DIMENSIONS OF A DERANGED CANTOR
SET WITH SPECIFIC CONTRACTION RATIOS

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ABSTRACT. We investigate a deranged Cantor set (a generalized
Cantor set) using the similar method to find the dimensions of
cookie-cutter repeller. That is, we will use a Gibbs measure which
is a weak limit of a subsequence of discrete Borel measures to find
the dimensions. The deranged Cantor set that will be considered
is a generalized form of a perturbed Cantor set (a variation of the
symmetric Cantor set) and a cookie-cutter repeller.

1. Introduction

We define a deranged Cantor set [2]. Let $I_\phi = [0,1]$. We can ob-
tain the left subinterval $I_{\tau,1}$ and the right subinterval $I_{\tau,2}$ of $I_\tau$ deleting
middle open subinterval of $I_\tau$ inductively for each $\tau \in \{1,2\}^n$, where
$n = 0,1,2,\cdots$. Consider $E_n = \bigcup_{\tau \in \{1,2\}^n} I_\tau$. Then $(E_n)$ is a decreasing
sequence of closed sets. For each $n$, we put $|I_{\tau,1}|/|I_\tau| = c_{\tau,1}$ and
$|I_{\tau,2}|/|I_\tau| = c_{\tau,2}$ for all $\tau \in \{1,2\}^n$, where $|I|$ denotes the
diameter of $I$. We call $F = \bigcap_{n=0}^\infty E_n$ a deranged Cantor set. We note
that if $c_{\tau,1} = a_{n+1}$ and $c_{\tau,2} = b_{n+1}$ for all $\tau \in \{1,2\}^n$ for each $n$ then
$F = \bigcap_{n=0}^\infty E_n$ is called a perturbed Cantor set [1]. We recall the $s$-
dimensional Hausdorff measure of $F$:

$$H^s(F) = \lim_{\delta \to 0} H^s_\delta(F),$$

where $H^s_\delta(F) = \inf\{\sum_{n=1}^{\infty} |U_n|^s : \{U_n\}_{n=1}^{\infty}$ is a $\delta$-cover of $F\}$, and the
Hausdorff dimension of $F$:

$$\dim_H(F) = \sup\{s > 0 : H^s(F) = \infty\}$$

$$= \inf\{s > 0 : H^s(F) = 0\}$$

(see [3]).

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Also we recall the $s$-dimensional packing measure of $F$:

$$p^s(F) = \inf \{ \sum_{n=1}^{\infty} P^s(F_n) : \bigcup_{n=1}^{\infty} F_n = F \},$$

where $P^s(F_n) = \lim_{\delta \to 0} P^s_\delta(F_n)$ and $P^s_\delta(E) = \sup\{ \sum_{n=1}^{\infty} |U_n|^s : \{U_n\} \text{ is a } \delta\text{-packing of } E \}$, and the packing dimension of $F$:

$$\dim_p(F) = \sup\{ s > 0 : p^s(F) = \infty \} = \inf\{ s > 0 : p^s(F) = 0 \} \quad ([3]).$$

We introduce functions $h^s(F) = \liminf_{n \to \infty} \sum_{\sigma \in \{1,2\}^n} |I_{\sigma}|^s$ and $q^s(F) = \limsup_{n \to \infty} \sum_{\sigma \in \{1,2\}^n} |I_{\sigma}|^s$ for $s \in (0,1)$ and a deranged Cantor set $F$. Clearly $h^s(F)$ and $q^s(F)$ are decreasing functions for $s$.

Using $h^s$ and $q^s$, we define the lower Cantor dimension and the upper Cantor dimension of a deranged Cantor set $F$ by $\dim_{\text{C}}(F) = \sup\{ s > 0 : h^s(F) = \infty \}$ and $\dim_{\text{C}}(F) = \sup\{ s > 0 : q^s(F) = \infty \}$. Then $\dim_{\text{C}}(F) = \inf\{ s > 0 : h^s(F) = 0 \}$ and $\dim_{\text{C}}(F) = \inf\{ s > 0 : q^s(F) = 0 \}$ since $h^s(F)$ and $q^s(F)$ are decreasing functions for $s$. We note $\dim_{\text{C}}$ and $\dim_{\text{C}}$ are just functions whose domains are the class of the deranged Cantor sets. We note that if $c_\tau$ are given, then a deranged Cantor set is determined. We also note that a perturbed Cantor set and a cookie-cutter repeller are special examples of deranged Cantor sets. We are now ready to study the ratio geometry of the deranged Cantor set.

