POSTNIKOV SECTIONS AND GROUPS OF SELF PAIR HOMOTOPY EQUIVALENCES

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Abstract. In this paper, we apply the concept of the group $\mathcal{E}(X, A)$ of self pair homotopy equivalences of a CW-pair $(X, A)$ to the Postnikov system. By using a short exact sequence related to the group of self pair homotopy equivalences, we obtain the following result: for any Postnikov section $X_n$ of a CW-complex $X$, the group $\mathcal{E}(X_n, X)$ of self pair homotopy equivalences on the pair $(X_n, X)$ is isomorphic to the group $\mathcal{E}(X)$ of self homotopy equivalences on $X$. As a corollary, we have, $\mathcal{E}(K(\pi, n), M(\pi, n)) \cong \mathcal{E}(\pi, n))$ for each $n \geq 1$, where $K(\pi, n)$ is an Eilenberg-McLane space and $M(\pi, n)$ is a Moore space.

1. Introduction

If $X$ is a based topological space, let $\mathcal{E}(X)$ denote the set of homotopy classes of self homotopy equivalences of $X$. Then $\mathcal{E}(X)$ is a group with group operation given by the composition of homotopy classes. The group $\mathcal{E}(X)$ is a fundamental object in the homotopy theory and has been studied extensively by several authors; for instances, M. Arkowitz [1], K. Maruyama [6], J. Rutter [7], N. Sawashita [8] and A. Sieradski [9], et al..

Let $\mathcal{E}(X, A)$ denote the set of pair homotopy classes of self pair homotopy equivalences of a CW-pair $(X, A)$. Then it is a group, a homotopy invariant and this concept is a generalization of that of the group $\mathcal{E}(Y)$ for a CW-complex $Y$. Moreover, for a CW-pair $(X, A)$, there exists a exact sequence

$$1 \rightarrow \mathcal{E}(X, A; id_A) \rightarrow \mathcal{E}(X, A) \rightarrow \mathcal{E}(A),$$

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where $\mathcal{E}(X, A; id_A)$ is the subgroup of $\mathcal{E}(X, A)$ which consists of the pair homotopy classes of the self pair homotopy equivalences such that the restriction to $A$ is the identity on $A$ ([5]). In this paper, we show that for a pair $(X_n, X)$, the sequence (1) becomes a split short exact sequence, where $X_n$ be the $n$-th Postnikov section of a CW-complex $X$. We also show that $\mathcal{E}(X_n, X; id_X)$ is trivial. By the exactness, we obtain the following main results:

**Theorem.** Let $X$ be a CW-complex and $\{X_n\}$ the Postnikov system of $X$. Then for any section $X_n$, the group $\mathcal{E}(X_n, X)$ is isomorphic to $\mathcal{E}(X)$.

**Corollary.** For each $n \geq 1$, $\mathcal{E}(K(\pi, n), M(\pi, n))$ is isomorphic to $\mathcal{E}(M(\pi, n))$, where $K(\pi, n)$ is an Eilenberg-McLane space and $M(\pi, n)$ is a Moore space.

2. The groups of self pair homotopy equivalences and certain exact sequences

In this section, we will introduce some definitions and some theorems in [5] with brief proofs, which are needed to develop our assertion.

In the category of pairs, the “objects” are maps $(X_1, *) \to (X_2, *)$ and “morphism” from $\alpha : X_1 \to X_2$ to $\beta : Y_1 \to Y_2$ is a pair of maps $(f_1, f_2)$ such that the diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\alpha} & X_2 \\
\downarrow f_1 & & \downarrow f_2 \\
Y_1 & \xrightarrow{\beta} & Y_2
\end{array}
$$

is commutative, i.e., $\beta f_1 = f_2 \alpha$. A homotopy of $(f_1, f_2)$ is just a pair of homotopies $(f_{1t}, f_{2t})$ such that $\beta f_{1t} = f_{2t} \alpha$. This category reduces to the category of ordinary pairs of spaces (with base point) if we restrict ourselves to maps $\alpha$ which are inclusions. If $(f_1, f_2)$ is homotopic to $(g_1, g_2)$ by the homotopy $(f_{1t}, f_{2t})$, we denote by

