ON THE SPECIAL FINSLER METRIC

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Abstract. Given a Riemannian manifold \((M, \alpha)\) with an almost Hermitian structure \(f\) and a non-vanishing covariant vector field \(b\), consider the generalized Randers metric \(L = \alpha + \beta\), where \(\beta\) is a special singular Riemannian metric defined by \(b\) and \(f\). This metric \(L\) is called an \((a, b, f)\)-metric. We compute the inverse and the determinant of the fundamental tensor \((g_{ij})\) of an \((a, b, f)\)-metric. Then we determine the maximal domain \(D\) of \(TM \setminus O\) for an \((a, b, f)\)-manifold where a \(\gamma\)-local Finsler structure \(L\) is defined. And then we show that any \((a, b, f)\)-manifold is quasi-C-reducible and find a condition under which an \((a, b, f)\)-manifold is C-reducible.

1. Introduction

Let \(M\) be a smooth \(2m\)-dimensional manifold. We will consider a Finsler metric \(L = \alpha + \beta\), where \(\alpha\) is a Riemannian metric on \(M\) and \(\beta\) is a singular Riemannian metric on \(M\). We call such a Finsler metric a generalized Randers metric. In case where \(\beta\) is a 1-form on \(M\), \(L\) is a usual Randers metric.

We denote a point of \(M\) by \(x = (x^i)\) and a tangent vector at that point \(x\) by \(y = (y^j)\). Let \(\alpha(x, y) = (a_{ij}(x)y^i y^j)^{1/2}\) be a Riemannian metric on \(M\). Given an almost Hermitian structure \(f^i_j(x)\) of \((M, \alpha)\) and a non-vanishing covariant vector field \(b_i(x)\) on \(M\), we have a singular Riemannian metric \(\beta(x, y) = (b_{ij}(x)y^i y^j)^{1/2}\), where \(b_{ij} = b_i b_j + f_i f_j\) and \(f_i = b_i f^i\). Such \(L = \alpha + \beta\) is an interesting example of a generalized Randers metric, which we call an \((a, b, f)\)-metric. For the further study about \((a, b, f)\)-metrics, we refer to [3] and [4].

Received November 6, 2002.
2000 Mathematics Subject Classification: Primary 53B40; Secondary 53C60, 58B20.
Key words and phrases: Finsler metric, generalized Randers metric, \((a, b, f)\)-metric, Rizza manifold, C-reducible.

Partially supported by the University of Seoul, 2000.
Note that a manifold with an \((a, b, f)\)-metric becomes a Rizza manifold. A Rizza manifold \((M, L, f)\) is by definition a Finsler manifold \((M, L)\) with an almost complex structure \(f_j^i(x)\) satisfying the condition
\[
L(x, \phi \theta(y)) = L(x, y),
\]
where \(\phi \theta = \cos \theta \cdot \delta_j^i + \sin \theta \cdot f_j^i\). But an \((a, b, f)\)-metric is not a \(y\)-global Finsler metric. And so we have to restrict a domain in the tangent bundle \(T(M)\) over \(M\), say, \(\{y : \beta(y) \neq 0\}\). In section 4, we show that the \(n \times n\) Hessian matrix \((g_{ij}) := (\frac{1}{2}L^2)_{y_i y_j}\) is positive definite on \(\{y : \beta(y) \neq 0\}\) by checking the sign of determinant of \((g_{ij})\). For this purpose, we compute the determinant of \((g_{ij})\).

It is interesting and valuable to study Finsler space with some important tensors of special form. For example, M. Matsumoto[6] initiated the study of a Finsler metric whose Cartan tensor \(A_{ijk} := \frac{L^2}{\beta}(L^2)_{y_i y_j y_k}\) satisfies
\[
A_{ijk} = \mathcal{G}_{(ijk)}\{Q_{ij} R_k\},
\]
where \(Q_{ij}\) is a symmetric Finsler tensor field satisfying \(Q_{ij} y^j = 0\) and \(R_k\) is assumed to satisfy \(R_k y^k = 0\). Here we use the notation \(\mathcal{G}_{(ijk)}\) to denote the summation of the cyclic permutation of indices \(i, j, k\), i.e.,
\[
\mathcal{G}_{(ijk)}\{S_{ijk}\} = S_{ijk} + S_{jki} + S_{kij}.
\]
In case \(R_k = A_k\) with \(A_k := g^{ij} A_{ijk}\), the Finsler manifold is called quasi-C-reducible. Furthermore, if \(Q_{ij} = \frac{1}{n+1} h_{ij}\) where \(h_{ij}\) is the angular metric \(h_{ij} := g_{ij} - L_i L_j\), we call the Finsler manifold to be \(C\)-reducible. In section 4, we show that any \((a, b, f)\)-manifold is quasi-C-reducible and find a sufficient condition that an \((a, b, f)\)-manifold is \(C\)-reducible. To get \(A_k\), we compute the inverse \((g^{ij})\) of \((g_{ij})\).