2. Main results

In this section, $F$ means a deranged Cantor set determined by $\{c_\tau\}$ with $\tau \in \{1,2\}^n$ where $n = 1,2,\cdots$. Hereafter we only consider a deranged Cantor set whose contraction ratios $c_{\tau,1}$, $c_{\tau,2}$ and gap ratios $d_{\tau}(\equiv 1 - (c_{\tau,1} + c_{\tau,2}))$ are uniformly bounded away from 0.

**Theorem 1.** Assume that for all $\tau, \sigma \in \{1,2\}^k$ where $k$ is any integer,

$$\frac{c_{\tau,l_1}c_{\tau,l_2}\cdots c_{\tau,l_1,l_2,\cdots,l_m}}{c_{\sigma,l_1}c_{\sigma,l_2}\cdots c_{\sigma,l_1,l_2,\cdots,l_m}} \geq B$$

for all $m$ where $B > 0$ and $\sum_{\sigma \in \{1,2\}^n} |I_\sigma|^s > a$ for some $s > 0$ and for all $n$ where some $a > 0$. Then $H^s(F) > 0$.

**Proof.** We may assume that $c_{\tau}, d_{\tau}(\equiv 1 - (c_{\tau,1} + c_{\tau,2})) > \alpha > 0$ for some small $\alpha$ for all $\tau$. By the assumption, there exists $a > 0$ such that $\sum_{\sigma \in \{1,2\}^n} |I_\sigma|^s > a$ for all $n$. Let $l(I_\sigma)$ be the set of left end points of $I_\sigma$ for $\sigma \in \{1,2\}^n$ where $n = 1,2,\cdots$. 

For each $n$ and any set $A$, we define a discrete measure

$$
\mu_s^n(A) = \frac{\sum_{\tau : \tau \subseteq (I_{\tau}) \cap A \not\subset \emptyset, \tau \in \{1, 2\}^n} |I_\tau|^s}{\sum_{\tau \in \{1, 2\}^n} |I_\tau|^s},
$$

where $n = 1, 2, \ldots$.

Then $\mu_s^n$ is a Borel measure on $[0, 1]$ whose support is in $F = \bigcap_{n=1}^\infty E_n$. Clearly $\mu_s^n([0, 1]) = 1$. By the weak convergence theorem of Borel measure, there is a weak limit $\mu_s$ (which is a Borel measure) of a subsequence of $(\mu_s^n)$ supported by $F$. Now, for $k \leq n$ and $\sigma \in \{1, 2\}^k$,

$$
\mu_s^n(I_\sigma) = \frac{\sum_{\tau : \tau \subseteq (I_{\tau}) \cap I_\sigma \not\subset \emptyset, \tau \in \{1, 2\}^n} |I_\tau|^s}{\sum_{\tau \in \{1, 2\}^n} |I_\tau|^s} \leq \frac{|I_\sigma|^s}{\alpha B^s}.
$$

Considering a suitable open interval containing $I_\sigma$, we easily obtain

$$
\frac{\mu_s(I_\sigma)}{|I_\sigma|^s} \leq \frac{1}{\alpha B^s}.
$$

Let $x \in F = \bigcap_{n=1}^\infty E_n$. Then there is a sequence $(I_{\sigma_n})_{n=1}^\infty$, where $\sigma_n \in \{1, 2\}^n$ such that $\bigcap_{n=1}^\infty I_{\sigma_n} = \{x\}$. Given a small positive number $r$, there exists $n$ such that $|I_{\sigma_{n+1}}| \leq r < |I_{\sigma_n}|$. Since $d_{j+1}|I_{\sigma_j}| \geq \alpha |I_{\sigma_n}| > \alpha r$ for $0 \leq j \leq n$, $B_{\alpha r}(x) \subset \bigcup_{\tau \not\subset \emptyset, \tau \in \{1, 2\}^n} |I_\tau|^c$, where $B_{\alpha r}(x)$ is the ball of radius $\alpha r$ with center $x$. Thus $\mu_s(B_{\alpha r}(x)) \leq \mu_s(I_{\sigma_n})$.

