A LOCAL APPROXIMATION METHOD
FOR THE SOLUTION OF K–POSITIVE
DEFINITE OPERATOR EQUATIONS

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ABSTRACT. In this paper we extend the definition of K-positive
definite operators from linear to Fréchet differentiable operators.
Under this setting, we derive from the inverse function theorem a
local existence and approximation results corresponding to those of
Theorems 1 and 2 of the authors [8], in an arbitrary real Banach
space. Furthermore, an asymptotically K-positive definite operator
is introduced and a simplified iteration sequence which converges
to the unique solution of an asymptotically K-positive definite op-
erator equation is constructed.

1. Introduction

Let $H_1$ be a dense subspace of a Hilbert space, $H$. An operator $T$ with
domain $D(T) \supseteq H_1$ is called continuously $H_1$ invertible if the range of $T$,
$R(T)$, with $T$ considered an operator restricted to $H_1$ is dense in $H$ and
$T$ has a bounded inverse on $R(T)$. Let $H$ be a complex and separable
Hilbert space and $A$ be a linear unbounded operator defined on a dense
domain $D(A)$ in $H$ with the property that there exists a continuously
$D(A)$–invertible closed linear operator $K$ with $D(A) \subseteq D(K)$, and a
constant $c > 0$ such that

$$\langle Au, Ku \rangle \geq c||Ku||^2, \quad u \in D(A),$$

then $A$ is called $K$–positive definite ($Kpd$) (see e.g. [13]). If $K = I$ (the
identity operator) inequality (1.1) reduces to $\langle Au, u \rangle \geq c||u||^2$, and in
this case, $A$ is called positive definite. If in addition $c = 0$, $A$ is called
positive operator (or accretive operator). Positive definite operators have
been studied by various authors (see, e.g. [1, 2, 3, 6, 7, 15]. It is clear that

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the class of $Kpd$ operators contains among others, the class of positive
definite operators, and also contains the class of invertible operators
(when $K = A$ as its subclasses. Furthermore, Petryshyn [13] remarked
that for a proper choice of $K$, the ordinary differential operators of odd
order, the weakly elliptic partial differential operators of odd order, are
members of the class of $Kpd$ operators. Moreover, if the operators are
bounded, the class of $Kpd$ operators forms a subclass of symmetrically
operators studied by Reid [15].

In [13], Petryshyn proved the following theorem.

**Theorem P.** If $A$ is a $Kpd$ operator and $D(A) = D(K)$, then there
exists a constant $\alpha > 0$ such that for all $u \in D(K)$,

$$||Au|| \leq \alpha ||Ku||.$$ 

Furthermore, the operator $A$ is closed, $R(A) = H$ and the equation
$Au = f$, $f \in H$, has a unique solution.

In the case that $K$ is bounded and $A$ is closed, F. E. Browder [3]
obtained a result similar to the second part of Theorem P.

In [8], the authors extended the notion of a $K$-positive definite ($Kpd$)
operator to real separable Banach spaces, $X$. In particular, if $X$ is a
real separable Banach space with a strictly convex dual, we proved that
the equation $Au = f$, $f \in X$, where $A$ is a $Kpd$ operator with the same
domain as $K$ has a unique solution. Furthermore, if $X = Lp$ (or $l_p$), $p \geq
2$, and is separable, we constructed an iteration process which converges
strongly to this solution.

Precisely, the following theorems were proved in [8].

**Theorem CA1.** Let $X$ be a real separable Banach space with a
strictly convex dual and let $A$ be a $Kpd$ operator with $D(A) = D(K)$.
Suppose that for all $x, y \in D(K)$,

$$\langle Ax, j(Ky) \rangle = \langle Kx, j(j(Ay)) \rangle,$$

then there exists a constant $\omega > 0$ such that for $x \in D(A)$,

$$||Ax|| \leq \omega ||Kx||.$$ 

Furthermore, the operator $A$ is closed, $R(A) = X$ and the equation
$Ax = h$, for each $h \in X$, has a unique solution.

