FIXED POINTS OF SEQUENTIALLY
CONDENSING OPERATORS

IN-SOOK KIM

ABSTRACT. Introducing the concept of a sequentially condensing operator in a more general framework, we give a new fixed point theorem for sequentially condensing operators in Banach spaces, with the aid of attractors.

1. Introduction

A fixed point theorem for condensing operators goes back to B. N. Sadovskii [8]; see also [1, 2]. S. J. Daher [3] gave the following definition of a sequentially condensing operator which generalizes the one of a condensing operator in [8].

DEFINITION 1. Let \( E \) be a Banach space and \( \alpha \) the Hausdorff measure of noncompactness on \( E \). A continuous operator \( f : E \to E \) is said to be sequentially condensing if for every bounded \( Kf \subset E \) with \( \alpha(Kf) > 0 \) and \( f(Kf) \) bounded, \( f \) satisfies the condition:

\[
\alpha(f(Kf)) < \alpha(Kf).
\]

For the notation of \( Kf \), see Definition 3 below. Here \( \alpha(A) \) of a set \( A \) is the infimum of the numbers \( \varepsilon > 0 \) such that \( A \) can be covered by a finite number of closed balls in \( E \) with radius \( \varepsilon \).

A fixed point theorem was established in [3] as follows:

THEOREM A. Suppose that \( K \subset B \subset S \subset E \) are convex sets in a Banach space \( E \) such that \( K \) is nonempty and compact, \( B \) is relatively
open in \( S \), and \( S \) is bounded and closed in \( E \). Let \( f : S \to E \) be a sequentially condensing operator such that \( f^j(B) \subset S \) for all \( j \geq 0 \) and \( K \) attracts compact sets of \( B \). Then \( f \) has a fixed point.

In this paper, we introduce the concept of a sequentially \( \gamma \)-condensing operator in a more general framework to extend Theorem A. For this, we mainly follow the basic idea of the proofs of related results in [3] and [5], although we salvage the proofs in [3], for instance, see Lemma 1 and Lemma 4 below.

**Definition 2.** Let \( E \) be a topological vector space and \( M \) a collection of nonempty subsets of \( E \) containing all precompact subsets of \( E \) with the property that for any \( M \in M \), the closure \( \overline{M} \) and the convex hull \( \text{co} M \) belong to \( M \). A nonnegative real-valued function \( \gamma : M \to [0, \infty) \) is called a measure of noncompactness on \( E \) if the following conditions hold for any \( M \in M \):

1. \( \gamma(\overline{M}) = \gamma(M) = \gamma(\text{co} M) \); and
2. if \( M \subset E \) is precompact, then \( \gamma(M) = 0 \).

The measure \( \gamma \) of noncompactness on \( E \) is said to be regular provided that \( \gamma(M) = 0 \) if and only if \( M \) is precompact; see [7].

**Definition 3.** Let \( X \) be a nonempty subset of a locally convex topological vector space \( E \) and \( f : X \to E \) an operator. For a nonempty compact subset \( K \) of \( X \) let

\[
Kf := \bigcup_{n \geq 0} \{ K_n : K_0 = \text{co} K, K_n = \text{co} f(K \cap K_{n-1}) \}.
\]

Given a measure \( \gamma \) of noncompactness on \( E \), a continuous operator \( f : X \to E \) is said to be sequentially \( \gamma \)-condensing provided that if \( Kf \) is any subset of \( X \) such that \( \gamma(Kf) \leq \gamma(f(Kf)) \), then \( f(Kf) \) is relatively compact.

In case where \( \alpha \) is the Hausdorff measure of noncompactness on a Banach space \( E \), a sequentially condensing operator \( f : E \to E \) is sequentially \( \alpha \)-condensing. Also some examples of sequentially \( \alpha \)-condensing operator can be found in [3].

2. Main result

We begin with the following lemma which is a corrected version of [3, Lemma 3].
LEMMA 1. Let $X$ be a nonempty convex subset of a locally convex topological vector space, $f : X \to X$ an operator, and $K$ a nonempty compact subset of $X$. Then $\overline{\overline{coKf}} = \overline{coK \cup \overline{co}f(Kf)}$.