Then

$$
\frac{\mu_s(B_{\alpha r}(x))}{(\alpha r)^s} \leq \frac{\mu_s(I_{\sigma_n})}{\alpha^s |I_{\sigma_{n+1}}|^s} \leq \frac{1}{\alpha B^s \alpha^{2s}}.
$$

Then

$$
\limsup_{r \to 0} \frac{\mu_s(B_r(x))}{r^s} \leq \frac{1}{\alpha B^s \alpha^{2s}}.
$$

Thus $H^s(F) > 0$ by the Hausdorff density theorem (Proposition 4.9 [3] or Proposition 2.2 [4]).

**THEOREM 2.** Assume that for all $\tau, \sigma \in \{1, 2\}^k$ where $k$ is any integer,

$$
\frac{c_{\tau, d_1}c_{\tau, d_2} \cdots c_{\tau, d_{l_2}} \cdots c_{\sigma, d_1}c_{\sigma, d_2} \cdots c_{\sigma, d_{l_2}} \cdots c_{\sigma, l_m}}{c_{\tau, d_1}c_{\sigma, d_2} \cdots c_{\tau, l_2} \cdots c_{\sigma, l_2} \cdots c_{\tau, l_m}} \geq B
$$

for all $m$ where $B > 0$ and $\sum_{\sigma \in \{1, 2\}^n} |I_\sigma|^s < b$ for some $s > 0$ and for all $n$ where some $b < \infty$. Then $p^s(F) < \infty$.

**Proof.** We may assume that $c_\tau, d_\tau (= 1 - (c_{\tau, 1} + c_{\tau, 2})) > \alpha > 0$ for some small $\alpha$ for all $\tau$. By the assumption, there exists $b < \infty$ such that
\[
\sum_{\sigma \in \{1,2\}^n} |I_{\sigma}|^s < b \text{ for all } n. \text{ Let } l(I_{\sigma}) \text{ be the set of left end points of } I_{\sigma} \text{ for } \sigma \in \{1,2\}^n \text{ where } n = 1, 2, \ldots.
\]

For each \(n\) and any set \(A\), we define a discrete measure
\[
\mu^n_s(A) = \frac{\sum_{(\tau : l(I_\tau) \cap A \neq \emptyset, \tau \in \{1,2\}^n)} |I_\tau|^s}{\sum_{\tau \in \{1,2\}^n} |I_\tau|^s},
\]
where \(n = 1, 2, \ldots\).

Then \(\mu^n_s\) is a Borel measure on \([0,1]\) whose support is in \(F = \bigcap_{n=1}^{\infty} E_n\). Clearly \(\mu^n_s([0,1]) = 1\). By the weak convergence theorem of Borel measure, there is a weak limit \(\mu_s\) (which is a Borel measure) of a subsequence of \((\mu^n_s)\) supported by \(F\). Using a similar argument with the proof in Theorem 1, we obtain \(\frac{\mu^n_s(I_{\sigma})}{|I_{\sigma}|^s} \geq \frac{1}{bb^{-s}}\). Noting that \(I_{\sigma}\) is a compact set, we easily obtain \(\frac{\mu_s(I_{\sigma})}{|I_{\sigma}|^s} \geq \frac{1}{bb^{-s}}\).

Let \(x \in F = \bigcap_{n=1}^{\infty} E_n\). Then there is a sequence \((I_{\sigma_n})_{n=1}^{\infty}\), where \(\sigma_n \in \{1,2\}^n\) such that \(\bigcap_{n=1}^{\infty} I_{\sigma_n} = \{x\}\).