2. Preliminaries

Let \((M, \alpha)\) be a \(2m\)-dimensional Riemannian manifold and let \(f_j^i(x)\) be an almost Hermitian structure of \((M, \alpha)\). For a non-vanishing covariant vector field \(b_i(x)\) on \(M\), we have a singular Riemannian metric
\[
\beta(x, y) = (h_{ij}(x) y_i y_j)^{1/2},
\]
where \(b_{ij} = b_i b_j + f_i f_j\), \(f_i = b_i f_i\) and we consider a generalized Randers metric \(L = \alpha + \beta\). Such a generalized Randers metric \(L = \alpha + \beta\) is called an \((a, b, f)\)-metric and \((M, L)\) an \((a, b, f)\)-manifold.

Recall the definition of a \(y\)-global Finsler metric \(F\) on \(M\).
Definition 2.1. A \( y \)-global Finsler metric on \( M \) is a function \( F : TM \to \mathbb{R} \) such that

(P1) Nonnegativity: \( F \geq 0 \) on \( TM \).
(P2) Regularity: \( F \) is smooth on \( TM \setminus O \).
(P3) Absolute homogeneity: \( F(x, \lambda y) = |\lambda|F(x, y) \) for all \( \lambda \in \mathbb{R} \).
(P4) Strong convexity: The \( n \times n \) Hessian matrix \( (g_{ij}) := ((\frac{1}{2}F^2)_{ij}) \) is positive definite at every point of \( TM \setminus O \).

Note that for the most important physical applications, the assumptions are too restrictive. And so we have to consider a \( y \)-local Finsler structure \( F \) defined only on a domain \( \mathcal{D} \) of \( TM \setminus O \) with \( \mathcal{D} \cap T_x M \neq \emptyset \) for every \( x \in M \).

Now we find the maximal domain \( \mathcal{D} \) of \( TM \setminus O \) for \((a, b, f)\)-metric. Because \( L(y) = \alpha(y) + \beta(y) \) is positive for any \( y \in TM \setminus O \) and both \( \alpha \) and \( \beta \) are regular away from \( \{y : \beta(y) = 0\} = \ker B \) with \( B = (b_{ij}) \), our possible domain \( \mathcal{D} \) is the complement \( \mathcal{C}(\ker B) \) of \( \ker B \). In section 4, we show that \( (g_{ij}) \) is positive definite on \( \mathcal{C}(\ker B) \), i.e., all the eigenvalues of \( (g_{ij}) \) are positive on \( \mathcal{C}(\ker B) \).

We use the following lemma extensively in the next section. For its proof, see [1].

Lemma 2.1. Let \( (P_{ij}) \) be a real symmetric non-singular matrix with the inverse \( (P'^{ij}) \). And let \( (Q_{ij}) = (P_{ij} \pm c_{ij}c_{ij}) \) with \( 1 \pm c^2 \neq 0 \) and \( c^2 := c_iP^{ij}c_j \). Then the matrix \( (Q_{ij}) \) is non-singular and its inverse is \( (Q'^{ij}) = (P'^{ij} \pm \frac{1}{1 \pm c^2}c'i c') \) where \( c' = P'^{ij}c_j \) and \( \det(Q_{ij}) = (1 \pm c^2) \det(P_{ij}) \).

3. The computation of the determinant and the inverse of \((g_{ij})\)

In this section, we compute the inverse and the determinant of the fundamental tensor \((g_{ij})\) of \((a, b, f)\)-metric. Here we assume that \( y \in \mathcal{C}\ker B \).