Proof. Since $Kf = \bigcup_{n \geq 0} K_n$, where $K_0 = coK$ and $K_n = cof(K_{n-1})$, we have $f(Kf) \subset Kf$ and $cof(Kf) \supset \bigcup_{n \geq 1} K_n$. From $coK \cup \overline{co}f(Kf) \supset K \cup f(Kf)$ it follows that

$$\overline{co(K \cup \overline{co}f(Kf))} \supset \overline{coKf} \supset \overline{co}f(Kf)$$

and

$$\overline{coKf} \supset \overline{co}f(Kf).$$

Therefore we conclude that $\overline{coKf} = \overline{co(K \cup \overline{co}f(Kf))}$. This completes the proof. \qed

For our aim, we need several auxiliary facts one of which is given in [3, Lemma 6], with the aid of attractors.

DEFINITION 4. Let $E$ be a metrizable locally convex topological vector space. For a given continuous operator $f : E \to E$, we say that a set $K \subset E$ attracts a set $H \subset E$ if for any $\varepsilon > 0$, there is an integer $N(H, \varepsilon)$ such that $f^n(H) \subset B_\varepsilon(K)$ for all $n \geq N(H, \varepsilon)$, where $B_\varepsilon(K)$ is the $\varepsilon$-neighborhood of $K$. If $K$ attracts each compact set $H \subset E$, we say that $K$ attracts compact sets of $E$; see [5]. In particular, $K$ attracts points of $B \subset E$ if for every $x \in B$ and for every $\varepsilon > 0$, there is an integer $N(x, \varepsilon)$ such that $f^n(x) \in K + B_\varepsilon(0)$ for all $n \geq N(x, \varepsilon)$.

LEMMA 2. Suppose that $K \subset B \subset S \subset E$ are subsets of a metrizable locally convex topological vector space $(E, d)$ such that $K$ is nonempty and compact, $B$ is convex and relatively open in $S$, and $S$ is closed in $E$. Let $f : S \to E$ be a continuous operator such that $f^j(B) \subset S$ for all $j \geq 0$ and $K$ attracts points of $B$. Then there is a nonempty compact set $C$ with $C \subset K$ such that $f(C) = C$.

Proof. Fix $x \in B$ and let

$$C = \bigcap_{i \geq 1} \bigcup_{n \geq i} f^n(x).$$

Since $K$ attracts points of $B$, for every $\varepsilon > 0$, there is a positive integer $N(\varepsilon)$ such that

$$\{f^n(x) : n \geq N(\varepsilon)\} \subset K + B_\varepsilon(0).$$
Hence it follows from the closedness of $K$ that
\[ \bigcap_{\varepsilon > 0} \bigcup_{i \geq 1} f^n(x) \subseteq \bigcap_{\varepsilon > 0} K + B_\varepsilon(0) \subseteq \bigcap_{\varepsilon > 0} K + B_\varepsilon(0) = \overline{K} = K. \]

Thus, $C \subseteq K$ and $C$ is compact because $C$ is a closed subset of the compact set $K$. Moreover, since $f$ is continuous, we have
\[ f(C) \subseteq \bigcap_{n \geq 1} f(\bigcup_{i \geq 1} f^n(x)) \subseteq \bigcap_{n \geq 1} \bigcup_{i \geq 1} f^{n+1}(x) = C. \]

Now it remains to show that $C \subseteq f(C)$. Let $z \in C$ be arbitrary and set $A_i = \bigcup_{n \geq 1} f^n(x)$ for $i \geq 1$. Then there exists a subsequence $\{f^{n_i}(x)\}$ of $\{f^n(x)\}$ with $f^{n_i}(x) \in A_i$ such that
\[ z = \lim_{i \to \infty} f^{n_i}(x). \]

In fact, since $z \in A_i$ for all $i \in \mathbb{N}$, if $i = 1$, there is an integer $n_1 \geq 1$ such that
\[ d(f^{n_1}(x), z) < 1; \]
if $i = n_1$, there is an integer $n_2$ with $n_2 > n_1$ such that
\[ d(f^{n_2}(x), z) < \frac{1}{2}. \]

By induction, there exists a subsequence $\{f^{n_i}(x)\}$ with $f^{n_i}(x) \in A_i$ such that
\[ d(f^{n_i}(x), z) < \frac{1}{i} \quad \text{for every } i \in \mathbb{N} \]
and so $\lim_{i \to \infty} f^{n_i}(x) = z$.