**Theorem CA2.** Suppose $X = Lp$ or $l_p$, $p \geq 2$, and is separable.
Suppose $A : D(A) \subseteq X \to X$ is a $Kpd$ operator with $D(A) = D(K) =$
$R(K)$ and that for all $x, y \in D(A), \langle Ax, j(Ky) \rangle = \langle Kx, j(Ay) \rangle$. Define the sequence $\{x_n\}$ iteratively by

\begin{align*}
(1.2) \quad & x_0 \in D(K) \\
(1.3) \quad & x_{n+1} = x_n + t_n K^{-1} r_n, \ n \geq 0, \\
(1.4) \quad & t_n = \frac{\langle Br_n, j(Kr_n) \rangle}{(p-1)||Br_n||^2}, \text{ where } B = KAK^{-1}
\end{align*}

and

\begin{align*}
(1.5) \quad & r_n = f - Ax_n, \quad f \in R(K).
\end{align*}

If $A$ and $K$ commute, then $\{x_n\}_{n=1}^\infty$ converges strongly to the unique solution of $Ax = f$ in $X$.

In [10], the authors extended the above result to a larger space, the $q$-uniformly smooth Banach spaces.

Let $K$ be a subset of a real Banach space $E$. A map $T : K \to K$ is called a strict contraction if there exists $k \in [0, 1)$ such that $\|Tx - Ty\| \leq k \|x - y\|$, and it is called nonexpansive if, for arbitrary $x, y \in K$, $\|Tx - Ty\| \leq \|x - y\|$. The map $T$ is called pseudocontractive if, for each $x, y \in K$, there exists $j(x - y) \in J(x - y)$ such that

\begin{align*}
\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2.
\end{align*}

In 1972, Goebel and Kirk [11] introduced a class of mappings generalizing the class of nonexpansive operators.

Let $K$ be a nonempty subset of a normed space $E$. A mapping $T : K \to K$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\}, k_n \geq 1$, such that $\lim_{n \to \infty} k_n = 1$, and $\|T^nx - T^ny\| \leq k_n \|x - y\|$ for each $x, y$ in $K$ and for each integer $n \geq 1$.

Later in 1993, Bruck et. al. introduced and studied another class of asymptotic nonexpansive maps. A mapping $T : K \to K$ is called asymptotically nonexpansive in the intermediate sense (see e.g., Bruck et. al. [5]) provided $T$ is uniformly continuous and

\begin{align*}
\lim_{n \to \infty} \sup_{x,y \in K} \{ \sup_{x,y \in K} (\|T^nx - T^ny\| - \|x - y\|) \} \leq 0.
\end{align*}

Asymptotic pseudocontractive operators have also been introduced and studied, first by Schu (see e.g., [16]) and then by a host of other authors, as a generalization of asymptotic nonexpansive maps. $T : K \to K$ is called asymptotically pseudocontractive if there exists a sequence $\{k_n\}$, $k_n \geq 1$, $\lim k_n = 1$ such that

\begin{align*}
\langle T^nx - T^ny, j(x - y) \rangle \leq k_n \|x - y\|^2
\end{align*}
for each $x, y \in K$.

It is easy to see that asymptotically pseudocontractive maps include the asymptotic nonexpansive ones. These classes of maps have been studied by various authors.

Motivated by Goebel and Kirk [11], Bruck et. al. [5] and Schu [16], we now introduce the class of asymptotically $K$-positive definite operators.

**Definition 1.1.** Let $X$ be a Banach space and let $A$ be a linear unbounded operator defined on a dense domain, $D(A)$, in $X$. The operator $A$ will be called asymptotically $K$-positive definite $Kpd$ if there exist a continuously $D(A)$-invertible closed linear operator $K$ with $D(K) \supseteq D(A) \supseteq R(A)$, and a constant $c > 0$ such that for $j(Ku) \in J(Ku)$,

$$
\langle K^{n-1}Au, j(K^n u) \rangle \geq c k_n \|K^n u\|^2,
$$

$u \in D(A),$

where $\{k_n\}$ is a real sequence such that $k_n \geq 1, \lim_{n \to \infty} k_n = 1$.

It is our purpose in this paper to extend the notion of a $kpd$ operator to Fréchet differentiable operators. Under this setting, a local existence theorem and an iterative scheme which converges to the unique solution of the $Kpd$ operator equation in an arbitrary Banach spaces, are derived from the inverse function theorem. Moreover, we introduce and study a new notion-asymptotically $K$-positive definite operators.

