B.-Y. CHEN INEQUALITIES FOR SUBMANIFOLDS IN GENERALIZED COMPLEX SPACE FORMS

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ABSTRACT. Some B.-Y. Chen inequalities for different kind of submanifolds of generalized complex space forms are established.

1. Introduction

According to Nash’s immersion theorem every $n$-dimensional Riemannian manifold admits an isometric immersion into Euclidean space $\mathbb{E}^{n(n+1)(3n+11)/2}$. Thus, one becomes able to consider any Riemannian manifold as a submanifold of Euclidean space; and this provides a natural motivation for the study of submanifolds of Riemannian manifolds. To find simple relationships between the main intrinsic invariants and the main extrinsic invariants of a submanifold is one of the basic interests of study in the submanifold theory. Gauss-Bonnet Theorem, Isoperimetric inequality and Chern-Lashof Theorem provide relations between extrinsic and intrinsic invariants for a submanifold in a Euclidean space.

In [2], B.-Y. Chen established a sharp inequality for a submanifold in a real space form involving intrinsic invariants, namely the sectional curvatures and the scalar curvature of the submanifold; and the main extrinsic invariant, namely the squared mean curvature.

On the other hand, A. Gray introduced the notion of constant type for a nearly Kähler manifold ([6]), which led to definitions of $RK$-manifolds $\bar{M}(c, \alpha)$ of constant holomorphic sectional curvature $c$ and constant type $\alpha$ ([10]) and generalized complex space forms $\bar{M}(f_1, f_2)$ ([7]). We have
the inclusion relation $\tilde{M}(c) \subset \tilde{M}(c, \alpha) \subset \tilde{M}(f_1, f_2)$, where $\tilde{M}(c)$ is the complex space form of constant holomorphic sectional curvature $c$.

Thus it is worthwhile to study relationships between intrinsic and extrinsic invariants of submanifolds in a generalized space form. In this paper, we establish several such relationships for slant, totally real and invariant submanifolds in generalized complex space forms, complex space forms and $RK$-manifolds. The paper is organized as follows. Section 2 is preliminary in nature. It contains necessary details about generalized complex space form and its submanifolds. In section 3, we obtain a basic inequality for a submanifold in a generalized complex space form involving intrinsic invariants, namely the scalar curvature and the sectional curvatures of the submanifold on left hand side and the main extrinsic invariant, namely the squared mean curvature on the right hand side. Then, we apply this result to get a B.-Y. Chen inequality between Chen's $\delta$-invariant and squared mean curvature for $\theta$-slant submanifolds in a generalized complex space form. Particular cases are put in a table in concise form. Next, we establish another general inequality for submanifolds of generalized complex space forms and then using it we obtain a B.-Y. Chen's inequality between Chen's $\delta(n_1, \cdots, n_k)$-invariant and squared mean curvature for slant submanifolds. In last, we again list particular cases in a table.

2. Preliminaries

Let $\tilde{M}$ be an almost Hermitian manifold with an almost Hermitian structure $(J, \langle \cdot, \cdot \rangle)$. An almost Hermitian manifold becomes a nearly Kähler manifold ([6]) if $(\tilde{\nabla}_X J) X = 0$, and becomes a Kähler manifold if $\tilde{\nabla} J = 0$ for all $X \in T\tilde{M}$, where $\tilde{\nabla}$ is the Levi-Civita connection of the Riemannian metric $\langle \cdot, \cdot \rangle$. An almost Hermitian manifold with $J$-invariant Riemannian curvature tensor $\tilde{R}$, that is,

$$\tilde{R}(JX, JY, JZ, JW) = \tilde{R}(X, Y, Z, W), \quad X, Y, Z, W \in T\tilde{M},$$

is called an $RK$-manifold ([10]). All nearly Kähler manifolds belong to the class of $RK$-manifolds.

The notion of constant type was first introduced by A. Gray for a nearly Kähler manifold ([6]). An almost Hermitian manifold $\tilde{M}$ is said to have (pointwise) constant type if for each $p \in \tilde{M}$ and for all $X, Y, Z \in T_p\tilde{M}$ such that

$$\langle X, Y \rangle = \langle X, Z \rangle = \langle X, JY \rangle = \langle X, JZ \rangle = 0, \quad \langle Y, Y \rangle = 1 = \langle Z, Z \rangle$$
we have
\[ \hat{R}(X, Y, X, Y) - \hat{R}(X, Y, JX, JY) = \hat{R}(X, Z, X, Z) - \hat{R}(X, Z, JX, JZ). \]