Since $K$ attracts points of $B$, we have
\[ d(f^{n_i-1}(x), K) \to 0 \quad \text{as } i \to \infty, \]
where $d(y, K) = \inf\{d(y, p) : p \in K\}$. The compactness of $K$ implies that
\[ K \cap \{f^{n_i-1}(x) : i \in \mathbb{N}\} \neq \emptyset; \]
see [5, Lemma 4]. Hence there exists a subsequence of $\{f^{n_i-1}(x)\}$ which converges to some point $u \in K$. Without loss of generality we may suppose that the subsequence of $\{f^{n_i-1}(x)\}$ is again denoted by $\{f^{n_i-1}(x)\}$.
Since \( f \) is continuous, we have \( f(u) = z \). From \( f^{n_i-1}(x) \in A_{i-1} \) it follows that
\[
    u = \lim_{i \to \infty} f^{n_i-1}(x) \in \bigcap_{i \geq 2} A_{i-1} = C
\]
and hence \( z = f(u) \in f(C) \). It has been proved that \( C \subset f(C) \). This completes the proof. \( \square \)

Now the proof of the next lemma is exactly the same as the one given for sequentially condensing operators in Banach spaces; see [3, Lemma 5].

**Lemma 3.** Let \( S \) be a nonempty, convex and closed subset of a locally convex topological vector space \( E \). If \( f : S \to E \) is a sequentially \( \gamma \)-condensing operator and \( r : E \to S \) is a retraction of \( E \) on \( S \), then the composition \( r \circ f \) is sequentially \( \gamma \)-condensing.

The following result is a modification of [5, Theorem 4]. Recall that a topological vector space \( E \) is quasi-complete if every bounded, closed subset of \( E \) is complete.

**Lemma 4.** Suppose that \( K \subset B \subset S \subset E \) are convex sets in a quasi-complete metrizable locally convex topological vector space \( E \) such that \( K \) is nonempty and compact, \( B \) is relatively open in \( S \), and \( S \) is closed in \( E \). Let \( f : S \to E \) be a sequentially \( \gamma \)-condensing operator such that \( f^j(B) \subset S \) for all \( j \geq 0 \) and \( K \) attracts points of \( B \), where \( \gamma \) is a regular measure of noncompactness on \( E \). Then there exists a compact convex set \( A \) with \( A \subset S \) and \( A \cap B \neq \emptyset \) such that \( f^j(A \cap B) \subset A \) for all \( j \geq 0 \).

**Proof.** Let \( C \) be a nonempty compact set in \( K \) such that \( f(C) = C \), as in Lemma 2. Define
\[
    \mathcal{F}(S) := \{ L \subset S : C \subset L, \ L \text{ is convex and closed,} \ f^j(L \cap B) \subset L \text{ for all } j \geq 0 \}.
\]
Then \( S \in \mathcal{F}(S) \). Take \( A = \bigcap \{ L : L \in \mathcal{F}(S) \} \). Note that \( C \subset A \subset S \), \( A \) is a nonempty, convex and closed set, and
\[
    f^j(A \cap B) \subset f^j(L \cap B) \subset L \quad \text{for all } j \geq 0 \text{ and for all } L \in \mathcal{F}(S).
\]
Since \( f^j(A \cap B) \subset A \) for all \( j \geq 0 \), we have \( A \in \mathcal{F}(S) \).

Consider \( C \subset K \subset B \subset S \) and an operator \( g := r \circ f : S \to E \to S \), where \( r \) is a retraction of \( E \) on \( S \). The existence of such a retraction
$r$ is assured since $S$ is a convex closed subset of a metrizable locally convex topological vector space; see [4]. By Lemma 3, $g$ is sequentially $\gamma$-condensing. Define

$$G(S) := \{G \subset S : C \subset G, \ G \text{ is convex and closed, } g(G) \subset C\}.$$ 

We can take $H = \bigcap\{G : G \in G(S)\}$. Then $H$ is convex and closed, $C \subset H \subset S$, and $g(H) \subset H$. Thus, $H \in G(S)$. Observe that $g$ is continuous and $g(C) = C$. Then $Cg = \bigcup_{n \geq 0} C_n$ and $C_{i-1} \subset C_i$ for all $i \geq 1$, where $C_0 = \text{co}C$ and $C_n = \text{co}g(C_{n-1})$. Furthermore, $Cg$ is convex, $C \subset Cg \subset H \subset S$, and $g(Cg) \subset g(Cg) \subset Cg$ by Lemma 1. Hence $Cg \in G(S)$ and therefore $Cg = H$.