$$(f_{1t}, f_{2t}) : (f_1, f_2) \simeq (g_1, g_2).$$

We denote by $[f_1, f_2]$ the homotopy class of the morphism $(f_1, f_2) : \alpha \to \beta$ and by $\Pi(\alpha, \beta)$ the set of all homotopy classes from $\alpha$ to $\beta$. $(f_1, f_2)$ is called a homotopy equivalent morphism, or simply a homotopy equivalence if there is a morphism $(g_1, g_2)$ such that $(g_1, g_2) \circ (f_1, f_2) \simeq (id_{X_1}, id_{X_2})$ and $(f_1, f_2) \circ (g_1, g_2) \simeq (id_{Y_1}, id_{Y_2})$. Such morphism $(g_1, g_2)$ is called a homotopy inverse of $(f_1, f_2)$. Furthermore, $(f_1, f_2)$ is called
a self homotopy equivalent morphism, or simply a self homotopy equivalence if $\alpha = \beta$ and a self pair homotopy equivalent morphism, or simply a self pair homotopy equivalence if $\alpha = \beta = i : A \to X$ is the inclusion.

**Definition 2.1.** For a given object $\alpha$, we define the subset $\mathcal{E}(\alpha)$ of $\Pi(\alpha, \alpha)$ by

$$\mathcal{E}(\alpha) = \{[f_1, f_2] \in \Pi(\alpha, \alpha) \mid (f_1, f_2) \text{ is a homotopy equivalence}\}.$$ 

Especially, for a CW-pair $(X, A)$, if $\alpha = i : A \to X$ is the inclusion, we denote $\mathcal{E}(i)$ by $\mathcal{E}(X, A)$. If $(f_1, f_2)$ is a morphism from the inclusion $i$ to itself, then $f_1|_A = f_2$. Thus we can consider the morphism $(f_1, f_2)$ as the pair map $f_1 : (X, A) \to (X, A)$. So the group $\mathcal{E}(X, A)$ is just the group of pair homotopy equivalences, i.e.,

$$\mathcal{E}(X, A) = \{[f] \mid f : (X, A) \to (X, A) \text{ is a pair homotopy equivalence}\}.$$ 

We also define the subset $\mathcal{E}(X, A; id_A)$ by

$$\mathcal{E}(X, A; id_A) = \{[id_A, f] \in \mathcal{E}(X, A) \mid id_A \text{ is the identity on } A\}.$$ 

These sets are all groups, homotopy invariants in the category of pairs and generalizations of several concepts of the group of self homotopy equivalences.

**Theorem 2.2.** Let $\alpha : X_1 \to X_2$ be an object in the category of pairs. Then the set $\mathcal{E}(\alpha)$ has a group structure induced by the composition of morphisms.

**Proof.** Let $[f_1, f_2]$ and $[g_1, g_2]$ be elements of $\mathcal{E}(\alpha)$. Then

$$[f_1, f_2] \circ [g_1, g_2] = [f_1 g_1, f_2 g_2] \in \mathcal{E}(\alpha),$$

since $(f_1 g_1, f_2 g_2)$ is a self homotopy equivalent morphism on $\alpha$. For each $[f_1, f_2] \in \mathcal{E}(\alpha)$, let $(h_1, h_2)$ be a homotopy inverse morphism of $(f_1, f_2)$. Then $[h_1, h_2]$ is the inverse element of $[f_1, f_2]$. Moreover, $[id_{X_1}, id_{X_2}]$ is the identity element of $\mathcal{E}(\alpha)$. \hfill $\Box$

**Theorem 2.3.** If $\alpha$ and $\beta$ have same homotopy type, then $\mathcal{E}(\alpha)$ and $\mathcal{E}(\beta)$ are isomorphic.