An \( RK \)-manifold \( \hat{M} \) has (pointwise) constant type if and only if there is a differentiable function \( \alpha \) on \( \hat{M} \) satisfying ([10])
\[ \hat{R}(X, Y, X, Y) - \hat{R}(X, Y, JX, JY) = \alpha \{ \langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2 - \langle X, JY \rangle^2 \} \]
for all \( X, Y \in T\hat{M} \). Furthermore, \( \hat{M} \) has global constant type if \( \alpha \) is constant. The function \( \alpha \) is called the constant type of \( \hat{M} \). An \( RK \)-manifold of constant holomorphic sectional curvature \( c \) and constant type \( \alpha \) is denoted by \( \hat{M}(c, \alpha) \). For \( \hat{M}(c, \alpha) \) it is known that ([10])
\[ 4\hat{R}(X, Y)Z = (c + 3\alpha) \{ \langle Y, Z \rangle X - \langle X, Z \rangle Y \} + \langle X, JZ \rangle JY - \langle Y, JZ \rangle JX + 2 \langle X, JY \rangle JZ \]
for all \( X, Y, Z \in T\hat{M} \). If \( c = \alpha \) then \( \hat{M}(c, \alpha) \) is a space of constant curvature. A complex space form \( \hat{M}(c) \) (a Kähler manifold of constant holomorphic sectional curvature \( c \)) belongs to the class of almost Hermitian manifolds \( \hat{M}(c, \alpha) \) (with the constant type zero).

An almost Hermitian manifold \( \hat{M} \) is called a \textit{generalized complex space form} \( \hat{M}(f_1, f_2) \) ([7]) if its Riemannian curvature tensor \( \hat{R} \) satisfies
\[
\hat{R}(X, Y)Z = f_1 \{ \langle Y, Z \rangle X - \langle X, Z \rangle Y \} + f_2 \{ \langle X, JZ \rangle JY - \langle Y, JZ \rangle JX + 2 \langle X, JY \rangle JZ \}
\]
for all \( X, Y, Z \in T\hat{M} \), where \( f_1 \) and \( f_2 \) are smooth functions on \( \hat{M} \).

The Riemannian invariants are the intrinsic characteristics of a Riemannian manifold. Here, we recall a number of Riemannian invariants ([4]) in a Riemannian manifold. Let \( M \) be a Riemannian manifold and \( L \) be a \( r \)-plane section of \( T_p M \). Choose an orthonormal basis \( \{ e_1, \cdots, e_r \} \) for \( L \). Let \( K_{ij} \) denote the sectional curvature of the plane section spanned by \( e_i \) and \( e_j \) at \( p \in M \). The scalar curvature \( \tau \) of the \( r \)-plane section \( L \) is given by
\[
\tau(L) = \sum_{1 \leq i < j \leq r} K_{ij}.
\]
Given an orthonormal basis \( \{ e_1, \cdots, e_n \} \) for \( T_p M \), \( \tau_1 \cdots \tau \) will denote the scalar curvature of the \( r \)-plane section spanned by \( e_1, \cdots, e_r \). If \( L \) is a 2-plane section then \( \tau(L) \) reduces to the sectional curvature \( K \) of the plane section \( L \). We denote by \( K(\pi) \) the sectional curvature of \( M \) for a plane section \( \pi \) in \( T_p M, p \in M \). The scalar curvature \( \tau(p) \) of \( M \) at \( p \)
is the scalar curvature of the tangent space of \( M \) at \( p \). Thus, the scalar curvature \( \tau \) at \( p \) is given by

\[
\tau (p) = \sum_{i<j} K_{ij},
\]

where \( \{e_1, \cdots, e_n\} \) is an orthonormal basis for \( T_pM \) and \( K_{ij} \) is the sectional curvature of the plane section spanned by \( e_i \) and \( e_j \) at \( p \in M \). Chen’s \( \delta \)-\textit{invariant} is defined by the following identity

\[
\delta_M(p) = \tau(p) - \inf\{K(\pi) \mid \pi \text{ is a plane section} \subset T_pM\},
\]

which is certainly an intrinsic character of \( M \).

For an integer \( k \geq 0 \), we denote by \( S(n,k) \) the finite set which consists of \( k \)-tuples \( (n_1, \cdots, n_k) \) of integers \( \geq 2 \) satisfying \( n_1 < n \) and \( n_1 + \cdots + n_k \leq n \). Denote by \( S(n) \) the set of all (unordered) \( k \)-tuples with \( k \geq 0 \) for a fixed positive integer \( n \). For each \( k \)-tuple \( (n_1, \cdots, n_k) \in S(n) \), we B.-Y. Chen introduced a Riemannian invariant \( \delta (n_1,\cdots,n_k) \) defined by

\[
\delta (n_1,\cdots,n_k) (p) = \tau(p) - \inf\{\tau (L_1) + \cdots + \tau (L_k)\},
\]

where \( L_1, \cdots, L_k \) run over all \( k \) mutually orthogonal subspaces of \( T_pM \) such that \( \dim L_j = n_j, \ j = 1, \cdots, k \). For each \( (n_1, \cdots, n_k) \in S(n) \) we put

\[
a(n_1, \cdots, n_k) = \frac{1}{2} n(n-1) - \frac{1}{2} \sum_{j=1}^{k} n_j(n_j-1),
\]

\[
b(n_1, \cdots, n_k) = \frac{n^2 \left(n+k-1-\sum_{j=1}^{k} n_j\right)}{2 \left(n+k-\sum_{j=1}^{k} n_j\right)}.
\]

For more details we refer to [4] and corresponding references therein.