Now we will prove that $H$ is compact. $\text{co}C = C_0 \subset C_1 = \text{co}g(C_0) \subset \text{co}g(Cg)$ implies that $Cg = \text{co}(\text{co}C \cup \text{co}g(Cg)) = \text{co}g(Cg)$ by Lemma 1. The definition of a measure of noncompactness on $E$ implies that

$$\gamma(Cg) = \gamma(Cg) = \gamma(\text{co}g(Cg)) = \gamma(g(Cg)).$$

Since $g$ is sequentially $\gamma$-condensing, $g(Cg)$ is relatively compact. Notice that in a quasi-complete Hausdorff topological vector space the notions of precompactness and relative compactness coincide; see [9]. From the regularity of $\gamma$ it follows that $\gamma(Cg) = \gamma(g(Cg)) = 0$ and hence $Cg$ is compact. Thus, $H$ is compact.

Finally, since $f(H \cap B) \subset f(H) \subset H$ and so inductively $f^j(H \cap B) \subset H$ for all $j \geq 0$, we have $H \in F(S)$ and so $A \subset H$, by definition of $A$. Since $H$ is compact, the closed set $A$ is compact. In particular, $A \in F(S)$ implies that $A$ is convex and $f^j(A \cap B) \subset A$ for all $j \geq 0$. This completes the proof. \qed

**Remark 1.** In the proof of Lemma 4, to show that $H$ is compact, we mainly follow the basic line of the one of [3, Lemma 7] in case of Banach spaces, with minor correction, say, $Cg = \text{co}g(Cg)$ instead of $Cg = \text{co}g(Cg)$, although we use a more direct method to prove that $A \in F(S)$.

The following lemma which is originally a result of W.A. Horn holds also for metrizable locally convex topological vector spaces, observing that every convex closed subset of a metrizable locally convex topological vector space is a retract; see [6, Theorem 6] and [4, Theorem 4.1].

**Lemma 5.** Suppose that $S_0 \subset S_1 \subset S_2$ are convex subsets of a metrizable locally convex topological vector space $E$ such that $S_0$ and
$S_2$ are compact and $S_1$ is relatively open in $S_2$. Let $f : S_2 \to E$ be a continuous operator such that there exists an integer $m > 0$ with $f^j(S_1) \subset S_2$ for $0 \leq j \leq m - 1$ and $f^j(S_1) \subset S_0$ for $m \leq j \leq 2m - 1$. Then $f$ has a fixed point.

Using the previous results, we prove a new fixed point theorem for sequentially $\gamma$-condensing operators in an accurate process, based on the proof of [5, Theorem 5].

**Theorem 1.** Suppose that $K \subset B \subset S \subset E$ are convex sets in a Banach space $(E, \| \cdot \|)$ such that $K$ is nonempty and compact, $B$ is relatively open in $S$, and $S$ is closed in $E$. Let $f : S \to E$ be a sequentially $\gamma$-condensing operator such that $f^j(B) \subset S$ for all $j \geq 0$ and $K$ attracts compact sets of $B$, where $\gamma$ is a regular measure of noncompactness on $E$. Then $f$ has a fixed point.

