ON THE UNIQUENESS OF ENTIRE FUNCTIONS

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ABSTRACT. In this paper, we study the uniqueness of entire functions and prove the following result: Let $f(z)$ and $g(z)$ be two nonconstant entire functions, $n \geq 7$ a positive integer, and let $a$ be a nonzero finite complex number. If $f^{n}(z)(f(z) - 1)f'(z)$ and $g^{n}(z)(g(z) - 1)g'(z)$ share a CM, then $f(z) \equiv g(z)$. The result improves the theorem due to ref. [3].

1. Introduction and notations

Let $f(z)$ be a nonconstant meromorphic function in the whole complex plane. We use the following standard notation of value distribution theory,

$$T(r, f), m(r, f), N(r, f), \overline{N}(r, f), \cdots$$

(see Hayman [1], Yang [2]). We denote by $S(r, f)$ any function satisfying

$$S(r, f) = o\{T(r, f)\},$$

as $r \rightarrow +\infty$, possibly outside of a set with finite measure.

Let $a$ be a finite complex number. We denote by $N_{k}(r, \frac{1}{f-a})$ the counting function for zeros of $f(z) - a$ with multiplicity at most $k$, and by $\overline{N}_{k}(r, \frac{1}{f-a})$ the corresponding one for which multiplicity is not counted. Let $N_{k}(r, \frac{1}{f-a})$ be the counting function for zeros of $f(z) - a$ with multiplicity at least $k$ and $\overline{N}_{k}(r, \frac{1}{f-a})$ the corresponding one for which multiplicity is not counted. Set $N_{k}(r, \frac{1}{f-a}) = \overline{N}(r, \frac{1}{f-a}) + \overline{N}(2(r, \frac{1}{f-a}) + \cdots + \overline{N}_{k}(r, \frac{1}{f-a})$. We define

$$\delta_{2}(a, f) = 1 - \lim_{r \rightarrow +\infty} \frac{N_{2}(r, \frac{1}{f-a})}{T(r, f)}.$$
Let $g(z)$ be a meromorphic function, $a$ be a complex number. If $f(z) - a$ and $g(z) - a$ have the same zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share a CM.

In [3], Fang and Hong proved

**Theorem A.** Let $f(z)$ and $g(z)$ be two nonconstant entire functions, $n \geq 11$ be a positive integer, $a$ be a nonzero finite complex number. If $f^n(z)(f(z) - 1)f'(z)$ and $g^n(z)(g(z) - 1)g'(z)$ share a CM, then $f(z) \equiv g(z)$.

In this paper, using different method from [3], we have proved that Theorem A remains valid for $n \geq 7$.

**Theorem 1.** Let $f(z)$ and $g(z)$ be two nonconstant entire functions, $n \geq 7$ be a positive integer, $a$ be a nonzero finite complex number. If $f^n(z)(f(z) - 1)f'(z)$ and $g^n(z)(g(z) - 1)g'(z)$ share a CM, then $f(z) \equiv g(z)$.

2. Some lemmas

For the proof of Theorem 1 we need the following lemmas.

**Lemma 1 ([4]).** Let $f(z)$ be a meromorphic function. Then

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Here $a_n(\neq 0), a_{n-1}, \cdots, a_0$ are constants.

**Lemma 2 ([5, 6]).** Let $f_j(z) (j = 1, 2, \cdots, p)$ be linearly independent meromorphic functions, $p$ a positive integer. If

$$\sum_{j=1}^{p} f_j(z) \equiv 1,$$

then for $1 \leq j \leq p$

$$T(r, f_j) \leq \sum_{i=1}^{p} N(r, \frac{1}{f_i}) + N(r, f_j) + N(r, W) - \sum_{i=1}^{p} N(r, f_i) - N(r, \frac{1}{W}) + S(r),$$

where $W(f_1, f_2, \cdots, f_p)$ is the Wronskian determinant of $f_j(z) (j = 1, 2, \cdots, p)$,

$$S(r) = o(T(r)), (r \to \infty, r \notin E).$$
Here
\[ T(r) = \max_{1 \leq j \leq p} \{ T(r, f_j) \}, \]
and \( E \) is a set of finite measure.

By Lemma 2 we can easily obtain

**Lemma 3.** Let \( f_j(z) \ (j = 1, 2, \ldots, p) \) be linearly independent transcendental entire functions, \( p \) a positive integer. If
\[ \sum_{j=1}^{p} f_j(z) \equiv 1, \]
then for \( 1 \leq j \leq p \)
\[ T(r, f_j) \leq \sum_{i=1}^{p} N_{p-1}(r, \frac{1}{f_i}) + S(r). \]

Here \( S(r) \) is the same as in Lemma 2.

**Lemma 4.** Let \( f_j(z)(j = 1, 2, 3) \) be transcendental entire functions.
If \( f_1(z) + f_2(z) + f_3(z) \equiv 1, \) then
\[ \delta_2(0, f_1) + \delta_2(0, f_2) + \delta_2(0, f_3) \leq 2. \]

Here \( \delta_2(0, f_j) = 1 - \lim_{r \to \infty} \frac{N_2(r, \frac{1}{f_j})}{T(r, f_j)} \) \( (j = 1, 2, 3) \).

