HYERS-ULAM-RASSIAS STABILITY OF A QUADRATIC TYPE FUNCTIONAL EQUATION

SANG HAN LEE AND KIL-WOUNG JUN

ABSTRACT. In this paper, we prove the stability of a quadratic type functional equation

\[ a^2 f \left( \frac{x + y + z}{a} \right) + a^2 f \left( \frac{x - y + z}{a} \right) + a^2 f \left( \frac{x + y - z}{a} \right) \]
\[ + a^2 f \left( \frac{-x + y + z}{a} \right) = 4f(x) + 4f(y) + 4f(z). \]

1. Introduction

In 1940, S. M. Ulam ([9]) posed the following question on the stability of homomorphisms: Given a group \( G_1 \), a metric group \( G_2 \) with a metric \( d \) and a number \( \epsilon > 0 \), does there exist a \( \delta > 0 \) such that if a mapping \( f : G_1 \to G_2 \) satisfies the inequality

\[ d(f(xy), f(x)f(y)) < \delta \]

for all \( x, y \in G_1 \), then a homomorphism \( h : G_1 \to G_2 \) exists with

\[ d(f(x), h(x)) < \epsilon \]

for all \( x \in G_1 \)? This question became a source of the stability theory in the Hyers-Ulam sense. The case of approximately additive mappings was solved by D. H. Hyers ([1]) under the assumption that \( G_1 \) and \( G_2 \) are Banach spaces.

In 1978, Th. M. Rassias ([7]) generalized the result of Hyers as follows: Let \( f : X \to Y \) be a mapping between Banach spaces and let \( 0 \leq p < 1 \) be fixed. If \( f \) satisfies the inequality

\[ \|f(x + y) - f(x) - f(y)\| \leq \theta(||x||^p + ||y||^p) \]

Received August 2, 2002.
2000 Mathematics Subject Classification: Primary 39B72.
Key words and phrases: quadratic functional equation, stability.
for some $\theta \geq 0$ and all $x,y \in X$, then there exists a unique additive mapping $A : X \to Y$ such that

$$||A(x) - f(x)|| \leq \frac{2\theta}{2 - 2p} ||x||^p$$

for all $x \in X$. If, in addition, $f(tx)$ is continuous in $t$ for each fixed $x \in X$, then $A$ is linear.

The quadratic function $f(x) = x^2$ is a solution of the following functional equations

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

and

$$f(x + y + z) + f(x - y + z) + f(x + y - z) + f(-x + y + z)$$

$$= 4f(x) + 4f(y) + 4f(z).$$

So, it is natural that each equation is called a quadratic functional equation. In particular, every solution of the quadratic functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ is said to be a quadratic function.

In this paper we deal with a quadratic type functional equation

$$a^2 f \left( \frac{x + y + z}{a} \right) + a^2 f \left( \frac{x - y + z}{a} \right) + a^2 f \left( \frac{x + y - z}{a} \right)$$

$$+ a^2 f \left( \frac{-x + y + z}{a} \right) = 4f(x) + 4f(y) + 4f(z).$$

Throughout this paper $a$ is a nonzero real constant.

2. Solutions of a quadratic type functional equation

Throughout this section $X$ and $Y$ will be real linear spaces. Given a function $f : X \to Y$, consider the following equation

$$a^2 f \left( \frac{x + y + z}{a} \right) + a^2 f \left( \frac{x - y + z}{a} \right) + a^2 f \left( \frac{x + y - z}{a} \right)$$

$$+ a^2 f \left( \frac{-x + y + z}{a} \right) = 4f(x) + 4f(y) + 4f(z).$$
LEMMA 1. If an even function $f : X \to Y$ satisfies (2.1) for all $x, y, z \in X$ and $f(0) = 0$, then $f$ is quadratic.