Let \( M \) be an \( n \)-dimensional submanifold in a manifold \( \tilde{M} \) equipped with a Riemannian metric \( \langle , \rangle \). The Gauss and Weingarten formulas are given respectively by \( \tilde{\nabla}_X Y = \nabla_X Y + \sigma (X,Y) \) and \( \tilde{\nabla}_X N = -A_N X + \nabla^\perp_X N \) for all \( X, Y \in TM \) and \( N \in T^\perp M \), where \( \nabla, \tilde{\nabla} \) and \( \nabla^\perp \) are Riemannian, induced Riemannian and induced normal connections in \( M \), \( M \) and the normal bundle \( T^\perp M \) of \( M \) respectively, and \( \sigma \) is the
second fundamental form related to the shape operator $A_N$ in the direction of $N$ by $\langle \sigma (X,Y), N \rangle = \langle A_N X, Y \rangle$. The mean curvature vector $H$ is expressed by $nH = \text{trace}(\sigma)$. The submanifold $M$ is totally geodesic in $\tilde{M}$ if $\sigma = 0$.

In a submanifold $M$ of an almost Hermitian manifold, for a vector $0 \neq X_p \in T_pM$, the angle $\theta (X_p)$ between $JX_p$ and the tangent space $T_pM$ is called the Wirtinger angle of $X_p$. If the Wirtinger angle is independent of $p \in M$ and $X_p \in T_pM$, then $M$ is called a slant submanifold ([1]).

We put $JX = PX + FX$ for $X \in TM$, where $PX$ (resp. $FX$) is the projection of $JX$ on $TM$ (resp. $T^2M$). Slant submanifolds of almost Hermitian manifolds are characterized by the condition $P^2 + \lambda^2 I = 0$ for some real number $\lambda \in [0,1]$. Invariant and anti-invariant ([11]) (or totally real) submanifolds are slant submanifolds with $\theta = 0$ ($P = 0$) and $\theta = \pi/2$ ($P = 0$) respectively. For more details about slant submanifolds we refer to [1].

3. B.-Y. Chen inequalities

First we state the following algebraic lemma from [2] for later uses.

**Lemma 3.1.** If $a_1, \cdots, a_n, a_{n+1}$ are $n + 1$ ($n \geq 2$) real numbers such that

$$\left( \sum_{i=1}^{n} a_i \right)^2 = (n - 1) \left( \sum_{i=1}^{n} a_i^2 + a_{n+1} \right),$$

then $2a_1a_2 \geq a_{n+1}$, with equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

Let $M$ be a submanifold in an almost Hermitian manifold $\tilde{M}$. Let $\pi \subset T_pM$ be a plane section at $p \in M$. Then

$$\Theta (\pi) = \langle Pe_1, e_2 \rangle^2$$

is a real number in $[0, 1]$, which is independent of the choice of orthonormal basis $\{e_1, e_2\}$ of $\pi$. Moreover, if $\tilde{M}$ is a generalized complex space form, then Gauss equation becomes ([8])

$$R(X, Y, Z, W) = f_1 \{ \langle Y, Z \rangle \langle X, W \rangle - \langle X, Z \rangle \langle Y, W \rangle \}
+f_2 \{ \langle X, PZ \rangle \langle PY, W \rangle - \langle Y, PZ \rangle \langle PX, W \rangle + 2 \langle X, PY \rangle \langle PZ, W \rangle \}
+ \langle \sigma (X, W), \sigma (Y, Z) \rangle - \langle \sigma (X, Z), \sigma (Y, W) \rangle$$

for all $X, Y, Z, W \in TM$, where $R$ is the curvature tensors of $M$. Thus, we are able to state the following Lemma.
LEMMA 3.2. In an $n$-dimensional submanifold in a generalized complex space form $M(f_1, f_2)$, the scalar curvature and the squared mean curvature satisfy

$$2\tau = n(n - 1)f_1 + 3f_2 \|P\|^2 + n^2 \|H\|^2 - \|\sigma\|^2,$$

where

$$\|P\|^2 = \sum_{i,j=1}^{n} (e_i, Pe_j)^2 \quad \text{and} \quad \|\sigma\|^2 = \sum_{i,j=1}^{n} \langle \sigma(e_i, e_j), \sigma(e_i, e_j) \rangle.$$