**Proof.** Suppose that $\alpha : X_1 \to X_2$ and $\beta : Y_1 \to Y_2$ have the same homotopy type by a homotopy equivalent morphism $(e_1, e_2) : \alpha \to \beta$ with the homotopy inverse morphism $(e'_1, e'_2) : \beta \to \alpha$. Define $\Psi : \mathcal{E}(\alpha) \to \mathcal{E}(\beta)$ by

$$\Psi[f_1, f_2] = [(e_1, e_2) \circ (f_1, f_2) \circ (e'_1, e'_2)].$$

Then $\Psi$ is an isomorphism. \hfill $\Box$
REMARK. Let $X$ be a CW-complex and $\alpha : * \to X$ the constant map. Then we have $E(\alpha) = E(X)$. Similarly, if $\alpha : X \to *$ is a constant map, then we have $E(\alpha) = E(X)$. Moreover, for the identity map $id_X : X \to X$, we have $E(id_X) = E(X)$.

Now we fit three groups $E(X, A; id_A), E(X, A)$ and $E(A)$ together into an exact sequence.

**Theorem 2.4.** For a CW-pair $(X, A)$, there exists an exact sequence as follows:

$$1 \to E(X, A; id_A) \to E(X, A) \to E(A).$$

**Proof.** Let $\Phi : E(X, A; id_A) \to E(X, A)$ be the inclusion. Then it is trivial that $\Phi$ is a monomorphism. Define $\Psi : E(X, A) \to E(A)$ by

$$\Psi[f_1, f_2] = [f_1]$$

for $[f_1, f_2] \in E(X, A)$. Then $\Psi$ is well-defined. Let $[f_1, f_2] = [g_1, g_2] \in E(X, A)$. Then there exists a homotopy $(F|_A, F) : i \times id_I \to i$ such that $(F|_A, F) : (f_1, f_2) \simeq (g_1, g_2)$, where $i : A \to X$ is the inclusion and $id_I$ is the identity on the unit interval $[0, 1]$ . Since $F|_A : f_1 \simeq g_1$, we have

$$\Psi[f_1, f_2] = [f_1] = [g_1] = \Psi[g_1, g_2].$$

Furthermore, $\Psi$ is a homomorphism, since the group operations of $E(X, A)$ and $E(A)$ are induced by the composition of maps.

Now we show the exactness at $E(X, A)$. The image of $\Phi$ is contained in the kernel of $\Psi$, since

$$\Psi \Phi[id_A, f] = \Psi[id_A, f] = [id_A] \in E(A).$$

Thus it remains for us to show that the kernel of $\Psi$ is contained in the image of $\Phi$. That is, each element $[f_1, f_2] \in E(X, A)$ such that $[f_1] = [id_A] \in E(A)$ belongs to $E(X, A; id_A)$. Let $[f_1, f_2]$ be such an element. Since $f_1 \simeq id_A$ relative to $*$ in $A$, there exists a homotopy $H : A \times I \to A$ such that $H(a, 0) = f_1(a), H(a, 1) = a$ and $H(*, t) = *$. Then the map $f_2 \sqcup H : X \times 0 \sqcup A \times I \to X$ defined by $(f_2 \sqcup H)|_{X \times 0} = f_2$ and $(f_2 \sqcup H)|_{A \times I} = iH$ has an extension $F : X \times I \to X$. Let $\bar{f} = F(\cdot, 1)$. Then, for each $a \in A$, we have

$$\bar{f}(a) = F(a, 1) = H(a, 1) = a.$$

So $(id_A, \bar{f})$ is a morphism from $i$ to itself, where $i : A \to X$ is the inclusion. But $(f_1, f_2)$ is homotopic to $(id_A, \bar{f})$ by the homotopy $(H, F)$ in the category of pairs. Therefore, $[f_1, f_2] = [id_A, \bar{f}] \in E(X, A; id_A)$.

**Definition 2.5.** The CW-pair $(X, A)$ is called the self-homotopy equivalence extendable pair if for every homotopy equivalence $f : A \to A$,
there exists a homotopy equivalence $\bar{f} : X \to X$ such that $(f, \bar{f}) : i \to i$ is a self homotopy equivalent morphism in the category of pairs, where $i : A \to X$ is the inclusion. In this case, $\bar{f}$ is called a homotopy equivalence extension of $f$.