2. Preliminaries

Let $E$ be a real normed linear space with dual $E^*$. We denote by $J$ the normalized duality mapping from $E$ to $2^{E^*}$ defined by

$$
Jx = \{f \in E^* : \langle x, f \rangle = ||x||^2 = ||f||^2\},
$$

where $\langle ., . \rangle$ denotes the generalized duality pairing. It is well known that if $E^*$ is strictly convex then $J$ is single-valued and if $E$ is uniformly smooth (equivalently if $E^*$ is uniformly convex) then $J$ is uniformly continuous on bounded subsets of $E$. We shall denote the single-valued duality mapping by $j$. The modulus of smoothness of $E$ is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$
\rho_E(\tau) := \sup\left\{ \frac{||x + y|| + ||x - y||}{2} - 1 : ||x|| = 1, ||y|| = \tau \right\}.
$$

$E$ is said to be uniformly smooth if $\lim_{\tau \to 0^+} \frac{\rho_E(\tau)}{\tau} = 0$.

**Lemma 2.1.** (see, e.g., [14]) Let $E$ be a real uniformly smooth Banach space and let $J$ be the normalized duality map on $E$. Then for any given
$x, y \in E$, the following inequality holds:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x) \rangle + \max\{\|x\|, 1\}\|y\|\|b(||y||), \forall j(x) \in J(x),$$

where $b$ is a continuous nondecreasing function satisfying the conditions: $b(0) = 0$, $b(ct) \leq cb(t), \forall c \geq 1$, where $b$ is a continuous nondecreasing function satisfying the conditions: $b(0) = 0$, $b(ct) \leq cb(t), \forall c \geq 1$.

3. Main results

Now, we state the inverse function theorem and sketch its proof. We derive from the proof of the theorem that the iteration scheme in Theorem 2 of [8] converges to the unique solution of $Ax = f$ in an arbitrary real Banach space, provided $\|f - Ax_0\|$ is sufficiently small.

**Theorem 3.1.** (The inverse function Theorem) Suppose $X, Y$ are Banach spaces and $A : X \to Y$ is such that $A$ has uniformly continuous Fréchet derivatives in a neighborhood of some point $x_0$ of $X$. Then if $A'(x_0)$ is a linear homeomorphism of $X$ onto $Y$, then $A$ is a local homeomorphism of a neighborhood $U(x_0)$ of $x_0$ to a neighborhood of $A(x_0)$.

**Proof.** Let $A(x_0) = y_0$. We first determine $\rho$ so that $A(x_0 + \rho) = y$ provided $\|y - y_0\|$ is sufficiently small, or equivalently

(3.1) $$A(x_0 + \rho) - A(x_0) = y - y_0.$$  

Since $A$ is $C^1$ at $x_0$ and $A'(x_0)$ is invertible, then (3.1) and Taylor’s Theorem imply that $A'(x_0)\rho + R(x_0, \rho) = y - y_0$, i.e.,

$$\rho = [A'(x_0)]^{-1}(y - y_0) - R(x_0, \rho),$$

where the remainder

$$R(x_0, \rho) = A(x_0 + \rho) - A(x_0) - A'(x_0)\rho = o(||\rho||).$$

We show that (3.1) has one and only one solution for $||\rho||$ sufficiently small, by proving that the operator

$$T \rho = [A'(x_0)]^{-1}\{y - y_0 - R(x_0, \rho)\}$$
is a contraction mapping of a sphere $S(0, \epsilon)$ in $X$ into itself, for some $\epsilon$ sufficiently small. For any $\rho_1, \rho_2 \in S(0, \epsilon)$,

$$\begin{align*}
A'(x_0)(T\rho_2 - T\rho_1) &= R(x_0, \rho_1) - R(x_0, \rho_2) \\
&= A(x_0 + \rho_1) - A(x_0 + \rho_2) - A'(x_0)(\rho_1 - \rho_2) \\
&= \int_0^1 \{A'(x_0 + t\rho_1 + (1 - t)\rho_2) - A'(x_0)\}(\rho_1 - \rho_2)dt.
\end{align*}$$

