(I\) a nilpotent ideal of \(R\). By an argument of J. Stock (cf. [12, p.451]), one can construct a strongly \(\pi\)-regular ring \(R\) of bounded index 2, while \(J(R)\) is not \(T\)-nilpotent. Moreover, \(J(R)\) is not a nilpotent ideal. Let \(D\) be a division ring and let \(R = \{(x_1, \cdots, x_n, y, y, \cdots) | x_i \in M_i(D), n \in N, y \in D\}\) where \(y\) is treated as a scalar matrix of proper size when multiplied with \(x_i\). By [14, Example 2.3], \(R\) is a strongly \(\pi\)-regular ring not of bounded index, while its Jacobson radical is nilpotent. For Morita context over strongly \(\pi\)-regular rings with nilpotent Jacobson radicals, we now observe the following fact.

THEOREM 7. Let \(T\) be the ring of a Morita context \((A, B, M, N, \psi, \phi)\) with zero pairings. Then \(T\) is a strongly \(\pi\)-regular ring with nilpotent Jacobson radical if and only if so are \(A\) and \(B\).

Proof. One direction is obvious. Conversely, assume now that \(A\) and \(B\) are strongly \(\pi\)-regular rings with nilpotent Jacobson radicals. In view of Lemma 1, \(J(T) \cong \begin{pmatrix} J(A) & N \\ M & J(B) \end{pmatrix}\). Suppose that \(J(A)^s = 0\) and \(J(B)^t = 0\) for some \(s, t > 0\). Given any \(\begin{pmatrix} a & n \\ m & b \end{pmatrix} \in J(T)\), similarly to
the consideration in Theorem 2, we have \( \begin{pmatrix} a & n \\ m & b \end{pmatrix} \begin{pmatrix} a & n \\ m & b \end{pmatrix}^{2(s+t)} = 0 \). Hence

\[ J(T)^{2(s+t)} = 0. \] So the Jacobson radical of \( T \) is nilpotent. Given any \( x \in T \), there is a positive integer \( k \) such that

\[
\begin{align*}
(x + J(T)) & \quad (T/J(T)) \\
& = (x + J(T))^{k+1} (T/J(T)) \\
& = (x + J(T))^{2(s+t)k+1} (T/J(T)).
\end{align*}
\]

So we have a \( y + J(T) \in T/J(T) \) such that

\[ (x + J(T))^k = (x + J(T))^{2(s+t)k+1} (y + J(T)), \]

whence \( x^k - x^{2(s+t)k+1} y \in J(T) \). Therefore \( (x^k - x^{2(s+t)k+1} y)^{2(s+t)} = 0 \).

Consequently, \( x^{2(s+t)k} = x^{2(s+t)k+1} z \) for a \( z \in T \). This yields that \( T \) is a strongly \( \pi \)-regular ring, as required.

\( \square \)

**COROLLARY 8.** Let \( T \) be the ring of a Morita context \( (A, B, M, N, \psi, \phi) \) with zero pairings. If \( A \) and \( B \) are right (left) artinian, then \( T \) is strongly \( \pi \)-regular.

**Proof.** Inasmuch as \( A \) and \( B \) are right (left) artinian, they are strongly \( \pi \)-regular rings with nilpotent Jacobson radicals. The proof is completed by Theorem 7.

\( \square \)

**COROLLARY 9.** Let \( T \) be the ring of a Morita context \( (A, B, M, N, \psi, \phi) \) with zero pairings. If \( A \) and \( B \) are regular P.I. rings, then \( T \) is a strongly \( \pi \)-regular ring.

**Proof.** Since \( A \) is a regular ring, we claim that every projective right \( A \)-module has the finite exchange property. By \[12, Corollary 4.12\], \( A \) is strongly \( \pi \)-regular rings. Likewise, \( B \) is strongly \( \pi \)-regular. Clearly, \( J(A) = 0 \) and \( J(B) = 0 \). Thus the result follows from Theorem 7.

\( \square \)

A ring \( R \) is said to be right (left) quasi-duo if every maximal right (left) ideal is two-sided. Clearly, right (left) duo rings and weakly right (left) duo rings are all right (left) quasi-duo. By \[13, Proposition 4.3\], every P-exchange ring with all idempotents central is right (left) quasi-duo.

**THEOREM 10.** Let \( T \) be the ring of a Morita context \( (A, B, M, N, \psi, \phi) \) with zero pairings. Then \( T \) is a right (left) quasi-duo strongly \( \pi \)-regular ring if and only if so are \( A \) and \( B \).
Proof. It suffices to show that the result holds for right quasi-duo rings. Suppose that $T$ is a right quasi-duo strongly $\pi$-regular ring. Now we construct a map $\theta : T \to A$ given by \[
\begin{pmatrix} a & n \\ m & b \end{pmatrix} \mapsto a \text{ for any } \begin{pmatrix} a & n \\ m & b \end{pmatrix} \in T.\] Because of zero pairings, we claim that $\theta$ is a ring epimorphism. Since every factor ring of right quasi-duo strongly $\pi$-regular ring is again a right quasi-duo strongly $\pi$-regular ring, $A \cong T/\text{Ker}(\theta)$ is a right quasi-duo strongly $\pi$-regular ring. Likewise, $B$ is also a right quasi-duo strongly $\pi$-regular ring.