The following proposition gives a homotopical property of homotopy equivalence extensions.

**Proposition 2.6.** Let $(X, A)$ be a homotopy equivalence extendable pair, and $f$ and $g$ self homotopy equivalences on $A$. If $f$ and $g$ are homotopic relative to $\ast$, then there are homotopy equivalence extensions $\bar{f}$ and $\bar{g}$ of $f$ and $g$ respectively such that $(f, \bar{f})$ and $(g, \bar{g})$ are homotopic in the category of pairs.

**Proof.** Let $H : A \times I \to A$ be a homotopy between $f$ and $g$. Then we have $H(a, 0) = f(a)$, $H(a, 1) = g(a)$ and $H(\ast, t) = \ast$. Since $(X, A)$ is a homotopy equivalence extendable pair, there exists a homotopy equivalence extension $\bar{f} : X \to X$ of $f$. Define $\bar{f} \sqcup iH : X \times 0 \sqcup A \times I \to X$ by $(\bar{f} \sqcup iH)|_{X \times 0} = \bar{f}$ and $(\bar{f} \sqcup iH)|_{A \times I} = iH$, where $i : A \to X$ is the inclusion. Then it is well-defined, since $\bar{f}(a) = f(a) = H(a, 0)$, for each $a \in A$. Since the inclusion $i : A \to X$ is a cofibration, the map $(\bar{f} \sqcup H)$ has an extension $\bar{H} : X \times I \to X$. Define $\bar{g} : X \to X$ by $\bar{g}(x) = \bar{H}(x, 1)$. Then $\bar{g}(a) = \bar{H}(a, 1) = H(a, 1) = g(a)$. So $(g, \bar{g})$ is a morphism. Since $\bar{g}$ is homotopic to $\bar{f}$ by the homotopy $\bar{H}$, $\bar{g}$ is a self homotopy equivalence. Furthermore, we have $(\bar{H}, H) : (\bar{f}, f) \simeq (g, \bar{g})$, since $\bar{H} \circ i = i \circ H$, where $i : A \to X$ is the inclusion. Therefore, $\bar{g}$ is a homotopy equivalence extension of $g$. \hfill \Box

**3. Proof of the main theorem**

Let $X$ be a CW-complex, $X_n$ be the $n$-th Postnikov section of $X$ and $i_n : X \to X_n$ the inclusion. It is a well-known fact that $X_n$ can be obtained by attaching $(i+1)$-cells $(i > n)$ to $X$, so that $X_n$ kills the homotopy groups $\pi_i(X)$ for $i > n$. Thus for every $n \geq 1$, $X_n$ has the following properties:

(a) $(X_n, X)$ is a relative CW-complex with cells in dimensions $\geq n+2$;
(b) $\pi_i(X_n) = 0$ if $i > n$;
(c) $i_n^* : \pi_i(X_n) \to \pi_i(X)$ is an isomorphism if $i \leq n$.

Now we introduce the following proposition needed in this section.

**Proposition 3.1.** ([3], p131) Suppose $S$ is a set of integers and $(Y, X)$ is a relative CW-complex such that if $e_\alpha \subset Y - X$ is a cell, dim
\( e_\alpha \in S. \) Suppose that \( \pi_{i-1}(Z, \ast) = 0 \) for any \( i \in S. \) Then any map \( f : X \to Z \) admits an extension \( \overline{f} : Y \to Z: \)

\[
\begin{array}{c}
Y \\
i \uparrow \ \\ \ \ \ \ \ \ \overline{f} \\
X \quad \to \quad Z \\
\end{array}
\]

**Proposition 3.2.** For each Postnikov section \( X_n, n \geq 1, \) the CW-pair \((X_n, X)\) is a homotopy equivalence extendable pair.