For \( L = \alpha + \beta \), we have

\[
g_{ij} = \frac{L}{\alpha}a_{ij} + \frac{L}{\beta}b_i b_j + \frac{L}{\beta} f_i f_j + L_i L_j - \frac{L}{\alpha} \alpha_i \alpha_j - \frac{L}{\beta} \beta_i \beta_j,
\]

where \( \alpha_i = \frac{\partial \alpha}{\partial y^i} \), \( \beta_i = \frac{\partial \beta}{\partial y^i} \), \( L_i = \alpha_i + \beta_i \). We put \( \alpha^i = a^{ir} \alpha_r \), \( \beta^i = a^{ir} \beta_r \), \( b^i = a^{ij} b_j \) and \( f^i = a^{ij} f_j \). Then we can apply Lemma 2.1 to \((g_{ij})\) five times.
PROPOSITION 3.1. For the fundamental tensor \( (g_{ij}) \) of an \((a, b, f)\)-
metric \( L = \alpha + \beta \), the determinant of \((g_{ij})\) is

\[
\det(g_{ij}) = \frac{L\gamma}{\alpha\beta} \det A
\]

and the inverse \((g^{ij})\) of \((g_{ij})\) is given by

\[
g^{ij} = \frac{\alpha}{L} a^{ij} - \frac{\alpha^2}{\gamma L} b^{ij} + \frac{\alpha^2\gamma}{L^3} \alpha^i \alpha^j - \frac{\alpha}{L^2} (\alpha^i \beta^j + \alpha^j \beta^i) + \frac{\alpha^2}{L\gamma} \beta^i \beta^j,
\]

where \( A = \left( \frac{\gamma}{\alpha} a_{ij} \right) \), \( \gamma = \beta + b^2 \alpha \), \( b^{ij} = b^i b^j + f^i f^j \).

Proof. First, we set

\[
P_{ij} = -\frac{L}{\alpha} a_{ij}, \quad c_{1i} = \sqrt{\frac{\gamma}{\beta}} b_i \quad \text{and} \quad (Q_1)_{ij} = \frac{L}{\alpha} a_{ij} + \frac{L}{\beta} b_i b_j.
\]

Note that \( c_1^2 = c_{1i} P^{ij} c_{1j} = \frac{\gamma}{\beta} b^2 \), where \( b^2 = a^{ij} b_i b_j \) and \((a^{ij})\) is the inverse of \((a_{ij})\). And note also that \( b^2 = a^{ij} b_i b_j \) is positive, because \((a_{ij})\) is positive definite. In particular, the quantity \( 1 + c_1^2 = \frac{\gamma}{\beta} > 0 \), where \( \gamma = \beta + b^2 \alpha > 0 \). By Lemma 2.1, we have

\[
\det Q_1 = \frac{\gamma}{\beta} \det \left( \frac{L}{\alpha} a_{ij} \right) = \frac{\gamma}{\beta} \det A,
\]

\[
(Q_1)_{ij} = \frac{\alpha}{L} a^{ij} - \frac{\alpha^2}{\gamma L} b^i b^j.
\]

Secondly, let

\[
(Q_2)_{ij} = (Q_1)_{ij} + \frac{L}{\beta} f_i f_j, \quad c_{2i} = \sqrt{\frac{L}{\beta}} f_i
\]

and apply Lemma 2.1 in the same way. Then we have \( c_2^2 = c_{2i} (Q_1)_{ij} c_{2j} = \frac{\gamma}{\beta} b^2, 1 + c_2^2 = \frac{\gamma}{\beta} > 0 \). And Lemma 2.1 says that

\[
\det Q_2 = \frac{\gamma}{\beta} \det Q_1 = \frac{\gamma^2}{\beta^2} \det A,
\]

\[
(Q_2)_{ij} = \frac{\alpha}{L} a^{ij} - \frac{\alpha^2}{\gamma L} b^i b^j.
\]

Thirdly, let

\[
(Q_3)_{ij} = (Q_2)_{ij} + L_i L_j, \quad c_{3i} = L_i.
\]
Then we have \( c_3^2 = c_{3i}(Q_2)_{ij} c_{3j} = 1 \), 1 + \( c_3^2 = 2 \). And by Lemma 2.1,

\[
\det Q_3 = \frac{2\gamma}{\beta^2} \det A,
\]

\[
(Q_3)_{ij} = \frac{\alpha}{L} a_{ij} - \frac{\alpha^2}{\gamma L} b_{ij} - \frac{1}{2L^2} y^i y^j.
\]