For a submanifold $M$ in a real space form $R^m(c)$, B.-Y. Chen ([2]) gave the following

$$\delta_M \leq \frac{n^2(n - 2)}{2(n - 1)} \|H\|^2 + \frac{1}{2}(n + 1)(n - 2)c,$$

He ([3]) also established the basic inequality for submanifold $M$ in a complex space form $CP^m(4c)$ (respectively, $CH^m(4c)$, the complex hyperbolic space) of constant holomorphic sectional curvature $4c$ as follows:

$$\delta_M \leq \frac{n^2(n - 2)}{2(n - 1)} \|H\|^2 + \frac{1}{2}(n^2 + 2n - 2)c,$$

(respectively, $\delta_M \leq \frac{n^2(n - 2)}{2(n - 1)} \|H\|^2 + \frac{1}{2}(n + 1)(n - 2)c$).

Now, we prove the following basic inequality for later uses.

THEOREM 3.3. Let $M$ be an $n$-dimensional submanifold isometrically immersed in a $2m$-dimensional generalized complex space form $\tilde{M}(f_1, f_2)$. Then, for each point $p \in M$ and each plane section $\pi \subset T_pM$, we have

$$\tau - K(\pi) \leq \frac{n^2(n - 2)}{2(n - 1)} \|H\|^2 + \frac{1}{2}(n + 1)(n - 2) f_1 + \frac{3}{2} f_2 \|P\|^2 - 3f_2 \Theta(\pi).$$

Equality in (11) holds at $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \cdots, e_n\}$ of $T_pM$ and an orthonormal basis $\{e_{n+1}, \cdots, e_{2m}\}$ of $T_p^\perp M$ such that (a) $\pi = \text{Span}\{e_1, e_2\}$ and (b) the shape operators $A_r \equiv A_{e_r}$, $r = n + 1, \cdots, 2m$, take the following forms:

$$(12) \quad A_{n+1} = \begin{pmatrix}
\lambda & 0 & 0 & \cdots & 0 \\
0 & \mu & 0 & \cdots & 0 \\
0 & 0 & \lambda + \mu & \cdots & 0 \\
0 & 0 & 0 & \cdots & \lambda + \mu
\end{pmatrix},$$
\begin{equation}
A_r = \begin{pmatrix}
c_r & d_r & 0 & \cdots & 0 \\
d_r & -c_r & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\end{pmatrix}, \quad r = n + 2, \cdots, 2m.
\end{equation}

Proof. Let \( \pi \subset T_pM \) be a plane section. Choose an orthonormal basis \( \{e_1, e_2, \cdots, e_n\} \) for \( T_pM \) and \( \{e_{n+1}, \cdots, e_{2m}\} \) for the normal space \( T_p^\perp M \) at \( p \) such that \( \pi = \text{Span} \{e_1, e_2\} \) and the normal vector \( e_{n+1} \) is in the direction of the mean curvature vector \( H \). We rewrite (10) as
\begin{equation}
\frac{1}{n-1} \left( \sum_{i=1}^{n} \sigma_{i}^{n+1} \right) = \sum_{i=1}^{n} \left( \sigma_{i}^{n+1} \right)^2 + \sum_{i \neq j} \left( \sigma_{ij}^{n+1} \right)^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} \left( \sigma_{ij}^{r} \right)^2 + \rho,
\end{equation}
where
\begin{equation}
\rho = 2\tau - \frac{n^2 (n-2)}{n-1} \|H\|^2 - n (n-1) f_1 - 3f_2 \|P\|^2
\end{equation}
and \( \sigma_{ij}^{r} = \langle \sigma (e_i, e_j), e_r \rangle, \ i, j \in \{1, \cdots, n\}; \ r \in \{n+1, \cdots, 2m\} \). Now, applying Lemma 3.1 to (14), we obtain
\begin{equation}
2\sigma_{11}^{n+1} \sigma_{22}^{n+1} \geq \sum_{i \neq j} \left( \sigma_{ij}^{n+1} \right)^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} \left( \sigma_{ij}^{r} \right)^2 + \rho.
\end{equation}
From (9) it also follows that
\begin{equation}
K(\pi) = \sigma_{11}^{n+1} \sigma_{22}^{n+1} - \left( \sigma_{12}^{n+1} \right)^2 + \sum_{r=n+2}^{2m} \left( \sigma_{11}^{r} \sigma_{22}^{r} - \left( \sigma_{12}^{r} \right)^2 \right) + f_1 + 3f_2 \Theta(\pi).
\end{equation}
Thus, from (16) and (17) we have
\begin{equation}
K(\pi) \geq f_1 + 3f_2 \Theta(\pi) + \frac{\rho}{2} + \sum_{r=n+1}^{2m} \sum_{j>2} \sum_{i,j} \left( \sigma_{ij}^{r} \right)^2 + \frac{1}{2} \sum_{i \neq j>2} \left( \sigma_{ij}^{n+1} \right)^2
\begin{equation}
+ \frac{1}{2} \sum_{r=n+2}^{2m} \sum_{i,j>2} \left( \sigma_{ij}^{r} \right)^2 + \frac{1}{2} \sum_{r=n+2}^{2m} \left( \sigma_{11}^{r} + \sigma_{22}^{r} \right)^2.
\end{equation}
In view of (15) and (18), we get (11).
If the equality in (11) holds, then the inequalities given by (16) and (18) become equalities. In this case, we have