**Proof.** Let \( f : X \to X \) be a self map. Consider the map \( i_n f : X \to X_n. \) Since \( X_n - X \) has cells in dimensions \( \geq n + 2 \) and \( \pi_{i+1}(X_n) = 0 \) for any \( i \geq n, \) \( i_n f \) has an extension \( \overline{f} : X_n \to X_n \) by the Proposition 3.1:

\[
\begin{array}{c}
X_n \\
i_n \uparrow \ \ \ \ \ \ \ \ \ \overline{f} \\
X \quad \to \quad X \\
\end{array}
\]

Thus \((f, \overline{f}) : i_n \to i_n\) is a morphism in the category of pairs. Let us show that \( \overline{f} \) is a self homotopy equivalence. Since \( f \) is a self homotopy equivalence, there exists a homotopy inverse \( g \) and a homotopy \( H : X \times I \to X \) such that \( H(x, 0) = (f \circ g)(x), H(x, 1) = x \) and \( H(\ast, t) = \ast. \) Let \( \overline{g} \) be an extension of \( g \) constructed in the above manner. Define a map

\[
\overline{f} \circ \overline{g} \sqcup i_n H \sqcup \text{id}_{X_n} : X_n \times 0 \sqcup X \times I \sqcup X_n \times 1 \to X_n
\]

by \( \overline{f} \circ \overline{g} \sqcup i_n H \sqcup \text{id}_{X_n} | x_n \times 0 = \overline{f} \circ \overline{g}, \overline{f} \circ \overline{g} \sqcup i_n H \sqcup \text{id}_{X_n} | X \times I = i_n H \) and \( \overline{f} \circ \overline{g} \sqcup i_n H \sqcup \text{id}_{X_n} | x_n \times 1 = \text{id}_{X_n}. \) Since \( X_n \times I - (X_n \times 0 \sqcup X \times I \sqcup X_n \times 1) \) has cells of the form \( e_\alpha^i \times e^1, \) where \( e_\alpha^i \subset X_n - X \) and \( e^1 = I - \{0, 1\}. \) But \( X_n - X \) has cells in dimensions \( \geq n + 2. \) So \( X_n \times I - (X_n \times 0 \sqcup X \times I \sqcup X_n \times 1) \) has cells in dimensions \( \geq n + 3. \) Since \( \pi_{i+1}(X_n) = 0 \) for \( i > n, \) the map \( \overline{f} \circ \overline{g} \sqcup i_n H \sqcup \text{id}_{X_n} \) has an extension \( \overline{H} : X_n \times I \to X_n \) by Proposition 3.1. The extension \( \overline{H} \) is a homotopy between \( \overline{f} \circ \overline{g} \) and \( \text{id}_{X_n} \) relative to \( \ast \) in \( X_n. \) Similarly, \( \overline{g} \circ \overline{f} \) is homotopic to \( \text{id}_{X_n} \) relative to \( \ast \) in \( X_n. \)

Thus \( \overline{g} \) is a homotopy inverse of \( \overline{f} \) and \( \overline{f} \) is an equivalence extension of \( f. \)

**Remark 3.3.** In the proof of the above proposition, we have \( \overline{H}(i_n \times \text{id}_I) = i_n H \) since \( \overline{H} \) is an extension of \( i_n H. \) This means that \((H, \overline{H})\) is a homotopy between \((f, \overline{f}) \circ (g, \overline{g}) \) and \((\text{id}_X, \text{id}_{X_n})\) in the category of pairs. So we have \([f, \overline{f}] \in \mathcal{E}(X_n, X).\)
Theorem 3.4. Let \( X_n \) be the \( n \)-th Postnikov section of \( X \) for each \( n \geq 1 \). Then we have the following split short exact sequence:

\[
1 \to \mathcal{E}(X_n, X; id_X) \xrightarrow{\Phi} \mathcal{E}(X_n, X) \xrightarrow{\Psi} \mathcal{E}(X) \to 1.
\]

where \( \Phi \) is the inclusion and \( \Psi \) is a homomorphism defined by \( \Psi[f, \mathcal{F}] = [f] \).