Fourthly, let

\[
(Q_4)_{ij} = (Q_3)_{ij} - \frac{L}{\beta} \beta_i \beta_j, \quad c_{4i} = \sqrt{\frac{L}{\beta}} \beta_i.
\]

Then we have \( c_4^2 = c_{4i}(Q_3)_{ij} c_{4j} = \frac{1}{\beta} \left( b^2 \alpha - \frac{b^2 \alpha^2}{\gamma} - \frac{\beta^2}{2L} \right), 1 - c_4^2 = \frac{\beta(2L + \gamma)}{2L \gamma} > 0 \). And by Lemma 2.1,

\[
\det Q_4 = \frac{(2L + \gamma) e}{L \beta} \det A,
\]

\[
(Q_4)_{ij} = \frac{\alpha L}{\alpha} a_{ij} - \frac{\alpha^2}{\gamma L} b_{ij} - \frac{1}{(2L + \gamma)} y^i y^j
\]

\[- \frac{\alpha}{L(2L + \gamma) \beta} \left( a^i b_k y^j y^j + y^i y^j b_i k \alpha_{ij} \right)
\]

\[+ \frac{2\alpha^2}{(2L + \gamma) \beta^2} a^i b_k y^j y^m b_m n a_{nj}.\]

Finally, let

\[
g_{ij} = (Q_4)_{ij} - \frac{L}{\alpha} \alpha_i \alpha_j, \quad c_{5i} = \sqrt{\frac{L}{\alpha}} \alpha_i.
\]

Then we get \( c_5^2 = c_{5i}(Q_4)_{ij} c_{5j} = \frac{L + \gamma}{2L + \gamma} - \frac{L \beta}{\alpha(2L + \gamma)}, 1 - c_5^2 = \frac{L^2}{\alpha(2L + \gamma)} > 0 \). And by Lemma 2.1,

\[
\det(g_{ij}) = \frac{L^2}{\alpha(2L + \gamma)} \cdot \frac{(2L + \gamma) e}{L \beta} \det A = \frac{L \gamma}{\beta} \det A,
\]

\[
g_{ij} = \frac{\alpha}{L} a_{ij} - \frac{\alpha^2}{\gamma L} b_{ij} + \frac{\gamma}{L^3} y^i y^j
\]

\[- \frac{\alpha}{L^2 \beta} \left( a^i b_k y^j y^j + y^i y^j b_i k \alpha_{ij} \right) + \frac{\alpha^2}{L \beta^2} a^i b_k y^j y^m b_m n a_{nj}.\]

If we set \( \alpha^i = \frac{y^i}{\alpha} \) and \( \beta^i = \frac{a^i \rho}{\beta} \), then the last equation yields equation (3.1). \( \square \)
4. Theorems

In this section, with the aid of Proposition 3.1, we show the positivity of $g_{ij}$ and the quasi-C-reducibility of an $(a, b, f)$-metric and find a sufficient condition of being C-reducible.

Now we are ready to prove that $(g_{ij})$ is positive definite on $\mathcal{C}(\ker B)$. This implies that $\mathcal{C}(\ker B)$ is the maximal domain $\mathcal{D}$ of $TM \setminus O$ for an $(a, b, f)$-manifold where a $y$-local Finsler structure $L$ is defined.

**Theorem 4.1.** $(g_{ij})$ is positive definite on $\mathcal{C}(\ker B)$.

**Proof.** Consider a one-parameter family of the $(a, b, f)$-metric $L^\epsilon = \alpha + \epsilon \beta$ with $0 \leq \epsilon \leq 1$. Let $g^\epsilon$ be the fundamental tensor of $L^\epsilon$. For $\epsilon > 0$, by Proposition 3.1, we have

$$\det(g^\epsilon_{ij}) = \frac{L^\epsilon \gamma^\epsilon}{\epsilon \alpha \beta} \det A^\epsilon,$$

where $A^\epsilon = (\frac{L^\epsilon}{\alpha} a_{ij})$, $\gamma^\epsilon = \epsilon \beta + \epsilon^2 b^2 \alpha > 0$, and so $\det(g^\epsilon_{ij})$ is positive. In particular, none of the eigenvalues of $(g^\epsilon_{ij})$ can vanish. For $\epsilon = 0$, $L^0 = \alpha$ and all the eigenvalues of $(g^0_{ij}) = (g_{ij})$ are positive. Since $\det(g^\epsilon_{ij})$ is continuous for $\epsilon$, all the eigenvalues of $(g^\epsilon_{ij})$ are positive by the intermediate value theorem. And so all the eigenvalues of $(g_{ij})$ are positive. This means that $(g_{ij})$ is positive definite.