**Proof.** We consider two cases.

**Case 1.** \( f_1, f_2, f_3 \) are linearly independent functions. Then by lemma 3 we have
\[ T(r, f_j) \leq \sum_{i=1}^{3} N_2(r, \frac{1}{f_i}) + S(r) \leq \sum_{i=1}^{3} (1 - \delta_2(0, f_i)) T(r, f_i) + S(r). \]

Thus we obtain
\[ T(r) \leq \sum_{i=1}^{3} (1 - \delta_2(0, f_i)) T(r) + S(r). \]

That is
\[ \left( \sum_{i=1}^{3} \delta_2(0, f_i) - 2 \right) T(r) \leq S(r). \]

Hence we get
\[ (2.1) \quad \sum_{i=1}^{3} \delta_2(0, f_i) \leq 2. \]
Case 2. $f_1, f_2, f_3$ are linearly dependent functions. Without loss of
generality, we assume that $f_1, f_2$ are linearly independent functions and
that $f_3 = c_1 f_1 + c_2 f_2$, where $c_1, c_2$ are constants. Hence we have

$$(1 + c_1) f_1(z) + (1 + c_2) f_2(z) \equiv 1.$$ 

Obviously $1 + c_1 \neq 0, 1 + c_2 \neq 0$. Then by the same argument as do in
case 1 we obtain

$$\Theta(0, f_1) + \Theta(0, f_2) \leq 1.$$ 

Considering $\Theta(0, f_i) \geq \delta_2(0, f_i) \quad (i = 1, 2)$ we obtain

$$(2.2) \quad \sum_{i=1}^{3} \delta_2(0, f_i) \leq \Theta(0, f_1) + \Theta(0, f_2) + \delta_2(0, f_3) \leq 2.$$ 

The proof of the lemma is complete. \qed

3. Proof of Theorem 1

By the assumption of the theorem we know that either both $f$ and $g$
are two transcendental entire functions or both $f$ and $g$ are two polynomials.

We first assume that both $f$ and $g$ are transcendental entire functions.
Then by the assumption of the theorem we have

$$(3.1) \quad \frac{f^n (f - 1) f' - \alpha}{g^n (g - 1) g' - \alpha} = e^{h(z)},$$

where $h(z)$ is an entire function. Thus we obtain

$$(3.2) \quad \frac{f^n (f - 1) f'}{a} - \frac{e^{h(z)} g^n (g - 1) g'}{a} + e^{h(z)} \equiv 1.$$ 

We claim that either $\frac{e^{h(z)} g^n (g - 1) g'}{a}$ or $e^{h(z)}$ is not a transcendental
function. Suppose that both $\frac{e^{h(z)} g^n (g - 1) g'}{a}$ and $e^{h(z)}$ are transcendental
functions. Then we have

$$N_2 \left( r, \frac{1}{f^n (f - 1) f'} \right)$$

$$(3.3) \quad \leq N(r, \frac{1}{f}) + N(r, \frac{1}{f - 1}) + N(r, \frac{1}{f'}) + S(r, f)$$

$$\leq \frac{2}{n} \left( n N(r, \frac{1}{f}) + N(r, \frac{1}{f - 1}) + N(r, \frac{1}{f'}) \right)$$
\[ + (1 - \frac{2}{n}) \left( N(r, \frac{1}{f-1}) + N(r, \frac{1}{f'}) \right) + S(r, f). \]

\[ N \left( r, \frac{a}{f^n(f-1)f'} \right) \]

\[ = N(r, \frac{1}{f^n(f-1)f'}) + S(r, f) \]

\[ = nN(r, \frac{1}{f}) + N(r, \frac{1}{f-1}) + N(r, \frac{1}{f'}) + S(r, f). \]

By Lemma 1 we have

\[ (n + 1)T(r, f) = T(r, f^n(f - 1)) + S(r, f) \]

\[ \leq T(r, \frac{f^n(f-1)f'}{a}) + T(r, \frac{1}{f'}) + S(r, f) \]

\[ \leq T(r, \frac{f^n(f-1)f'}{a}) + T(r, f) + S(r, f), \]

thus we have

\[ nT(r, f) \leq T(r, \frac{f^n(f-1)f'}{a}) + S(r, f). \]

By Lemma 1 and (3.5) we have

\[ N(r, \frac{1}{f-1}) + N(r, \frac{1}{f'}) \]

\[ \leq N(r, \frac{1}{f-1}) + T(r, \frac{1}{f'}) + S(r, f) \]

\[ \leq 2T(r, f) + S(r, f) \leq \frac{2}{n}T(r, \frac{f^n(f-1)f'}{a}) + S(r, f). \]

Hence by (3.3)-(3.6) we obtain

\[ N_2(r, \frac{1}{f^n(f-1)f'}) \leq \frac{4n - 4}{n^2} T(r, \frac{f^n(f-1)f'}{a}) + S(r, \frac{f^n(f-1)f'}{a}). \]