Proof. By Theorem 2.4, we have the following exact sequence:

\[
1 \to \mathcal{E}(X_n, X; id_X) \xrightarrow{\Phi} \mathcal{E}(X_n, X) \xrightarrow{\Psi} \mathcal{E}(X).
\]

Thus it is sufficient to show that there is a homomorphism \( J : \mathcal{E}(X) \to \mathcal{E}(X_n, X) \) such that \( \Psi \circ J = id_{\mathcal{E}(X)} \). Let \([f]\) be an element in \( \mathcal{E}(X) \).

Then there is a homotopy equivalence extension \( \mathcal{F} \) of \( f \) by Proposition 3.2. Define \( J : \mathcal{E}(X) \to \mathcal{E}(X_n, X) \) by \( \mathcal{J} = [f, \mathcal{F}] \).

Let us show that \( J \) is well-defined. By Proposition 2.6 and Remark 3.3, it is sufficient to show that if \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \) are any two homotopy equivalence extensions of \( f \), then \((f, \mathcal{F}_0)\) and \((f, \mathcal{F}_1)\) are homotopic in the category of pairs. Define a map

\[
\mathcal{F}_0 \sqcup i_n f \sqcup \mathcal{F}_1 : X_n \times 0 \sqcup X \times I \sqcup X_n \times 1 \to X_n
\]

by \( \mathcal{F}_0 \sqcup i_n f \sqcup \mathcal{F}_1 |_{X_n \times 0} = \mathcal{F}_0, \mathcal{F}_0 \sqcup i_n f \sqcup \mathcal{F}_1 |_{X \times I} = i_n f \text{ and } \mathcal{F}_0 \sqcup i_n f \sqcup \mathcal{F}_1 |_{X_n \times 1} = \mathcal{F}_1 \), where \( i_n \) is the inclusion from \( X \) to \( X_n \). By Theorem 3.1, \( \mathcal{F}_0 \sqcup i_n f \sqcup \mathcal{F}_1 \) has an extension \( \mathcal{H} : X_n \times I \to X_n \). Since \( \mathcal{H}(i_n \times id_I) = i_n f \), the pair map \((f, \mathcal{H})\) is a homotopy between \((f, \mathcal{F}_0)\) and \((f, \mathcal{F}_1)\) in the category of pairs.

Moreover, \( \Psi \circ J = id_{\mathcal{E}(X)} \) by definitions of \( \Psi \) and \( J \).

Let us show that \( J \) is a homomorphism. Let \([f]\) and \([g]\) be elements in \( \mathcal{E}(X) \). Since

\[
J([f] \cdot [g]) = J[f \circ g] = [f \circ g, \mathcal{F} \circ g]
\]

and

\[
J[f] \cdot J[g] = [f, \mathcal{F}] \cdot [g, \mathcal{G}] = [f \circ g, \mathcal{F} \circ \mathcal{G}],
\]

we have to show that \((f \circ g, \mathcal{F} \circ g)\) is homotopic to \((f \circ g, \mathcal{F} \circ \mathcal{G})\) in the category of pairs. Let \( H : X \times I \to X \) be the map given by \( H(x, t) = f(g(x)) \) for \((x, t) \in X \times I \). Define a map

\[
\mathcal{F} \circ g \sqcup i_n H \sqcup \mathcal{F} \circ \mathcal{G} : X_n \times 0 \sqcup X \times I \sqcup X_n \times 1 \to X_n
\]

by \((\mathcal{F} \circ g \sqcup i_n H \sqcup \mathcal{F} \circ \mathcal{G})|_{X_n \times 0} = \mathcal{F} \circ g, (\mathcal{F} \circ g \sqcup i_n H \sqcup \mathcal{F} \circ \mathcal{G})|_{X \times I} = i_n H \) and \((\mathcal{F} \circ g \sqcup i_n H \sqcup \mathcal{F} \circ \mathcal{G})|_{X_n \times 1} = \mathcal{F} \circ \mathcal{G} \). By Proposition 3.1, the map \( \mathcal{F} \circ g \sqcup i_n H \sqcup \mathcal{F} \circ \mathcal{G} \) has an extension \( \mathcal{H} : X_n \times I \to X_n \). Since \( \mathcal{H}(i_n \times id_I) = \)
Theorem 3.5. Let $X_n$ be the $n$-th Postnikov section for $n \geq 1$. Then $\mathcal{E}(X_n, X)$ is isomorphic to $\mathcal{E}(X)$.