**Proof.** For every $\varepsilon > 0$, let

$$B_\varepsilon(K) = \left\{ y \in E : \| y - x \| < \varepsilon \text{ for some } x \in K \right\}.$$

Then $B_\varepsilon(K)$ is an open neighborhood of $K$ and convex, because of the convexity of $K$. Since $B$ is relatively open in $S$ and $K$ is compact, there is an $\varepsilon > 0$ such that

$$\overline{B_\varepsilon(K)} \cap S \subset B.$$

In fact, for every $x \in B$, there exists an $\varepsilon_x > 0$ such that $B_{\varepsilon_x}(x) \cap S \subset B$. By the compactness of $K$, there are $x_i \in K$ and $\varepsilon_i = \varepsilon_x$ for $i = 1, \ldots, n$ such that $K \subset \bigcup_{i=1}^n (B_{\varepsilon_i}(x_i) \cap S)$. Taking $\varepsilon = \min\left\{\frac{\varepsilon_1}{\varepsilon_1}, \ldots, \frac{\varepsilon_n}{\varepsilon_n}\right\}$, it is easy to verify that $\overline{B_\varepsilon(K)} \cap S \subset \bigcup_{i=1}^n (B_{\varepsilon_i}(x_i) \cap S) \subset B$.

Put $B_0 := B_\varepsilon(K) \cap S$. Then $B_0$ is relatively open in $S$ and $K \subset B_0 \subset S$. Since $f^j(B_0) \subset S$ for all $j \geq 0$ and $K$ attracts points of $B_0$, Lemma 4 implies that there is a compact convex set $A$ with $A \subset S$ such that $f^j(A \cap B_0) \subset A$ for all $j \geq 0$. Set

$$S_0 := \overline{B_\varepsilon(K)} \cap A, \quad S_1 := B_\varepsilon(K) \cap A, \quad \text{and} \quad S_2 := S \cap A.$$

Then $S_0 \subset S_1 \subset S_2$ are convex sets, the sets $S_0, S_2$ are compact, and $S_1$ is relatively open in $S_2$. Since $K$ attracts compact sets of $B$ and $H := \overline{B_\varepsilon(K)} \cap A$ is a compact set of $B$, there exists a positive integer $N(H, \varepsilon)$ such that $f^j(\overline{B_\varepsilon(K)} \cap A) \subset B_{\delta}(K)$ for all $j \geq N(H, \varepsilon)$ and so $f^j(S_1) \subset \overline{B_\varepsilon(K)}$. Hence it follows from $f^j(S_1) = f^j(A \cap B_0) \subset A$ that $f^j(S_1) \subset S_0$ for all $j \geq N(H, \varepsilon)$. Moreover, we have $f^j(S_1) = \overline{B_\varepsilon(K)} \cap A.$
\( f^j(A \cap B_0) \subset A = S_2 \) for \( 0 \leq j \leq N(H, \varepsilon) \). By Lemma 5, the restriction \( f|_{S_2} \) has a fixed point and so does \( f \). This completes the proof. \( \square \)

The following is a result of J. K. Hale and O. Lopes on condensing operators when \( \gamma \) is the Kuratowski measure of noncompactness on \( E \); see [5, Corollary 1].

**Corollary 1.** Let \( E, K, B, \) and \( S \) be as in Theorem 1. Let \( f : S \to E \) be a continuous operator that satisfies the condition:

\[
\gamma(f(A)) < \gamma(A) \quad \text{for every bounded } A \subset E
\]

with \( \gamma(A) > 0 \) and \( f(A) \) bounded,

where \( \gamma \) is the Hausdorff or Kuratowski measure of noncompactness on \( E \). If \( f^j(B) \subset S \) for all \( j \geq 0 \) and \( K \) attracts compact sets of \( B \), then \( f \) has a fixed point.

**Proof.** This is an immediate consequence of Theorem 1 since \( f \) is clearly sequentially \( \gamma \)-condensing and \( \gamma \) is regular; see [1]. \( \square \)

**Corollary 2.** Let \( E, K, B, \) and \( S \) be as in Theorem 1. Let \( f : S \to E \) be a continuous operator such that \( f^j(B) \subset S \) for all \( j \geq 0 \) and \( K \) attracts compact sets of \( B \). If \( f(S) \) is relatively compact, then \( f \) has a fixed point.

**Proof.** Apply Theorem 1 because \( f \) is sequentially \( \gamma \)-condensing, where \( \gamma \) is a regular measure of noncompactness on \( E \). \( \square \)

Thus, Theorem 1 includes many of known results, as well as Theorem A. For related results and its applications to flows, we refer to [5, 6].

**References**


Fixed points of sequentially condensing operators


Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Korea
E-mail: iskim@math.skku.ac.kr