\[
\begin{align*}
\sigma_{ij}^{n+1} &= 0, \quad \sigma_{2j}^{n+1} = 0, \quad \sigma_{ij}^{n+1} = 0, \quad i \neq j > 2; \\
\sigma_{1j}^{r+1} &= \sigma_{2j}^{r+1} = \sigma_{ij}^{r+1} = 0, \quad r = n + 2, \cdots, 2m; \quad i, j = 3, \cdots, n; \\
\sigma_{11}^{n+2} + \sigma_{22}^{n+2} &= \cdots = \sigma_{11}^{2m} + \sigma_{22}^{2m} = 0. 
\end{align*}
\]

(19)

Now, we choose \(e_1\) and \(e_2\) so that \(\sigma_{12}^{n+1} = 0\). Applying Lemma 3.1 we also have

\[
\sigma_{11}^{n+1} + \sigma_{22}^{n+1} = \sigma_{33}^{n+1} = \cdots = \sigma_{nn}^{n+1}.
\]

(20)

Thus, choosing a suitable orthonormal basis \(\{e_1, \cdots, e_{2m}\}\), the shape operators of \(M\) become of the forms given by (12) and (13). The converse is simple to observe. \(\Box\)

As an application, we prove a B.-Y. Chen inequality for \(\theta\)-slant submanifolds in a generalized complex space form.

**Theorem 3.4.** For an \(n\)-dimensional \((n > 2)\) \(\theta\)-slant submanifold \(M\) isometrically immersed in a \(2m\)-dimensional generalized complex space form \(M(f_1, f_2)\), at every point \(p \in M\) and each plane section \(\pi \subset T_pM\), we have

\[
\delta_M \leq \frac{n-2}{2} \left\{ \frac{n^2}{n-1} \|H\|^2 + (n+1) f_1 + 3f_2 \cos^2 \theta \right\}.
\]

(21)

Equality in (21) holds at \(p \in M\) if and only if the shape operators of \(M\) in \(M(f_1, f_2)\) at \(p\) take the forms given by (12) and (13).

**Proof.** Let \(M\) be an \(n\)-dimensional \(\theta\)-slant submanifold \(M\) in an almost Hermitian manifold with \(n \geq 3\), \(n = 2l\). Let \(p \in M\) and \(\{e_i, \sec \theta Pe_i\}, \ i = 1, \cdots, l\), be an orthonormal basis of \(T_pM\). Thus, we have \(\|P\|^2 = n \cos^2 \theta\). Choosing an orthonormal basis \(\{e_i, \sec \theta Pe_i\}\) for any plane section \(\pi \subset T_pM\), we have \(\Theta(\pi) = \cos^2 \theta\). Putting these values of \(\|P\|^2\) and \(\Theta(\pi)\) in (11), we get (21). \(\Box\)

In particular, the above Theorem provides the following Corollary.
**Corollary 3.5.** We have the following table:

<table>
<thead>
<tr>
<th>Manifold</th>
<th>Submanifold</th>
<th>Inequality</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{M}(f_1, f_2)$</td>
<td>totally real</td>
<td>$\delta_M \leq \frac{n-2}{2} \left{ \frac{n^2}{n-1} |H|^2 + (n+1) f_1 \right}$</td>
</tr>
<tr>
<td>$\tilde{M}(f_1, f_2)$</td>
<td>invariant</td>
<td>$\delta_M \leq \frac{n-2}{2} \left{ \frac{n^2}{n-1} |H|^2 + (n+1) f_1 + 3 f_2 \right}$</td>
</tr>
<tr>
<td>$\tilde{M}(c, \alpha)$</td>
<td>$\theta$-slant</td>
<td>$\delta_M \leq \frac{n-2}{8} \left{ \frac{4n^2}{n-1} |H|^2 + (n+1) (c+3\alpha) + 3(c - \alpha) \cos^2 \theta \right}$</td>
</tr>
<tr>
<td>$\tilde{M}(c, \alpha)$</td>
<td>totally real</td>
<td>$\delta_M \leq \frac{n-2}{8} \left{ \frac{4n^2}{n-1} |H|^2 + (n+1) (c+3\alpha) \right}$</td>
</tr>
<tr>
<td>$\tilde{M}(c, \alpha)$</td>
<td>invariant</td>
<td>$\delta_M \leq \frac{n-2}{8} \left{ \frac{4n^2}{n-1} |H|^2 + (n+4) c + 3 c \alpha \right}$</td>
</tr>
<tr>
<td>$\tilde{M}(c)$</td>
<td>$\theta$-slant</td>
<td>$\delta_M \leq \frac{n-2}{8} \left{ \frac{4n^2}{n-1} |H|^2 + (n+1 + 3 \cos^2 \theta) c \right}$</td>
</tr>
<tr>
<td>$\tilde{M}(c)$</td>
<td>totally real</td>
<td>$\delta_M \leq \frac{n-2}{8} \left{ \frac{4n^2}{n-1} |H|^2 + (n+1) c \right}$</td>
</tr>
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<td>$\tilde{M}(c)$</td>
<td>invariant</td>
<td>$\delta_M \leq \frac{n-2}{8} \left{ \frac{4n^2}{n-1} |H|^2 + (n+4) c \right}$</td>
</tr>
<tr>
<td>$R(c)$</td>
<td></td>
<td>$\delta_M \leq \frac{n-2}{2} \left{ \frac{n^2}{n-1} |H|^2 + (n+1) c \right}$</td>
</tr>
</tbody>
</table>

Let $M$ be a submanifold of an almost Hermitian manifold. For an $r$-plane section $L \subset T_p M$ and for any orthonormal basis $\{e_1, \cdots, e_r\}$ of $L$, we put

$$\Psi(L_j) = \sum_{1 \leq i < j \leq r} (Pe_i, e_j)^2.$$  \hfill (22)

**Lemma 3.6.** Let $M$ be an $n$-dimensional submanifold of a $2m$-dimensional generalized complex space form $\tilde{M}(f_1, f_2)$. Let $n_1, \cdots, n_k$ be integers $\geq 2$ satisfying $n_1 < n, n_1 + \cdots + n_k \leq n$. For $p \in M$, let $L_j$ be an $n_j$-plane section of $T_p M$, $j = 1, \cdots, k$. Then we have

$$\tau - \sum_{j=1}^k \tau(L_j) \leq b(n_1, \cdots, n_k) \|H\|^2 + a(n_1, \cdots, n_k) f_1 + \frac{3}{2} \left\{ \|P\|^2 - 2 \sum_{j=1}^k \Psi(L_j) \right\} f_2.$$  \hfill (23)

**Proof.** Choose an orthonormal basis $\{e_1, \cdots, e_n\}$ for $T_p M$ and $\{e_{n+1}, \cdots, e_{2m}\}$ for the normal space $T_p^\perp M$ such that the mean curvature
vector $H$ is in the direction of the normal vector to $e_{n+1}$. We put
\[ a_i = \sigma_{ii}^{n+1} = \langle \sigma(e_i, e_i), e_{n+1} \rangle, \quad i = 1, \cdots, n, \]
\[ b_1 = a_1, \quad b_2 = a_2 + \cdots + a_n, \quad b_3 = a_{n+1} + \cdots + a_{n+n_2}, \cdots, \]
\[ b_{k+1} = a_{n_1+\cdots+n_{k-1}+1} + \cdots + a_{n_1+n_2+\cdots+n_{k-1}+n_k}, \]
\[ b_{k+2} = a_{n_1+\cdots+n_k+1}, \cdots, b_{n+1} = a_n, \]
and denote by $D_j$, $j = 1, \cdots, k$ the sets
\[ D_1 = \{1, \cdots, n_1\}, \quad D_2 = \{n_1 + 1, \cdots, n_1 + n_2\}, \cdots, \]
\[ D_k = \{(n_1 + \cdots + n_{k-1}) + 1, \cdots, (n_1 + \cdots + n_{k-1}) + n_k\}. \]
Let $L_1, \cdots, L_k$ be $k$ mutually orthogonal subspaces of $T_p M$, $\dim L_j$ $n_j$, defined by
\[ L_1 = \text{span}\{e_1, \cdots, e_{n_1}\}, \quad L_2 = \text{span}\{e_{n_1+1}, \cdots, e_{n_1+n_2}\}, \cdots, \]
\[ L_k = \text{span}\{e_{n_1+\cdots+n_{k-1}+1}, \cdots, e_{n_1+\cdots+n_{k-1}+n_k}\}. \]

From (9) it follows that
\[ \tau(L_j) = \frac{1}{2} \left\{ n_j \left(n_j - 1\right)f_1 + 6f_2 \Psi(L_j) \right\} \]
\[ + \sum_{r=n+1}^{2m} \sum_{\alpha_j < \beta_j} \left[ \sigma_{\alpha_j}^r \sigma_{\beta_j}^r - (\sigma_{\alpha_j}^r)^2 \right]. \]
(24)