Considering \( n \geq 7 \) we get

\[ \lim_{r \to \infty} \frac{N_2(r, \frac{f^n(f-1)f'}{a})}{T(r, \frac{f^n(f-1)f'}{a})} \leq \frac{4n - 4}{n^2} \leq \frac{24}{49}. \]

Thus we have

\[ \delta_2(0, \frac{f^n(f-1)f'}{a}) \geq 1 - \frac{24}{49} = \frac{25}{49}. \]
Likewise, we have
\[ \delta_2(0, -\frac{e^{h(z)}g^n(g-1)g'}{a}) \geq \frac{25}{49}. \]

Obviously,
\[ \delta_2(0, e^{h(z)}) = 1. \]

Thus we have
\[ \delta_2(0, \frac{f^n(f-1)f'}{a}) + \delta_2(0, -\frac{e^{h(z)}g^n(g-1)g'}{a}) + \delta_2(0, e^{h(z)}) > 2. \]

On the other hand, by Lemma 4 we have
\[ \delta_2(0, \frac{f^n(f-1)f'}{a}) + \delta_2(0, -\frac{e^{h(z)}g^n(g-1)g'}{a}) + \delta_2(0, e^{h(z)}) \leq 2. \]

Thus we get a contradiction. Hence we prove that either \( \frac{e^{h(z)}g^n(g-1)g'}{a} \)

or \( e^{h(z)} \) is not a transcendental function. Next we consider two cases.

**Case 1.** \( \frac{e^{h(z)}g^n(g-1)g'}{a} \) is not a transcendental function. In this case we can easily obtain that \( -\frac{e^{h(z)}g^n(g-1)g'}{a} = 1. \) Hence we get \( g^n(g-1)g' = -\frac{a}{e^{h(z)}}. \) Thus by (3.2) we deduce that \( f^n(f-1)f' = -ae^{h(z)}, \) which is a contradiction.

**Case 2.** \( e^{h(z)} \) is not a transcendental function. In this case we can also easily obtain that \( e^{h(z)} = 1. \) By (3.2) we get \( f^n(f-1)f' = g^n(g-1)g', \) that is

\[ \left( \frac{f^{n+2}}{n+2} - \frac{f^{n+1}}{n+1} \right) = \left( \frac{g^{n+2}}{n+2} - \frac{g^{n+1}}{n+1} \right). \]

Hence we obtain

\[ \frac{f^{n+2}}{n+2} - \frac{f^{n+1}}{n+1} = \frac{g^{n+2}}{n+2} - \frac{g^{n+1}}{n+1} + c, \]

where \( c \) is a constant.

We claim that \( c = 0. \) If \( c \neq 0, \) then by Lemma 1 and \( n \geq 7 \) we have

\[ \Theta(0, \frac{f^{n+2}}{n+2} - \frac{f^{n+1}}{n+1}) + \Theta(c, \frac{f^{n+2}}{n+2} - \frac{f^{n+1}}{n+1}) = \Theta(0, \frac{f^{n+2}}{n+2} - \frac{f^{n+1}}{n+1}) + \Theta(0, \frac{g^{n+2}}{n+2} - \frac{g^{n+1}}{n+1}) \geq 2\left(1 - \frac{2}{n+2}\right) = \frac{2n}{n+2} \geq \frac{14}{9} > 1, \]
which contradicts \( \Theta(0, \frac{f^{n+2}}{n+2} - \frac{f^{n+1}}{n+1}) + \Theta(c, \frac{f^{n+2}}{n+2} - \frac{f^{n+1}}{n+1}) \leq 1 \). Thus we deduce that

\[
\frac{f^{n+2}}{n+2} - \frac{f^{n+1}}{n+1} = \frac{g^{n+2}}{n+2} - \frac{g^{n+1}}{n+1}.
\]

Let \( f/g = h \). If \( h \neq 1 \), then by (3.8) we have

\[
g = \frac{(n+2)(1+h+\cdots+h^n)}{(n+1)(1+h+\cdots+h^{n+1})}.
\]

Thus we deduce by Picard’s theorem that \( h(z) \) is a constant. Hence \( g \) is a constant, a contradiction. Therefore we deduce that \( h(z) \equiv 1 \), that is \( f(z) \equiv g(z) \).

Next we assume that both \( f \) and \( g \) are two polynomials. Then by \( f^n(f-1)f' \) and \( g^n(g-1)g' \) share a CM we have

\[
f^n(z)(f(z) - 1)f'(z) - a \equiv k[g^n(z)(g(z) - 1)g'(z) - a],
\]

where \( k \) is a constant.

Thus by (3.9) and \( n \geq 7 \) we deduce that there exists \( z_0 \) such that \( f(z_0) = g(z_0) = 0 \). Substituting this into (3.9) we get \( k = 1 \), that is \( f^n(z)(f(z) - 1)f'(z) \equiv g^n(z)(g(z) - 1)g'(z) \). In the following by using the same argument as in case 2 we get \( f(z) \equiv g(z) \). \( \square \)

References

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