Proof. Note that $f(-x) = f(x)$ for all $x \in X$ since $f$ is an even function. Putting $y = z = 0$ in (2.1) we have

$$a^2 f\left(\frac{x}{a}\right) = f(x)$$

for all $x \in X$. Using (2.2) in (2.1) we have

$$f(x + y + z) + f(x - y + z) + f(x + y - z) + f(-x + y + z) = 4f(x) + 4f(y) + 4f(z)$$

for all $x, y, z \in X$. Putting $z = 0$ in (2.3) we deduce $2f(x) + 2f(y) = f(x + y) + f(x - y)$ for all $x, y \in X$. This shows that $f$ is quadratic. □

LEMMA 2. If an odd function $f : X \to Y$ satisfies (2.1) for all $x, y, z \in X$, then $f$ is additive.

Proof. Note that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$ since $f$ is an odd function. Putting $y = z = 0$ in (2.1) we have

$$a^2 f\left(\frac{x}{a}\right) = 2f(x)$$

for all $x \in X$. Using (2.4) in (2.1) we have

$$f(x + y + z) + f(x - y + z) + f(x + y - z) + f(-x + y + z) = 2f(x) + 2f(y) + 2f(z)$$

for all $x, y, z \in X$. Putting $z = 0$ in (2.5) we deduce $f(x + y) = f(x) + f(y)$ for all $x, y \in X$. This shows that $f$ is additive. □

REMARK. In Lemma 2, an additive mapping $f$ is nonzero in general. But if $a$ is a rational number and $a \neq 2$ in (2.1), then $f \equiv 0$.

THEOREM 3. If a function $f : X \to Y$ satisfies (2.1) for all $x, y, z \in X$ and $f(0) = 0$, then there exist an additive mapping $A : X \to Y$ and a quadratic function $Q : X \to Y$ such that

$$f(x) = Q(x) + A(x)$$

for all $x \in X$. 
Proof. Let $A(x) := \frac{1}{2}(f(x) - f(-x))$ for all $x \in X$. Then $A(-x) = -A(x)$ and $A$ satisfies (2.1) for all $x, y, z \in X$. By Lemma 2, $A$ is additive.

Let $Q(x) := \frac{1}{2}(f(x) + f(-x))$ for all $x \in X$. Then $Q(0) = 0$, $Q(-x) = Q(x)$ and $Q$ satisfies (2.1) for all $x, y, z \in X$. By Lemma 1, $Q$ is quadratic. Clearly, we have $f(x) = Q(x) + A(x)$ for all $x \in X$. □

3. Stability of a quadratic type functional equation

Let $\mathbb{R}_+$ denote the set of nonnegative real numbers. Recall that a function $H : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is homogeneous of degree $p > 0$ if it satisfies $H(tu, tv, tw) = t^p H(u, v, w)$ for all nonnegative real numbers $t, u, v$ and $w$. Throughout this section $X$ and $Y$ will be a real normed linear space and a real Banach space, respectively. We may assume that $H$ is homogeneous of degree $p$. Given a function $f : X \to Y$, we set

$$Df(x, y, z) := a^2 f \left( \frac{x + y + z}{a} \right) + a^2 f \left( \frac{x - y + z}{a} \right) + a^2 f \left( \frac{x + y - z}{a} \right) + a^2 f \left( \frac{-x + y + z}{a} \right) - 4f(x) - 4f(y) - 4f(z)$$

for all $x, y, z \in X$.

Theorem 4. Assume that $\delta \geq 0$, $p \in (0, \infty) \setminus \{1\}$ and $\delta = 0$ when $p > 1$. Let an odd function $f : X \to Y$ satisfy

$$(3.1) \quad ||Df(x, y, z)|| \leq \delta + H(||x||, ||y||, ||z||)$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$(3.2) \quad ||f(x) - A(x)|| \leq \frac{1}{2}\delta + \frac{1}{|2 - 2p|} h(x)$$

for all $x \in X$, where $h(x) = \frac{1}{3} \left( H(||x||, ||x||, 0) + H(||2x||, 0, 0) \right)$.