We rewrite (10) as
\[ n^2 \|H\|^2 = \left(\|\sigma\|^2 + \eta\right) \gamma, \]
(25)

or equivalently,
\[ \left(\sum_{i=1}^{n} \sigma_{ii}^{n+1}\right)^2 = \gamma \left(\sum_{i=1}^{n} (\sigma_{ii}^{n+1})^2 + \sum_{i \neq j} (\sigma_{ij}^{n+1})^2 \right) \]
\[ + \sum_{r=n+1}^{2m} \sum_{i,j=1}^{n} (\sigma_{ij}^r)^2 + \eta \]
where
\[ \eta = 2\tau - 2b(n_1, \cdots, n_k) \|H\|^2 - n(n-1)f_1 - 3f_2 \|P\|^2, \]
(27)
\[ \gamma = n + k - \sum_{j=1}^{k} n_j. \]
(28)
The relation (26) implies that
\[
\left( \sum_{i=1}^{\gamma+1} b_i \right)^2 = \gamma \left[ \eta + \sum_{i=1}^{\gamma+1} b_i^2 + \sum_{i \neq j} (\sigma^{r+1}_{ij})^2 + \sum_{r=n+2}^{2m} \sum_{i=1}^{n} (\sigma^r_{ij})^2 - 2 \sum_{\alpha_1 < \beta_1} a_{\alpha_1} a_{\beta_1} - \cdots - 2 \sum_{\alpha_k < \beta_k} a_{\alpha_k} a_{\beta_k} \right],
\]
with \(\alpha_j, \beta_j \in D_j\), for all \(j = 1, \cdots, k\). Applying Lemma 3.1 to the above relation, we have
\[
\sum_{\alpha_1 < \beta_1} a_{\alpha_1} a_{\beta_1} + \cdots + \sum_{\alpha_k < \beta_k} a_{\alpha_k} a_{\beta_k} \geq \frac{1}{2} \left[ \eta + \sum_{i \neq j} (\sigma^{r+1}_{ij})^2 + \sum_{r=n+2}^{2m} \sum_{i=1}^{n} (\sigma^r_{ij})^2 \right],
\]
which implies that
\[
\sum_{j=1}^{k} \sum_{r=n+1}^{2m} \sum_{\alpha_j < \beta_j} \left[ \sigma^r_{\alpha_j \alpha_j} \sigma^r_{\beta_j \beta_j} - (\sigma^r_{\alpha_j \beta_j})^2 \right] \geq \frac{\eta}{2} + \frac{1}{2} \sum_{\alpha_j \beta_j} \left( \sigma^r_{\alpha_j \beta_j} \right)^2 + \sum_{r=n+2}^{2m} \sum_{\alpha_j \in D_j} \left( \sigma^r_{\alpha_j \alpha_j} \right)^2,
\]
where we denote by \(D^2 = (D_1 \times D_1) \cup \cdots \cup (D_k \times D_k)\). Thus, we have
\[
\frac{\eta}{2} \leq \sum_{j=1}^{k} \sum_{r=n+1}^{2m} \sum_{\alpha_j < \beta_j} \left[ \sigma^r_{\alpha_j \alpha_j} \sigma^r_{\beta_j \beta_j} - (\sigma^r_{\alpha_j \beta_j})^2 \right].
\]

From (6), (24), (27) and (29), we obtain (23). \(\Box\)

B.-Y. Chen gave a general inequality for submanifolds in real space forms as follows ([5]).

**Theorem 3.7.** For any \(n\)-dimensional submanifold \(M\) in a real space form \(R(c)\) and for any \(k\)-tuple \((n_1, \cdots, n_k) \in S(n)\), regardless of dimension and codimension, we have
\[
\delta (n_1, \cdots, n_k) \leq b (n_1, \cdots, n_k) ||H||^2 + a (n_1, \cdots, n_k) c.
\]

We extend the above result for submanifolds in a generalized complex space forms.