Next, we show that $(a, b, f)$-manifolds are quasi-C-reducible and we determine a sufficient condition under which $(a, b, f)$-manifolds are C-reducible. We start with the definitions of quasi-C-reducibility and of C-reducibility.

**Definition 4.1.** A Finsler manifold of dimension $n$, $n \geq 3$, is quasi-C-reducible if there exists a symmetric Finsler tensor field $Q_{ij}$ satisfying $Q_{ij}y^i = 0$ and $A_{ijk} = \mathcal{G}_{(ijk)} \{Q_{ij} A_k\}$, where $A_k := g^{ij} A_{ijk}$.

**Definition 4.2.** A Finsler manifold of dimension $n$, $n \geq 3$, is C-reducible if $A_{ijk}$ is in the form $A_{ijk} = \frac{1}{n+1} \mathcal{G}_{(ijk)} \{h_{ij} A_k\}$, where $h_{ij} := g_{ij} - L_i L_j$ is the angular metric of $L$.

Note that for $(a, b, f)$-metric, the Cartan tensor is

$$A_{ijk} := \frac{L}{4}(L^2)(y^i y^j y^k) = \frac{L}{2} (g_{ij}) y^k$$

$$= \frac{L}{2} \mathcal{G}_{(ijk)} \left\{ \left( \frac{\alpha_{ij}}{\alpha} - \frac{\beta_{ij}}{\beta} \right) (\alpha \beta_k - \beta \alpha_k) \right\}.$$
By Proposition 3.1, we get

\[ A_k = \frac{\lambda}{2} (\alpha \beta_k - \beta \alpha_k), \]

where \( \lambda = \left( \frac{n+1}{\alpha} - \frac{\alpha}{\beta^2} \right) \). Since \( \text{rank}(b_{ij}) = 2 \), \( \lambda \neq 0 \). And if we let

\[ Q_{ij} = \frac{L}{\lambda} \left( \frac{\alpha_{ij}}{\alpha} - \frac{\beta_{ij}}{\beta} \right), \]

we have \( A_{ijk} = \mathcal{G}_{(ijk)} \{ Q_{ij} A_k \} \). Because \( Q_{ij} \) is symmetric and \( Q_{ij} y^j = 0 \) by Euler’s theorem, we have

**Theorem 4.2.** (\( a, b, f \))-manifolds are quasi-C-reducible.

Since the angular metric \( h_{ij} \) for \( (a, b, f) \)-manifold is \( L \cdot (\alpha_{ij} + \beta_{ij}) \), we can conclude

**Theorem 4.3.** If an \( (a, b, f) \)-metric \( L = \alpha + \beta \) satisfies

\[ \frac{\alpha_{ij}}{\alpha} - \frac{\beta_{ij}}{\beta} = \frac{\lambda}{n+1} (\alpha_{ij} + \beta_{ij}), \]

or equivalently \( \beta^2 \alpha \alpha_{ij} = (n \gamma + \beta) \beta_{ij} \), then the \( (a, b, f) \)-manifold is C-reducible.

**Remark.** If \( A_i = 0 \) for a C-reducible manifold, then \( A_{ijk} = 0 \) immediately. And so the manifold is Riemannian. For a C-reducible \( (a, b, f) \)-manifold with \( A_i = 0 \), we can show that

\[ g_{ij}(x) = g_{pq}(x) f_i^p f_j^q. \]

In other words, such an \( (a, b, f) \)-manifold is an almost Hermitian manifold. For its proof, we refer the readers to [2].

**Acknowledgement.** The author would like to thank Prof. M. Hashiguchi for his valuable comments on the use of notations. This simplifies the appearance of many equations.

**References**


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