Hence

$$\begin{align*}
(3.2) & \quad \|T\rho_2 - T\rho_1\| \\
& \leq \int \|A'(x_0)^{-1}\|\|A'(x_0 + t\rho_1 + (1 - t)\rho_2) - A'(x_0)\|\|\rho_1 - \rho_2\|dt.
\end{align*}$$

Since $A$ is a $C^1$ mapping, the middle term of the last integral can be made arbitrarily small by choosing $\|\rho_1\|, \|\rho_2\|$ sufficiently small; and hence for some constant $0 \leq \alpha < 1$ (and independent of $y - y_0$) and sufficiently small $\epsilon > 0$, $\|T\rho_2 - T\rho_1\| \leq \alpha\|\rho_2 - \rho_1\|$ for all $\rho_1, \rho_2 \in S(0, \epsilon)$. Furthermore, $T$ maps $S(0, \epsilon)$ into itself. For, if $\|T\rho\| = \|T\rho - T(0)\| + \|T(0)\| \leq \alpha\|\rho\| + \|T(0)\|$ and $\|T(0)\| = \|A'(x_0)^{-1}(y - y_0)\| < (1 - \alpha)\epsilon$ provided $\|y - y_0\| < (1 - \alpha)\epsilon\|A'(x_0)^{-1}\|^{-1}$. Hence $T$ is a contraction map of $S(0, \epsilon)$ into itself. By the contraction mapping theorem, $T$ has a unique fixed point $\rho^*$ in $S(0, \delta)$ where $\delta \leq \epsilon$ is chosen so small that $A(S(0, \delta) \subset S(y_0, (1 - \alpha)\epsilon\|A'(x_0)^{-1}\|^{-1})$. Reversing the steps in the argument, one finds that $A(x_0 + \rho) = y$ has one and only one solution when $\|y - y_0\|$ and $\|\rho\|$ are sufficiently small. Also, $A^{-1}(y) = x$ is a well-defined and continuous mapping from a sphere $S(y_0, \eta)$ in $Y$ to $X$. \hfill \Box

**Corollary 3.2.** Under the conditions of Theorem 3.1, the iteration sequence

$$x_{n+1} = x_n + [A'(x_0)]^{-1}r_n, \quad r_n = [y - A(x_n)],$$

converges to the unique solution of $A(x) = y$ in $U(x_0)$.

**Proof.** Since the operator $T$ in the proof of Theorem 3.1 is a contraction map, the sequence $\rho_n = T\rho_{n-1}$ converges to the unique fixed point of $T$. From Theorem 3.1, for $\|y - A(x_0)\|$ sufficiently small, $A(x) = y$ has a unique solution $x = x_0 + \rho^*$, where $\rho^*$ is the limit of the sequence $\rho_0 = 0, \rho_{n+1} = T\rho_n$. It then follows that the sequence $x_n = x_0 + \rho_n$.
converges to $x_0 + \rho^*$, the unique solution of $A(x) = y$ in $U(x_0)$. Now,

$$x_n = x_0 + \rho_n = x_0 + T\rho_{n-1} = x_0 + [A'(x_0)]^{-1}[y - A(x_0)] - R(x_0, \rho_{n-1})$$
$$= x_0 + [A'(x_0)]^{-1}[y + A'(x_0)\rho_{n-1} - A(x_0 + \rho_{n-1})]$$
$$= x_0 + \rho_{n-1} + [A'(x_0)]^{-1}[y - A(x_{n-1})]$$
$$= x_{n-1} + [A'(x_0)]^{-1}[y - A(x_{n-1})].$$

Henceforth, an operator $A$ defined on a dense domain $D(A)$ of a real Banach space will be called $K$-positive definite if $A$ is Fréchet differentiable and there exist a continuously $D(A)$—invertible closed linear operator $K$ with $D(A) \subseteq D(K)$, and a constant $c > 0$ such that for $j \in J(Ku)$, we have

$$\langle Au, j \rangle \geq ||Ku||^2, \quad u \in D(A).$$

**Corollary 3.3.** Suppose $A$ is a $Kpd$ operator defined on a dense domain $D(A)$ of a real Banach space, $X$ with range $R(T)$ in $X$. If for some $x_0 \in X$, $A'(x_0)$ is a linear homeomorphism of $X$ onto $Y$, then $A$ is a linear homeomorphism of a neighborhood $U(x_0)$ of $x_0$ to a neighborhood of $A(x_0)$. Furthermore, if $||y - A(x_0)||$ is sufficiently small, the sequence $x_{n+1} = x_n + K^{-1}r_n$, where $r_n = [y - A(x_n)]$ converges to the unique solution of $A(x) = y$ in $U(x_0)$.