Proof. Note that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$ since $f$ is an odd function. Putting $y = z = 0$ in (3.1) and then replacing $x$ by $2x$ we have

$$(3.3) \quad \left| a^2 f \left( \frac{2x}{a} \right) - 2f(2x) \right| \leq \frac{1}{2} (\delta + H(||2x||, 0, 0))$$
for all $x \in X$. Putting $y = x$ and $z = 0$ in (3.1) we have

$$a^2 f \left( \frac{2x}{a} \right) - 4f(x) \leq \frac{1}{2} (\delta + H(||x||, ||x||, 0))$$

for all $x \in X$. By (3.3) and (3.4), we have

$$||f(2x) - 2f(x)|| \leq \frac{1}{2} \delta + h(x)$$

for all $x \in X$, where $h(x) = \frac{1}{4} (H(||x||, ||x||, 0) + H(||2x||, 0, 0))$.

We divide the remaining proof by two cases.

(I) The case $0 < p < 1$. By (3.5), we have

$$||f(x) - \frac{f(2x)}{2}|| \leq \frac{1}{4} \delta + \frac{1}{2} h(x)$$

for all $x \in X$. Using (3.6) we have

$$||f(\frac{2^n x}{2^n}) - f(\frac{2^{n+1} x}{2^{n+1}})|| = \frac{1}{2^n} \left| f(2^n x) - \frac{f(2 \cdot 2^n x)}{2} \right|$$

$$\leq \frac{1}{2^n} + \frac{1}{2} 2^{(p-1)n} h(x)$$

for all $x \in X$ and all positive integers $n$. By (3.7), we have

$$||f(\frac{2^m x}{2^m}) - \frac{f(2^n x)}{2^n}|| \leq \sum_{k=m}^{n-1} \frac{1}{2^{k+2}} \delta + \sum_{k=m}^{n-1} \frac{1}{2} 2^{(p-1)k} h(x)$$

for all $x \in X$ and all positive integers $m$ and $n$ with $m < n$. This shows that $\{f(\frac{2^n x}{2^n})\}$ is a Cauchy sequence for all $x \in X$ since the right side of (3.8) converges to zero when $m \to \infty$. Consequently, we can define a mapping $A : X \to Y$ by

$$A(x) := \lim_{n \to \infty} \frac{f(2^n x)}{2^n}$$

for all $x \in X$. Since $f(-x) = -f(x)$ for all $x \in X$, we have $A(-x) = -A(x)$ for all $x \in X$. Also, we get

$$||D A(x, y, z)|| = \lim_{n \to \infty} 2^{-n} ||D f(2^n x, 2^n y, 2^n z)||$$

$$\leq \lim_{n \to \infty} 2^{-n} \delta + 2^{(p-1)n} H(||x||, ||y||, ||z||)$$

$$= 0$$
for all \( x, y, z \in X \). By Lemma 2, it follows that \( A \) is additive. By (3.6) and (3.7), we have

\[
(3.9) \quad \left\| \frac{f(2^n x)}{2^n} - f(x) \right\| \leq \sum_{k=0}^{n-1} \frac{1}{2^{k+2}} \delta + \sum_{k=0}^{n-1} \frac{1}{2} 2^{(p-1)k} h(x)
\]

for all \( x \in X \) and all positive integers \( n \). Taking the limit in (3.9) as \( n \to \infty \), we get (3.2).

Now, let \( A' : X \to Y \) be another additive mapping satisfying (3.2). Then we have

\[
\left\| A(x) - A'(x) \right\| = 2^{-n} \left\| A(2^n x) - A'(2^n x) \right\|
\leq 2^{-n} \left( \left\| A(2^n x) - f(2^n x) \right\| + \left\| A'(2^n x) - f(2^n x) \right\| \right)
\leq 2^{-n} \delta + \frac{2}{|2 - 2^n|} 2^{(p-1)n} h(x)
\]

for all \( x \in X \) and all positive integers \( n \). Since

\[
\lim_{n \to \infty} \left( 2^{-n} \delta + \frac{2}{|2 - 2^n|} 2^{(p-1)n} h(x) \right) = 0,
\]

we can conclude that \( A(x) = A'(x) \) for all \( x \in X \). This proves the uniqueness of \( A \).