Proof. By Theorem 3.4, it is sufficient to show that $\mathcal{E}(X_n, X; \text{id}_X)$ is trivial. Let us show that $\mathcal{E}(X_n, X; \text{id}_X) = \{[\text{id}_X, \text{id}_{X_n}]\}$. Let $[\text{id}_X, \tilde{f}]$ be an element in $\mathcal{E}(X_n, X; \text{id}_X)$ and $H : X \times I \to X$ be the map given by $H(x, t) = x$ for $(x, t) \in X \times I$. Define $H' : X_n \times 0 \sqcup X \times I \sqcup X_n \times 1 \to X_n$ by $H'|_{X_n \times 0} = \tilde{f}$, $H'|_{X \times I} = \text{id}_X$, and $H'|_{X_n \times 1} = \text{id}_{X_n}$. By Proposition 3.1, $H'$ has an extension $\overline{H} : X_n \times I \to X_n$. So we have $\overline{H}(x, 0) = \tilde{f}$, $\overline{H}(x, 1) = \text{id}_{X_n}$, $\overline{H}(\ast, t) = \ast$, and $\overline{H}(\text{id}_X \times \text{id}_I) = \text{id}_X$. Therefore, the pair $(H, \overline{H})$ is a homotopy between $(\text{id}_X, \tilde{f})$ and $(\text{id}_X, \text{id}_{X_n})$. This implies $[\text{id}_X, \tilde{f}] = [\text{id}_X, \text{id}_{X_n}]$. □

The Eilenberg-Maclane space $K(\pi, n)$ can be obtained from the Moore space $M(\pi, n)$ by killing homotopy groups of order $\geq n + 1$. That is, $k(\pi, n) = M(\pi, n)_n$. Thus we have following corollary:

**Corollary 3.6.** For each $n \geq 1$, $\mathcal{E}(K(\pi, n), M(\pi, n))$ is isomorphic to $\mathcal{E}(M(\pi, n))$.

We know that $\mathcal{E}(K(\pi, n)) = \text{Aut}(\pi)$, where $\text{Aut}(\pi)$ is the group of automorphisms on $\pi$ [1]. Moreover, it is a well known fact that if a group $\pi$ is non abelian, then $\text{Aut}(\pi)$ is not trivial. Thus for such group $\pi$, $\mathcal{E}(K(\pi, n))$ is not trivial. But $\mathcal{E}(K(\pi, n), M(\pi, n); \text{id}_{M(\pi, n)})$ is always trivial by Theorem 3.4. So $\mathcal{E}(X_n, X; \text{id}_X)$ is not isomorphic to $\mathcal{E}(X_n)$ in general.

**Example 3.7.** It is well-known facts that $\mathcal{E}(\mathbb{R}P^n) \equiv \mathbb{Z}_2 \equiv \mathcal{E}(S^n)$ [1]. Since $\mathbb{R}P^2 = M(\mathbb{Z}_2, 1)$, $\mathbb{R}P^\infty = K(\mathbb{Z}_2, 1)$, $\mathbb{C}P^\infty = K(\mathbb{Z}_2, 2)$ and $S^2 = M(\mathbb{Z}_2, 2)$, we have

$$\mathcal{E}(\mathbb{R}P^\infty, \mathbb{R}P^2) \equiv \mathcal{E}(\mathbb{R}P^2) \equiv \mathbb{Z}_2$$

and

$$\mathcal{E}(\mathbb{C}P^\infty, S^2) \equiv \mathcal{E}(S^2) \equiv \mathbb{Z}_2.$$ 

More generally, since $S^n = M(\mathbb{Z}, n)$, we have

$$\mathcal{E}(K(\mathbb{Z}, n), M(\mathbb{Z}, n)) \equiv \mathcal{E}(K(\mathbb{Z}, n), S^n) \equiv \mathcal{E}(S^n) \equiv \mathbb{Z}_2.$$ 

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