**Theorem 3.8.** Given an \(n\)-dimensional \((n \geq 3)\) \(\theta\)-slant submanifold \(M\), of a \(2m\)-dimensional generalized complex space form \(\tilde{M}(f_1, f_2)\), we
422 Jeong-Sik Kim, Yeong-Moo Song and Mukut Mani Tripathi

have

\[
\delta(n_1, \ldots, n_k) \leq b(n_1, \ldots, n_k) \|H\|^2 + a(n_1, \ldots, n_k) f_1 + \frac{3}{2} \left( n - 2 \sum_{j=1}^{k} \left\lfloor \frac{n_j}{2} \right\rfloor \right) f_2 \cos^2 \theta.
\]

(31)

Proof. Let \( M \) be an \( n \)-dimensional \( \theta \)-slant submanifold \( M \) in an almost Hermitian manifold with \( n \geq 3 \), \( n = 2l \). Then, we have \( \|P\|^2 = n \cos^2 \theta \). Let \( L_1, \ldots, L_k \) be \( k \) mutually orthogonal subspaces of \( T_pM \) with \( \dim L_j = n_j \). Let \( \{e_1, \ldots, e_n\} \) be an orthonormal basis of \( T_pM \), such that

\[
L_1 = \text{span}\{e_1, \ldots, e_{n_1}\}, \quad L_2 = \text{span}\{e_{n_1+1}, \ldots, e_{n_1+n_2}\}, \ldots, \\
L_k = \text{span}\{e_{n_1+\ldots+n_{k-1}+1}, \ldots, e_{n_1+\ldots+n_{k-1}+n_k}\}.
\]

Choosing \( e_2 = \sec \theta Pe_1, \ldots, e_{2k} = \sec \theta Pe_{2l-1} \), for \( i = 1, 3, \ldots, 2l - 1 \), we get \( \langle Pe_i, e_{i+1} \rangle = \cos \theta \). Thus, it follows that \( \Psi(L_j) = \left[ \frac{n_j}{2} \right] \cos^2 \theta \) for all \( j = 1, \ldots, k \). Putting these values of \( \|P\|^2 \) and \( \Psi(L_j) \) in (23), we get (31).

Corollary 3.9. For a submanifold \( M \) in a manifold \( \tilde{M} \), we have the following table

<table>
<thead>
<tr>
<th>( \tilde{M} )</th>
<th>( M )</th>
<th>Inequality</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{M}(f_1, f_2) )</td>
<td>TR</td>
<td>( \delta(n_1, \ldots, n_k) \leq b(n_1, \ldots, n_k) |H|^2 + a(n_1, \ldots, n_k) f_1 + \frac{3}{2} \left( n - 2 \sum_{j=1}^{k} \left\lfloor \frac{n_j}{2} \right\rfloor \right) f_2 \cos^2 \theta )</td>
</tr>
<tr>
<td>( \tilde{M}(f_1, f_2) )</td>
<td>I</td>
<td>( \delta(n_1, \ldots, n_k) \leq b(n_1, \ldots, n_k) |H|^2 + a(n_1, \ldots, n_k) f_1 + \frac{3}{2} \left( n - 2 \sum_{j=1}^{k} \left\lfloor \frac{n_j}{2} \right\rfloor \right) f_2 \cos^2 \theta )</td>
</tr>
<tr>
<td>( \tilde{M}(c, \alpha) )</td>
<td>S</td>
<td>( \delta(n_1, \ldots, n_k) \leq b(n_1, \ldots, n_k) |H|^2 + a(n_1, \ldots, n_k) \left( c + \alpha \right) \left( \frac{c + \alpha}{4} \right) )</td>
</tr>
<tr>
<td>( \tilde{M}(c, \alpha) )</td>
<td>TR</td>
<td>( \delta(n_1, \ldots, n_k) \leq b(n_1, \ldots, n_k) |H|^2 + a(n_1, \ldots, n_k) \left( c + \alpha \right) \left( \frac{c + \alpha}{4} \right) )</td>
</tr>
<tr>
<td>( \tilde{M}(c, \alpha) )</td>
<td>I</td>
<td>( \delta(n_1, \ldots, n_k) \leq b(n_1, \ldots, n_k) |H|^2 + a(n_1, \ldots, n_k) \left( c + \alpha \right) \left( \frac{c + \alpha}{4} \right) )</td>
</tr>
<tr>
<td>( \tilde{M}(c) )</td>
<td>S</td>
<td>( \delta(n_1, \ldots, n_k) \leq b(n_1, \ldots, n_k) |H|^2 + a(n_1, \ldots, n_k) c )</td>
</tr>
<tr>
<td>( \tilde{M}(c) )</td>
<td>TR</td>
<td>( \delta(n_1, \ldots, n_k) \leq b(n_1, \ldots, n_k) |H|^2 + a(n_1, \ldots, n_k) c )</td>
</tr>
<tr>
<td>( M(c) )</td>
<td>I</td>
<td>( \delta(n_1, \ldots, n_k) \leq b(n_1, \ldots, n_k) |H|^2 + a(n_1, \ldots, n_k) c )</td>
</tr>
<tr>
<td>( R(c) )</td>
<td></td>
<td>( \delta(n_1, \ldots, n_k) \leq b(n_1, \ldots, n_k) |H|^2 + a(n_1, \ldots, n_k) c )</td>
</tr>
</tbody>
</table>

where “TR”, “I” and “S” stand for totally real, invariant and \( \theta \)-slant respectively.
References


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