(II) The case \( p > 1 \). Replacing \( x \) by \( \frac{x}{2} \) in (3.5) we have

\[
(3.10) \quad \left\| 2f(2^{-1} x) - f(x) \right\| \leq 2^{-p} h(x)
\]

for all \( x \in X \). Using (3.10) we have

\[
(3.11) \quad \left\| 2^n f(2^{-n} x) - 2^{n+1} f(2^{-(n+1)} x) \right\| \leq 2^{-p} 2^{(1-p)n} h(x)
\]

for all \( x \in X \) and all positive integers \( n \). By (3.10) and (3.11), we have

\[
\left\| 2^n f(2^{-n} x) - f(x) \right\| \leq \sum_{k=0}^{n-1} 2^{(1-p)k} 2^{-p} h(x)
\]

for all \( x \in X \) and all positive integers \( n \). The rest of the proof is similar to the corresponding part of the case \( 0 < p < 1 \). \( \square \)
THEOREM 5. Assume that \( \delta \geq 0 \), \( p \in (0, \infty) \setminus \{2\} \) and \( \delta = 0 \) when \( p > 2 \). Let an even function \( f : X \to Y \) satisfy (3.1) for all \( x, y, z \in X \) and \( f(0) = 0 \). Then there exists a unique quadratic function \( Q : X \to Y \) such that

\[
||f(x) - Q(x)|| \leq \frac{1}{4} \delta + \frac{1}{|4 - 2p|} h(x)
\]

for all \( x \in X \), where \( h(x) = \frac{1}{2} H(||x||, ||x||, 0) + \frac{1}{2} H(||2x||, 0, 0) \).

Proof. Putting \( y = x \) and \( z = 0 \) in (3.1) we have

\[
||a^2 f \left( \frac{2x}{a} \right) - 4f(x)|| \leq \frac{1}{2} (\delta + H(||x||, ||x||, 0))
\]

for all \( x \in X \). Putting \( y = z = 0 \) in (3.1) and then replacing \( x \) by \( 2x \) we have

\[
||a^2 f \left( \frac{2x}{a} \right) - f(2x)|| \leq \frac{1}{4} (\delta + H(||2x||, 0, 0))
\]

for all \( x \in X \). By (3.13) and (3.14), we have

\[
||f(2x) - 4f(x)|| \leq \frac{3}{4} \delta + h(x)
\]

for all \( x \in X \), where \( h(x) = \frac{1}{2} H(||x||, ||x||, 0) + \frac{1}{2} H(||2x||, 0, 0) \).

We divide the remaining proof by two cases.

(1) The case \( 0 < p < 2 \). By (3.15), we have

\[
||f(x) - \frac{f(2x)}{4}|| \leq \frac{3}{16} \delta + \frac{1}{4} h(x)
\]

for all \( x \in X \). Using (3.16) we have

\[
||\frac{f(2^n x)}{4^n} - \frac{f(2^{n+1} x)}{4^{n+1}}|| \leq \frac{1}{4^n} \left( ||f(2^n x) - \frac{f(2 \cdot 2^n x)}{4}|| \right)
\]

\[
\leq \frac{3}{16} 4^{-n} \delta + \frac{1}{4} 2^{(p-2)n} h(x)
\]

for all \( x \in X \) and all positive integers \( n \). By (3.16) and (3.17), we have

\[
||\frac{f(2^n x)}{4^n} - \frac{f(2^{n+k} x)}{4^{n+k}}|| \leq \sum_{k=m}^{n-1} \frac{3}{16} 4^{-k} \delta + \sum_{k=m}^{n-1} \frac{1}{4} 2^{(p-2)k} h(x)
\]
for all \( x \in X \) and all nonnegative integers \( m \) and \( n \) with \( m < n \). This shows that \( \{ f(2^n x) \} \) is a Cauchy sequence for all \( x \in X \). Consequently, we can define a function \( Q : X \to Y \) by

\[
Q(x) := \lim_{n \to \infty} \frac{f(2^n x)}{4^n}
\]

for all \( x \in X \). We have \( Q(0) = 0 \), \( Q(-x) = Q(x) \) and

\[
||DQ(x, y, z)|| = \lim_{n \to \infty} 4^{-n}||Df(2^n x, 2^n y, 2^n z)||
\]

\[
\leq \lim_{n \to \infty} (4^{-n} \delta + 2^{(n-2)n} H(||x||, ||y||, ||z||))
\]

\[
= 0
\]

for all \( x, y, z \in X \). By Lemma 1, it follows that \( Q \) is quadratic. Putting \( m = 0 \) in (3.18) and letting \( n \to \infty \) we have (3.12). The proof of the uniqueness of \( Q \) is similar to the proof of Theorem 4.