Given a small positive number \(r\), there exists \(n\) such that \(|I_{\sigma_{n+1}}| \leq r < |I_{\sigma_n}|\).

Then
\[
\frac{\mu_s(B_r(x))}{r^s} \geq \frac{\mu_s(I_{\sigma_{n+1}})}{|I_{\sigma_n}|^s}.
\]
Since \(|I_{\sigma_{n+1}}|/|I_{\sigma_n}| > \alpha > 0\) for all \(n\),
\[
\frac{\mu_s(B_r(x))}{r^s} \geq \frac{\mu_s(I_{\sigma_{n+1}})}{(\alpha)^s|I_{\sigma_{n+1}}|^s} \geq \frac{\alpha^s}{bb^{-s}}.
\]
Then
\[
\liminf_{r \to 0} \frac{\mu(B_r(x))}{r^s} \geq \liminf_{n \to \infty} \frac{\alpha^s}{bb^{-s}}.
\]
Thus \(p^s(F) < \infty\) by the packing density theorem (Proposition 2.2 [4]).

**Theorem 3.** If \(f\) is a positively oriented cookie-cutter map in the sense that \(f' > 1\), then the repeller of \(f\) satisfies the assumption of Theorems 1 and 2 (in this paper, we assume that the cookie-cutter map \(f\) is of differentiability of class \(C^2\)).

**Proof.** Given integers \(m\) and \(k\), consider \(I_{l_1,\ldots,l_m}\) and \(f^k\). If \(\tau, \sigma \in \{1,2\}^k\), then \((f^k)^{-1}(I_{l_1,\ldots,l_m}) \cap I_{\tau} = I_{\tau,l_1,\ldots,l_m}\) and \((f^k)^{-1}(I_{l_1,\ldots,l_m}) \cap I_{\sigma} = I_{\sigma,l_1,\ldots,l_m}\).

Using the mean value theorem, we easily obtain \(|I_{\tau}| = \frac{1}{(f^k)'(x)}\) and \(|I_{\sigma}| = \frac{1}{(f^k)'(y)}\) for some \(x \in I_{\tau}\) and \(y \in I_{\sigma}\). Similarly we obtain
dimensions of a deranged Cantor set with specific contraction ratios

\[ |I_{\tau_{l_1, \ldots, l_m}}| = \frac{|I_{\tau_{l_1, \ldots, l_m}}|}{(f^k)(z')} \quad \text{and} \quad |I_{\sigma_{l_1, \ldots, l_m}}| = \frac{|I_{\sigma_{l_1, \ldots, l_m}}|}{(f^k)(y')} \]

for some \( \tau \in I_{\tau_{l_1, \ldots, l_m}} \) and \( \tau' \in I_{\sigma_{l_1, \ldots, l_m}} \). Noting that there is a positive number \( B \) such that \( B^{-1} \leq \frac{(f^k)(z')}{(f^k)(z)} \leq B \) for all \( z, z' \in I_{\tau} \) where \( \tau \in \{1, 2\}^k \) (cf. [4]), we easily see that

\[ \sum_{\sigma_{l_1, \ldots, l_m}} I_{\sigma_{l_1, \ldots, l_m}} \frac{|I_{\sigma_{l_1, \ldots, l_m}}|}{|I_{\tau_{l_1, \ldots, l_m}}|} = B^2. \]

Further, we easily see that there are \( a > 0 \) and \( b < \infty \) such that \( a < \sum_{\sigma_{l_1, \ldots, l_m}} I_{\sigma_{l_1, \ldots, l_m}} \leq b \) for all \( n \) (cf. (5.21) in [4]) where \( s \) is the unique real number satisfying the equation that the topological pressure of \(-s \log f'\) is zero (cf. (5.5) and (5.19) in [4]).