Conversely, assume that $A$ and $B$ are both right quasi-duo strongly $\pi$-regular rings. It is well known that a ring $R$ is right quasi-duo if and only if so is $R/J(R)$. Thus $A/J(A)$ and $B/J(B)$ are both right quasi-duo rings. By using Lemma 1, we see that $T/J(T)$ is right quasi-duo. Furthermore, $T$ is also a right quasi-duo ring.

In view of [9, Lemma 6], $A/J(A)$ and $B/J(B)$ are both regular rings. Hence it follows by [13, Corollary 2.4] that they are abelian regular rings. This yields that $T/J(T)$ is an abelian regular ring, so it is unit-regular. Thus for any $x + J(T) \in T/J(T)$, we have an idempotent $e \in T/J(T)$ and unit $u \in T/J(T)$ such that $x + J(T) = eu$. Since $T$ is an exchange ring, idempotents can be lifted modulo $J(T)$. On the other hand, units can be lifted modulo $J(T)$. Therefore we have idempotent $f \in T$ and unit $v \in T$ such that $x = fv + r$ for some $r \in J(T)$.

Given any \[
\begin{pmatrix} a & n \\ m & b \end{pmatrix} \in J(T),\] then $a \in J(A)$ and $b \in J(B)$ by Lemma 1. As $A$ and $B$ are both strongly $\pi$-regular rings, there are positive integers $s, t$ such that $a^s = 0$ and $b^t = 0$. Analogously to the discussion in Theorem 2, we have \[
\begin{pmatrix} a & n \\ m & b \end{pmatrix}^{2(s+t)} = 0.\] That is, $J(T)$ is nil. According to [9, Corollary 14], we conclude that $T$ is a strongly $\pi$-regular ring.

Corollary 11. Let $T$ be the ring of a Morita context $(A, B, M, N, \psi, \phi)$ with zero pairings. If $A$ and $B$ are right (left) quasi-duo rings with all prime ideals right (left) primitive, then $T$ is a strongly $\pi$-regular ring.

Proof. By [13, Theorem 2.5], $A$ and $B$ are strongly $\pi$-regular rings. Thus we complete the proof by Theorem 10.

Corollary 12. A ring $R$ is a right (left) quasi-duo strongly $\pi$-regular ring if and only if so is the ring of all $n \times n$ lower triangular matrices over $R$. 
Proof. Suppose that the ring $T$ of all $n \times n$ lower triangular matrices over $R$ is a right (left) quasi-duo strongly $\pi$-regular ring. Then we have an idempotent $e \in T$ such that $R \cong eTe$. Thus $R$ is a strongly $\pi$-regular ring. According to [13, Proposition 2.1], $R$ is a right (left) quasi-duo ring, as required.

Conversely, assume now that $R$ is a right (left) quasi-duo strongly $\pi$-regular ring. Using Theorem 10, we show that the triangular matrix ring \[
\begin{pmatrix}
A & 0 \\
M & B
\end{pmatrix}
\] is a right (left) quasi-duo strongly $\pi$-regular ring if and only if so are $A$ and $B$. By induction, we get the result. \hfill \Box

Analogously, we deduce that a ring $R$ is a right (left) quasi-duo strongly $\pi$-regular ring if and only if so is the ring of all $n \times n$ upper triangular matrices over $R$.

Theorem 13. The following are equivalent:

1. $A_1, A_2$ and $A_3$ are right (left) quasi-duo strongly $\pi$-regular rings.

2. The formal triangular matrix ring $A = \begin{pmatrix}
A_1 & 0 & 0 \\
M_{21} & A_2 & 0 \\
M_{31} & M_{32} & A_3
\end{pmatrix}$ is right (left) quasi-duo strongly $\pi$-regular ring.

Proof. (2) $\Rightarrow$ (1) Clearly, $A_1, A_2$ and $A_3$ are all strongly $\pi$-regular rings. Since $A$ is right (left) quasi-duo, so is $A/J(A)$. One easily checks that $J(A) = \begin{pmatrix}
J(A_1) & 0 & 0 \\
M_{21} & J(A_2) & 0 \\
M_{31} & M_{32} & J(A_3)
\end{pmatrix}$; hence, $A/J(A) \cong A_1/J(A_1) \oplus A_2/J(A_2) \oplus A_3/J(A_3)$. It is straightforward that $A_1/J(A_1) \oplus A_2/J(A_2) \oplus A_3/J(A_3)$ is right (left) quasi-duo if and only if so are $A_1/J(A_1), A_2/J(A_2)$ and $A_3/J(A_3)$. Therefore $A_1, A_2$ and $A_3$ are right (left) quasi-duo.

(1) $\Rightarrow$ (2) Set $B = \begin{pmatrix}
A_2 & 0 \\
M_{32} & A_3
\end{pmatrix}$ and $M = \begin{pmatrix}
M_{21} \\
M_{31}
\end{pmatrix}$. By Theorem 10, $B$ is a right (left) quasi-duo strongly $\pi$-regular rings. Using Theorem 10 again, we get the result. \hfill \Box

Corollary 14. Let $A_1, A_2$ and $A_3$ be right (left) quasi-duo regular rings. Then the formal triangular matrix ring $A = \begin{pmatrix}
A_1 & 0 & 0 \\
M_{21} & A_2 & 0 \\
M_{31} & M_{32} & A_3
\end{pmatrix}$ is a strongly $\pi$-regular rings.
Proof. By [13, Theorem 2.7], every right (left) quasi-duo regular ring is strongly π-regular. It follows by Theorem 13 that A is a strongly π-regular ring.

References


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