(II) The case \( p > 2 \). Replacing \( x \) by \( \frac{x}{2} \) in (3.15) we have

(3.19)

\[
||4f(2^{-1} x) - f(x)|| \leq 2^{-p} h(x)
\]

for all \( x \in X \). Using (3.19) we have

(3.20)

\[
||4^n f(2^{-n} x) - 4^{n+1} f(2^{-(n+1)} x)|| \leq 2^{-p} 2^{(2-p)n} h(x)
\]

for all \( x \in X \). By (3.19) and (3.20), we have

\[
||4^n f(2^{-n} x) - f(x)|| \leq \sum_{k=0}^{n-1} 2^{(2-p)k} 2^{-p} h(x)
\]

for all \( x \in X \) and all positive integers \( n \). The rest of the proof is similar to the corresponding part of the case \( p < 2 \).

**Theorem 6.** Let \( \delta > 0 \) and \( p \in (0, \infty) \setminus \{1, 2\} \). Assume that \( \delta = 0 \) if \( p > 1 \) and \( ||(a^2 - 3)f(0)|| = 0 \) if \( p > 2 \). If a function \( f : X \to Y \) satisfy (3.1) for all \( x, y, z \in X \), then there exist a unique quadratic function \( Q : X \to Y \) and a unique additive mapping \( A : X \to Y \) such that

(3.21)

\[
||f(x) - f(0) - Q(x) - A(x)||
\]

\[
\leq \frac{3}{4} \delta + ||(a^2 - 3)f(0)|| + \frac{1}{|4 - 2^p|} h_1(x) + \frac{1}{|2 - 2^p|} h_2(x),
\]
\begin{equation}
\left\| \frac{f(x) + f(-x)}{2} - f(0) - Q(x) \right\| \leq \frac{1}{4} \delta + \frac{1}{4} (a^2 - 3) f(0) + \frac{1}{|4 - 2p|} h_1(x),
\end{equation}

and
\begin{equation}
\left\| \frac{f(x) - f(-x)}{2} - A(x) \right\| \leq \frac{1}{2} \delta + \frac{1}{|2 - 2p|} h_2(x)
\end{equation}

for all \( x \in X \), where \( h_1(x) = \frac{1}{4} H(||x||, ||x||, 0) + \frac{1}{4} H(||2x||, 0, 0) \) and \( h_2(x) = \frac{1}{4} \left( H(||x||, ||x||, 0) + H(||2x||, 0, 0) \right) \).

**Proof.** Let \( q_1(x) := \frac{1}{2} (f(x) + f(-x)) \) for all \( x \in X \). Then \( q_1(0) = f(0), q_1(-x) = q_1(x) \) and
\begin{align*}
||Dq_1(x, y, z)|| &\leq \delta + H(||x||, ||y||, ||z||)
\end{align*}

for all \( x, y, z \in X \).

Let \( q(x) := q_1(x) - q_1(0) \) for all \( x \in X \). Then \( q(0) = 0, q(-x) = q(x) \) and
\begin{align*}
||Dq(x, y, z)|| &= ||Dq_1(x, y, z) - (4a^2 - 12) q_1(0)|| \\
&\leq ||Dq_1(x, y, z)|| + ||(4a^2 - 12) q_1(0)|| \\
&\leq \delta + ||(4a^2 - 12) ||f(0)|| + H(||x||, ||y||, ||z||)
\end{align*}

for all \( x, y, z \in X \). By Theorem 5, there exists a unique quadratic function \( Q : X \to Y \) satisfying (3.22).