**Corollary 4.** Assume that for all \( \tau, \sigma \in \{1, 2\}^k \) where \( k \) is any integer,

\[ \frac{c_{l_1, l_2, \ldots, l_m}}{c_{l_1, l_2, \ldots, l_m}} \geq B \]

for all \( m \) where \( B > 0 \).

Then \( \dim_{C}(F) = \dim_{H}(F) \).

**Proof.** Suppose that \( 0 < s < \dim_{C}(F) \). Then \( h^s(F) = \infty \). Then there exists \( a > 0 \) such that \( \sum_{\sigma_{l_1, \ldots, l_m}} I_{\sigma} \leq a \) for all \( n \). By Theorem 1, \( H^s(F) > 0 \). On the other hand, \( H^s(F) \leq h^s(F) \).

**Corollary 5.** Assume that for all \( \tau, \sigma \in \{1, 2\}^k \) where \( k \) is any integer,

\[ \frac{c_{l_1, l_2, \ldots, l_m}}{c_{l_1, l_2, \ldots, l_m}} \geq B \]

for all \( m \) where \( B > 0 \).

Then \( \dim_{C}(F) = \dim_{p}(F) \).

**Proof.** Suppose that \( \dim_{C}(F) < s \). Then \( q^s(F) = 0 \). Then there exists \( b < \infty \) such that \( \sum_{\sigma_{l_1, \ldots, l_m}} I_{\sigma} \leq b \) for all \( n \). By Theorem 2, \( p^s(F) < \infty \). On the other hand, for \( 0 < t < \dim_{C}(F) \), we have \( p^t(F) = \infty \) using the Baire category theorem (cf. [1], Theorem 5).

**Corollary 6.** If \( f \) is a positively oriented cookie-cutter map in the sense that \( f' > 1 \), then \( 0 < H^s(F) \leq p^s(F) < \infty \) for the repeller \( F \) of \( f \) and the solution \( s \) of the equation \( P(-s \log f') = 0 \) where \( P(\phi) \) is the topological pressure of \( \phi \).

**Proof.** It follows from Theorems 1, 2 and 3.

**Remark 7.** In Corollary 6, we only considered the repeller of a positively oriented cookie-cutter map \( f \), but we also have the same result
for any cookie-cutter map. To avoid technical difficulties, we substitute \( X_\tau \) and \( \gamma_\tau \) for \( I_\tau \) and \( c_\tau \). Let \( X = [0, 1] \), \( X_1 = I_1 \) and \( X_2 = I_2 \). We define \( X_{i_1, \ldots, i_k} = F_{i_1} \circ \cdots \circ F_{i_k}(X) \), where \( f(x) = \begin{cases} F_1^{-1}(x) & \text{if } x \in X_1 \\ F_2^{-1}(x) & \text{if } x \in X_2, \end{cases} \)
and \( \gamma_{\tau, i} = \frac{|X_{\tau, i}|}{|X_\tau|} \), where \( \tau \in \{1, 2\}^n \) and \( i = 1 \) or \( 2 \). Then the repeller of any cookie-cutter map \( f \) satisfies the condition that for all \( \tau, \sigma \in \{1, 2\}^k \) where \( k \) is any integer,
\[
\frac{\gamma_{\tau, l_1} \gamma_{\tau, l_2} \cdots \gamma_{\tau, l_m}}{\gamma_{\sigma, l_1} \gamma_{\sigma, l_2} \cdots \gamma_{\sigma, l_m}} \geq B
\]
for all \( m \) where \( B > 0 \) with \( a < \sum_{\sigma \in \{1, 2\}^n} |X_\sigma|^s < b \) for some \( 0 < a < b < \infty \) for all \( n \) where \( s \) is the solution of the equation \( P(-s \log |f'|) = 0 \). Hence it follows from the same arguments with Corollary 6.

**Remark 8.** In the perturbed Cantor set, we note that
\[
\frac{c_{\tau, l_1} c_{\tau, l_2} \cdots c_{\tau, l_m}}{c_{\sigma, l_1} c_{\sigma, l_2} \cdots c_{\sigma, l_m}} = 1.
\]

**References**


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