**Proof.** $A'(x_0)$ satisfies the condition for $K$ in the definition of a $Kpd$ operator. Hence setting $K = A'(x_0)$ in Theorem 3.1, we are done. \[ \square \]

**Remark 3.4.** If $X$ is a separable Banach space, with a strictly convex dual and the operator $A$ is linear, a global existence result was obtained in the domain of $A$, $D(A)$ in Theorem 1 of [8].

**Remark 3.5.** The iteration scheme $\{x_n\}$ in Corollary 3.3 above corresponds to the one of Theorem 2 in [8] by setting $\iota_n \equiv 1$. In Theorem 2 of [8], the scheme $x_{n+1} = x_n + t_nK^{-1}$ converges globally to the unique solution of $A(x) = y$ in $L^p$ (or $l_p$), $p \geq 2$, while in Corollary 3.2 above the corresponding scheme converges locally to the unique solution of $A(x) = y$ in some neighborhood of a point $x_0$ in a real Banach space $X$. Furthermore, under this setting, the operator $A$ need not be linear but Fréchet differentiable.

By writing our iteration scheme in the form of Theorem CO [10], we prove the following Theorem for asymptotically $K$-positive definite operators in a uniformly convex Banach space.
THEOREM 3.6. Suppose $X$ is a real uniformly smooth Banach space. Suppose $A$ is an asymptotically $K$-positive definite operator defined in a neighborhood $U(x_0)$ of a real uniformly smooth Banach space, $X$. Define the sequence $\{x_n\}$ by $x_0 \in U(x_0)$, $x_{n+1} = x_n + r_n$, $n \geq 0$, $r_n = K^{-1}y - K^{-1}A(x_n)$, $y \in R(A)$. Then $\{x_n\}$ converges strongly to the unique solution of $A(x) = y \in U(x_0)$.

Proof. By the linearity of $K$ we obtain $Kr_{n+1} = Kr_n - Ar_n$. Using Lemma 2.1 and Definition 1.1, we obtain the following estimates:

$$
\begin{align*}
||K^nr_{n+1}||^2 &\leq ||K^nr_n - K^{n-1}Ar_n||^2 \\
&\leq ||K^nr_n||^2 - 2\langle K^{n-1}Ar_n, J(K^nr_n) \rangle \\
&\quad + \max\{||K^nr_n||, 1\}||K^{n-1}Ar_n||b(||K^{n-1}Ar_n||) \\
(3.3) &\leq ||K^nr_n||^2 - 2ck_n||K^nr_n||^2 \\
&\quad + \max\{||K^nr_n||, 1\}||K^{n-1}Ar_n||b(||K^{n-1}Ar_n||) \\
&\leq ||K^nr_n||^2 - 2ck_n||K^nr_n||^2 \\
&\quad + (||K^nr_n|| + 1)||K^{n-1}Ar_n||b(||K^{n-1}Ar_n||).
\end{align*}
$$

Since $A$ is Fréchet differentiable and by the properties of the function $b$, the quantity $||K^{n-1}Ar_n||b(||K^{n-1}Ar_n||)$ can be made as small as possible in a small neighborhood $U(x_0)$ of $X$. Infact there exists $c$ such that

$$
(3.4) ||K^{n-1}Ar_n||b(||K^{n-1}Ar_n||) \leq ck_n||K^nr_n||^2.
$$

Inequality (3.4) implies that the sequence $||K^nr_n||_{n=0}^\infty$ is monotone decreasing and hence converges to some real number $\beta \geq 0$. Inequalities (3.3) and (3.4) imply that

$$
\lim_{n \to \infty} ||K^nr_n|| = 0.
$$

Since $K$ is continuously $D(A)$-invertible, this implies that $r_n \to 0$. Since $A$ has a bounded inverse, this implies $x_n \to A^{-1}y$, the unique solution of $Ax = y$ in $U(x_0)$.

References


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