Let \( g(x) := \frac{1}{2} (f(x) - f(-x)) \) for all \( x \in X \). Then \( g(-x) = -g(x) \) and
\begin{align*}
||Dg(x, y, z)|| &\leq \delta + H(||x||, ||y||, ||z||)
\end{align*}

for all \( x, y, z \in X \). By Theorem 4, there exists a unique additive mapping \( A : X \to Y \) satisfying (3.23). Clearly, we have (3.21) for all \( x \in X \). \( \square \)

Define a function \( H : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \) by \( H(a, b, c) = (a^p + b^p + c^p)^\theta \) where \( \theta \geq 0 \) and \( p \in (0, \infty) \). Then \( H \) is homogeneous of degree \( p \). Thus we have the following corollaries.

**Corollary 7.** Assume that \( \delta \geq 0, p \in (0, \infty) \setminus \{1\} \) and \( \delta = 0 \) when \( p > 1 \). Let an odd function \( f : X \to Y \) satisfy
\begin{align*}
||Df(x, y, z)|| &\leq \delta + \theta(||x||^p + ||y||^p + ||z||^p)
\end{align*}

for all \( x, y, z \in X \). Then there exists a unique additive mapping \( A : X \to Y \) such that
\begin{align*}
||f(x) - A(x)|| &\leq \frac{1}{2} \delta + \frac{2 + 2p}{4(2 - 2p)} \theta||x||^p
\end{align*}

for all \( x \in X \).
COROLLARY 8. Assume that $\delta \geq 0$, $p \in (0, \infty) \setminus \{2\}$ and $\delta = 0$ when $p > 2$. Let an even function $f : X \to Y$ satisfy

$$||Df(x, y, z)|| \leq \delta + \theta(||x||^p + ||y||^p + ||z||^p)$$

for all $x, y, z \in X$ and $f(0) = 0$. Then there exists a unique quadratic function $Q : X \to Y$ such that

$$||f(x) - Q(x)|| \leq \frac{1}{4} \delta + \frac{4 + 2^p}{4|4 - 2^p|} \theta||x||^p$$

for all $x \in X$.

COROLLARY 9. Let $\delta \geq 0$ and $p \in (0, \infty) \setminus \{1, 2\}$. Assume that $\delta = 0$ if $p > 1$ and $||(a^2 - 3)f(0)|| = 0$ if $p > 2$. If a function $f : X \to Y$ satisfy

$$||Df(x, y, z)|| \leq \delta + \theta(||x||^p + ||y||^p + ||z||^p)$$

for all $x, y, z \in X$, then there exist a unique quadratic function $Q : X \to Y$ and a unique additive mapping $A : X \to Y$ such that

$$||f(x) - f(0) - Q(x) - A(x)||$$

$$\leq \frac{3}{4} \delta + ||(a^2 - 3)f(0)|| + \left(\frac{4 + 2^p}{4|4 - 2^p|} + \frac{2 + 2^p}{4|2 - 2^p|}\right) \theta||x||^p,$$

$$\frac{||f(x) + f(-x)||}{2} - f(0) - Q(x) \leq \frac{1}{4} \delta + ||(a^2 - 3)f(0)|| + \frac{4 + 2^p}{4|4 - 2^p|} \theta||x||^p,$$

and

$$\frac{||f(x) - f(-x)||}{2} - A(x) \leq \frac{1}{2} \delta + \frac{2 + 2^p}{4|2 - 2^p|} \theta||x||^p$$

for all $x \in X$.

References


SANG HAN LEE, DEPARTMENT OF CULTURAL STUDIES, CHUNGBUK PROVINCIAL UNIVERSITY OF SCIENCE & TECHNOLOGY, OXCHEON 373-807, KOREA
E-mail: shlee@ctech.ac.kr

KIL-WOUNG JUN, DEPARTMENT OF MATHEMATICS, CHUNGAM NATIONAL UNIVERSITY, TAEJEON 305-764, KOREA
E-mail: kwjun@math.chungnam.ac.kr